RAIRO. ANALYSE NUMÉRIQUE

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RAIRO. Analyse numérique, tome 11, nº 2 (1977), p. 135-144
<http://www.numdam.org/item?id=M2AN_1977__11_2_135_0>

R.A.I.R.O. Analyse Numérique/Numerical Analysis
(vol. 11, n° 2, 1977, p. 135 à 144)

ERROR ESTIMATES FOR ELASTO-PLASTIC PROBLEMS ([1])

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Communiqué par P. G. CIARLET

Abstract. — *Under some reasonable smoothness assumptions on the displacements, we are able to derive an error estimate of the form* $\|\sigma - \sigma_h\|_{L^2(\Omega)} \leqslant Ch$, *for the approximation of the stress field* σ *in some problems in elasto-plasticity.*

Using the same ideas, we also find a piecewise linear approximation of Mosolov's problem, for which we still get an $0(h)$ *error estimate.*

I. INTRODUCTION

In this paper, we consider the approximation of some stationary elastic-perfectly problems formalized by Duvaut-Lions [7]. Our main purpose is to derive error estimates for the approximation of the stress field σ given by a finite element method, appearing in Mercier [16]. The approximate problems we solve, however, will be in terms of the displacements, which are the natural variables for computation.

This work appears to parallel that of Johnson [12], who considered the derivation of error estimates for evolution problems in plasticity. In this stationary case, we are able to obtain improved error estimates over those derived in [12].

Using some ideas from Johnson [11], we are able to establish the existence of a displacement in $L^{\gamma}(\Omega)$ for a class of problems in stationary elasto-plasticity.

Finally, we apply the method to obtain error estimates for the elasto-plastic torsion, and Mosolov's problem.

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(*) This work was supported under NSF grant, MPS 74-05795.

II. PHYSICAL PROBLEM

Let us consider (as in [7]) a continuous medium $\Omega \subset \mathbf{R}^N$, submitted to body forces inside Ω, and to pressure loads on a part Γ_F of its boundary.

On the other part Γ_U, it is assumed to be fixed.

The stress field $\sigma \in K$, and the displacement field $u \in V$, are shown ([7]) to be solutions, if they exist, of the following relations :

$$(g(\sigma), \tau - \sigma) - (\varepsilon(u), \tau - \sigma) \geqslant 0 \qquad \forall \tau \in K; \tag{1}$$

$$(\sigma, \varepsilon(v)) = L(v) \qquad \forall v \in V; \tag{2}$$

with the following notation :

$$V = \{ v \in (H^1(\Omega))^N \mid v = 0 \text{ on } \Gamma_U \}$$

is the set of the admissible displacements.

$$K = \{ \tau \in Y \mid \tau(x) \in P \text{ a.e.} \}$$

is the convex set of plastically admissible stress fields, where

$$Y = \{ \tau \mid \tau_{ij} \in L^2(\Omega); \tau_{ij} = \tau_{ji}; i, j = 1, \ldots, N \}$$

and P is a fixed closed convex subset of $\mathbf{R}^{N^2}$.

We denote by $|.|$ the euclidean norm of $\mathbf{R}^{N^2}$, and observe that Y is a Hilbert space with the scalar product

$$(\tau, \sigma) = \int_\Omega \sum_{i,j=1}^{N} \sigma_{ij}\tau_{ij}\, dx,$$

and associated norm

$$\|\tau\| = \left(\int_\Omega |\tau|^2 \, dx \right)^{1/2}.$$

$\varepsilon : V \to Y$ is the strain operator given by

$$\varepsilon_{ij}(v) \equiv \frac{1}{2}\left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i} \right).$$

$L(v)$ is the work of the external loads in a "virtual" displacement $v \in V (L \in V')$.

$g : \mathbf{R}^{N^2} \to \mathbf{R}^{N^2}$ is an isomorphism representing the elasticity coefficients (the analogue of (1) in the elastic case would be $\varepsilon(u) = g(\sigma)$.

