RAIRO. ANALYSE NUMÉRIQUE

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RAIRO. Analyse numérique, tome 11, nº 3 (1977), p. 241-243
<http://www.numdam.org/item?id=M2AN_1977__11_3_241_0>

R.A.I.R.O. Analyse Numérique/Numerical Analysis
(vol. 11, n° 3, 1977, p. 241 à 243)

ON RATIONAL APPROXIMATIONS TO THE EXPONENTIAL (1)

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Communiqué par P.-A. RAVIART

Abstract. — *We give in this paper a simple characterization of the A-acceptability property for a family of rational approximations to e^{-z}. This result, which is implicitely contained in Norsett* [3] *is obtained here in a direct way. As a corollary, we obtain the results of Ehle* [1] *on Pade approximations to e^{-z}.*

Let $r(z) = \dfrac{1 + a_1 z + \ldots + a_n z^n}{1 + b_1 z + \ldots + b_n z^n}$, a_i, $b_i \in \mathbf{R}$, be a rational approximation to e^{-z}. The function $r(z)$ is said to be A-acceptable iff

$$|r(z)| \leqslant 1 \quad \text{for} \quad z \in \mathbf{C}\ \mathrm{Re}\, z \geqslant 0. \tag{1}$$

This paper is devoted to the proof of the following.

THEOREM : *Assume that the rational function $r(z)$ satisfies*

$$r(z) = e^{-z} + 0(z^{2n-1}) \quad (z \to 0) \tag{2}$$

Then $r(z)$ is A-acceptable iff

$$|a_n| \leqslant b_n \tag{3}$$

or

$$r(z) \quad \textit{is reducible}. \tag{4}$$

Proof : We set
$$p(z) = 1 + a_1 z + \ldots + a_n z^n$$
$$q(z) = 1 + b_1 z + \ldots + b_n z^n$$

(1) Manuscrit reçu le 4 janvier 1977.
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1. THE CONDITION IS NECESSARY

Assume that $r(z)$ is A-acceptable. Then the poles of $r(z)$ belong to the open half-plane $\operatorname{Re} z < 0$ and we have $\lim_{|z|\to\infty} |r(z)| \leqslant 1$. Moreover, if $r = p/q$ is irreducible, we have $b_n \geqslant 0$ (otherwise the polynomial q would have a positive root) and $|a_n| \leqslant b_n$.

2. THE CONDITION IS SUFFICIENT

Let us prove it by induction. We first notice that the result is clearly true for $n = 1$. Let us assume that the property is true for $n - 1$. We consider two cases :

1st case r is reducible or $a_n = b_n = 0$. Then it follows from (2) that r is the $(n - 1/n - 1) - th$ Padé approximation to e^{-z} and, from Hummel and Seebeck [2] and Padé [4] $r(z)$ may be written on the form $\dfrac{A_{n-1}(-z)}{A_{n-1}(z)}$ with

$$A_{n-1}(z) = \sum_{k=0}^{n-1} \frac{(2n-2-k)!\,(n-1)!}{(2n-2)!\,k!\,(n-1-k)!} z^k .$$

Therefore r satisfies the theorem hypothesis for $n - 1$ and it follows from the induction hypothesis that r is A-acceptable.

2nd case $b_n \neq 0$ *and* $|a_n| \leqslant b_n$. We set $(a, b) = (a_1, \ldots, a_n, b_1, \ldots, b_n)$ and $E_n = \{ (a, b) ; r(z) = e^{-z} + 0(z^{2n-1}), |a_n| \leqslant b_n \text{ and } b_n \neq 0 \}$.

The subset E_n of $\mathbf{R}^{2n}$ is convex and therefore connected.

For all $(a, b) \in E_n$ we have

$$|r(iy)|^2 = \frac{1 + \alpha_1 y^2 + \alpha_2 y^4 + \ldots + \alpha_n y^{2n}}{1 + \beta_1 y^2 + \beta_2 y^4 + \ldots + \beta_n y^{2n}} = 1 + 0(y^{2n-1})$$

and then we have $\alpha_1 = \beta_1, \ldots, \alpha_{n-1} = \beta_{n-1}$. From the inequality $|a_n| \leqslant b_n$ it follows that $\alpha_n \leqslant \beta_n$ and therefore we have for all $(a, b) \in E_n$:

$$|r(iy)| \leqslant 1 \quad \text{for all} \quad y \in \mathbf{R}. \tag{5}$$

And so r has no pole on the axis $\operatorname{Re} z = 0$. Therefore q cannot have a root on this axis : indeed $q(iy_0) = 0$ implies $q(-iy_0) = 0$ and $y_0 \neq 0$ for $q(0) = 1$; since iy_0 and $-iy_0$ cannot be poles of r, r may be written $r(z) = p_1(z)/q_1(z)$ with $d^0 p_1 \leqslant n - 2$ and $d^0 q_1 \leqslant n - 2$, which is incompatible with (2).

Now we set

$F_n = \{ (a, b) ; (a, b) \in E_n$ and the roots of q belong to the closed half plane $\operatorname{Re} z \leqslant 0 \}$

Since q has no root on the imaginary axis, we have

$F_n = \{ (a, b) ; (a, b) \in E_n$ and the roots of q belong to the open half plane $\operatorname{Re} z < 0 \}$

We notice that F_n is not empty; indeed the $(n - 1/n - 1) - th$ Padé approximation to e^{-z} is irreducible and it follows from the induction hypothesis that its denominator A_{n-1} has no pole in the half plane $\operatorname{Re} z \geqslant 0$. We set

$$p(z) = A_{n-1}(-z)(1 + z) \quad \text{and} \quad q(z) = A_{n-1}(z)(1 + z);$$

then we have $(a, b) \in F_n$.

From (6) and (7) it follows that F_n is an open and closed subset of E_n and then we have $F_n = E_n$. Therefore if $(a, b) \in E_n$, r is analytic in a neighbourhood of the half plane $\operatorname{Re} z \geqslant 0$; hence from the maximum principle and from (5) it follows that r is A-acceptable.

COROLLARY 1 : *The Padé approximations to* e^{-z} *of type* (n/n), $(n - 1/n)$, $(n - 2/n)$ *are A-acceptable.*

COROLLARY 2 : *The poles of Padé approximations to* e^{-z} *of type* (n/n), $(n - 1/n)$, $(n - 2/n)$ *belong to the open left half plane* $\operatorname{Re} z < 0$.

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