We make the following monotonicity hypothesis on g, i.e. there exists $\alpha > 0$ such that

$$J(\tau) \equiv \frac{1}{2}(g(\tau), \tau) \geqslant \alpha \|\tau\|^2 \qquad \forall \tau \in Y. \tag{3}$$

We note this implies a coercivity condition on the "complementary energy" $J(\tau)$.

Finally, we introduce the set of statically admissible stress fields

$$M = \{ \tau \in Y : (\tau, \varepsilon(v)) = L(v), \forall v \in V \}.$$

We choose $\tau \in K \cap M$ in (1). (We suppose the set $K \cap M$ is non empty.)

We then eliminate u, and we see that σ is the solution of the problem (P) : Find $\sigma \in K \cap M$ such that

$$J(\sigma) = \inf_{\tau \in K \cap M} J(\tau).$$

Using hypothesis (3), we have the existence and uniqueness of σ. We are not able to prove, in the general case, that there exists a $u \in V$ such that (σ, u) is a solution of (1), (2). However, in a slightly more restrictive case, we are able to prove the existence of a weak solution $u \in [L^{q'}(\Omega)]^N$ (see section IV).

For the derivation of error estimates, we will assume that u satisfies the regularity condition

$$u \in V \cap [H^2(\Omega)]^N \tag{4}$$

From the exact solutions, given by Mandel [14], we see that this hypothesis is not an unreasonable one, provided we are not near plastic collapse.

III. APPROXIMATION

Let us assume for simplicity that Ω is a bounded polytope. Corresponding to each value of a parameter h, $0 < h < 1$, let $\mathcal{T}_h$ be a regular triangularization of Ω by N-simplices T of sides less than or equal to h. Define $V_h \subset V$ as the subspace of functions in V which are continuous on Ω and linear on each T of $\mathcal{T}_h$, and $Y_h \subset Y$ as the subspace of tensors in Y which are constant on each $T \in \mathcal{T}_h$. For properties of such finite element spaces, we refer the reader to [5], [6]. We note that

$$\varepsilon : V_h \to Y_h. \tag{5}$$

Using the above notation, we define our approximate problem

(P_h) : Find $\sigma_h \in K \cap M_h$ such that

$$J(\sigma_h) = \inf_{\tau_h \in K \cap M_h} J(\tau_h),$$

where

$$M_h = \{ \tau_h \in Y_h : (\tau_h, \varepsilon(v_h)) = L(v_h), \forall v_h \in V_h \}.$$

Applying the results of [16], we know that there exists a unique solution σ_h to problem (P_h) and that it converges to σ as $h \to 0$. Our purpose, in this paper, is to derive an error estimate for $\|\sigma - \sigma_h\|$.

THEOREM 1 : *If* $u \in [H^2(\Omega)]^N$, *we have the error estimate*

$$\|\sigma - \sigma_h\| \leqslant Ch \|u\|_{[H^2(\Omega)]^N}$$

where C *is a constant independent of* h, u, *and* σ.

Proof : From (1), we get with $\tau = \sigma_h$

$$(g(\sigma), \sigma_h - \sigma) - (\varepsilon(u), \sigma_h - \sigma) \geqslant 0 \tag{6}$$

and from the definition of σ_h, we have

$$(g(\sigma_h), \tau_h - \sigma_h) \geqslant 0 \qquad \forall \tau_h \in K \cap M_h. \tag{7}$$

Writing $\tau_h - \sigma_h$ as $\tau_h - \sigma + \sigma - \sigma_h$, and adding (7) to (6), we get

$$(g(\sigma - \sigma_h), \sigma_h - \sigma) + (g(\sigma_h), \tau_h - \sigma) - (\varepsilon(u), \sigma_h - \sigma) \geqslant 0 \qquad \forall \tau_h \in K \cap M_h.$$

Hence, applying (3)

$$\alpha \|\sigma - \sigma_h\|^2 \leqslant (g(\sigma_h), \tau_h - \sigma) - (\varepsilon(u), \sigma_h - \sigma). \tag{8}$$

Since $\sigma_h \in M_h$, and $\sigma \in M$, we have

$$(\sigma - \sigma_h, \varepsilon(v_h)) = 0 \qquad \forall v_h \in V_h,$$

so that

$$(\varepsilon(u), \sigma_h - \sigma) = (\varepsilon(u - v_h), \sigma_h - \sigma) \qquad \forall v_h \in V_h.$$

Since

$$\begin{aligned}(\varepsilon(u - v_h), \sigma_h - \sigma) &\leqslant \|\varepsilon(u - v_h)\| \, \|\sigma_h - \sigma\| \\ &\leqslant \frac{1}{2\alpha} \|\varepsilon(u - v_h)\|^2 + \frac{\alpha}{2} \|\sigma_h - \sigma\|^2,\end{aligned}$$

we obtain, after collecting terms, that

$$\frac{\alpha}{2} \|\sigma - \sigma_h\|^2 \leqslant (g(\sigma_h), \tau_h - \sigma) + \frac{1}{2\alpha} \|\varepsilon(u - v_h)\|^2,$$

$$\forall v_h \in V_h, \quad \tau_h \in K \cap M_h. \tag{9}$$

We now choose $\tau_h = \Pi_h \sigma$ where Π_h denotes the projection of $Y \to Y_h$ associated with the norm $\|.\|$. Then

$$(\sigma - \tau_h, \gamma_h) = 0, \qquad \forall \gamma_h \in Y_h. \tag{10}$$

Applying (5) and using the fact that $\sigma \in M$, we see that

$$(\tau_h, \varepsilon(v_h)) = (\sigma, \varepsilon(v_h)) = L(v_h) \qquad \forall v_h \in V_h,$$

and hence $\tau_h \in M_h$. Since Y_h is a space of piecewise constants,

$$\tau_h |_T = \frac{1}{\text{meas}(T)} \int_T \sigma \, dx.$$

Then, since $\sigma \in P$ a.e., and P is closed and convex, we get $\tau_h \in P$ for all $T \in \mathscr{T}_h$. Thus $\tau_h \in K \cap M_h$, and from (10),

$$(g(\sigma_h), \tau_h - \sigma) = 0.$$

Thus (9) becomes

$$\|\sigma - \sigma_h\| \leqslant \frac{1}{\alpha} \|\varepsilon(u - v_h)\| \qquad \forall v_h \in V_h. \tag{11}$$

Using the continuity of ε and the well known approximation properties of the space V_h [5], we obtain

$$\|\sigma - \sigma_h\| \leqslant Ch \|u\|_{[H^2(\Omega)]^N}.$$

IV. REMARKS ON THE EXISTENCE OF A DISPLACEMENT

As in [7], we make the following additional hypotheses. Let $\|.\|$ be the L^∞ norm defined by

$$\|e\|_\infty \equiv \operatorname{ess\,sup}_{x\in\Omega} |e(x)|.$$

We assume

$$\exists \delta > 0 \textit{ and } \chi \in M \textit{ such that } \chi + e \in K \;,\; \forall e \in Y \textit{ with } \|e\|_\infty \leqslant \delta\,. \tag{12}$$

Furthermore, we shall restrict ourselves to the case where

$$\Gamma_F = \emptyset \text{ and where } L(v) = \int_\Omega f\, v\, dx,\ f \in [L^q(\Omega)]^N \text{ with } q = N.$$

Choosing $\chi_h = \Pi_h \chi$, we see that $\chi_h \in M_h$, and using the convexity of P, that χ_h belongs to the relative interior of K in Y_h. We may then apply the Kuhn-Tucker theorem [18] to show the existence of $u_h \in V_h$ such that

$$(g(\sigma_h), \tau_h - \sigma_h) - (\varepsilon(u_h), \tau_h - \sigma_h) \geqslant 0 \qquad \forall \tau_h \in K. \tag{13}$$

We now define $(D\tau)_i = - \sum_{j=1}^{N} \frac{\partial \tau_{ij}}{\partial x_j}$ and notice that $D : Y \to V'$ is the adjoint of ε.

Using the regularity we assumed on L, we see that the solution σ of (P) satisfies

$$- D\sigma + f = 0$$

in the distribution sense on Ω. Then

$$\sigma \in K_1 = \{ \tau \in Y : D\tau \in [L^q(\Omega)]^N \}.$$

We shall now prove the existence of a displacement u which satisfies the following relation

$$(g(\sigma), \tau - \sigma) - (u, D(\tau - \sigma)) \geqslant 0 \qquad \forall \tau \in K_1, \tag{14}$$

which can be considered as a weak formulation of (1).

THEOREM 2 : *Under hypothesis* (12), *the sequence* $\varepsilon(u_h)$ *is bounded in* $[L^1(\Omega)]^{N^2}$. *Hence a subsequence of* u_h *is converging weakly to* $u \in [L^{q'}(\Omega)]^N$ *when* $q' = \frac{N}{N-1}$ *and* (σ, u) *is a solution of* (14).

Proof : Let $e \in Y$ satisfy $\|e\|_\infty \leqslant \delta$ and let χ be as defined in (12). Since $\tau_h = \Pi_h e + \chi_h \in K$, we may use this choise of τ_h in (13) to obtain

$$(g(\sigma_h), \Pi_h e) + (g(\sigma_h), \chi_h - \sigma_h) - (\varepsilon(u_h), \Pi_h e) - (\varepsilon(u_h), \chi_h - \sigma_h) \geqslant 0. \quad (15)$$

Using the definition (10) of Π_h, we can replace $\Pi_h e$ by e everywhere in (15). Since χ_h and $\sigma_h \in M_h$, the last term of (15) is zero. Applying the continuity of g, we get

$$(e, \varepsilon(u_h)) \leqslant (g(\sigma_h), \chi_h - \sigma_h) + C\delta \|\sigma_h\| \quad (16)$$

Since Ω is bounded, σ_h being bounded in Y implies σ_h is also bounded in $[L^1(\Omega)]^{N^2}$. As (16) is true for all $e \in Y$ with $\|e\|_\infty \leqslant \delta$, we get

$$\|\varepsilon(u_h)\|_{[L^1(\Omega)]^{N^2}} \leqslant C.$$

We then apply a result of Strauss [19] to obtain

$$\|u_h\|_{[L^{q'}(\Omega)]^{N^2}} \leqslant C \|\varepsilon(u_h)\|_{[L^1(\Omega)]^{N^2}} \leqslant C.$$

From this, we deduce that a subsequence of u_h (which we still denote by u_h) is converging weakly to u in $[L^{q'}(\Omega)]^N$.

For any $\tau \in K_1$, we choose $\tau_h = \Pi_h \tau$ in (13) and obtain

$$(g(\sigma_h), \sigma_h) \leqslant (g(\sigma_h), \tau_h) - (\varepsilon(u_h), \tau_h - \sigma_h). \quad (17)$$

Now

$$(\varepsilon(u_h), \tau_h) = (\varepsilon(u_h), \tau) = (u_h, D\tau) \to (u, D\tau),$$

and since $\sigma_h \in M_h$,

$$(\varepsilon(u_h), \sigma_h) = (f, u_h) \to (f, u) = (D\sigma, u).$$

Also

$$(g(\sigma_h), \tau_h) = (g(\sigma_h), \tau) \to (g(\sigma), \tau),$$

because σ_h converges to σ, and g is continuous. In the same way $(g(\sigma_h), \sigma_h)$ converges to $(g(\sigma), \sigma)$. Hence letting $h \to 0$ in (17), we obtain (14), which is the desired result.

V. OTHER APPLICATIONS

5.1. Elastic-plastic torsion

This problem is usually formulated as the following minimization problem, where $N = 2$, [7] :

Find $u \in K$ minimizing

$$\frac{1}{2}\|\nabla v\|^2 - (f, v) \quad \text{over } K, \text{ where} \quad (18)$$

$$K = \{ v \in H_0^1(\Omega) : |\nabla v| \leqslant 1 \text{ a.e. in } \Omega \}, \text{ and}$$

$$\|\cdot\| = \|\cdot\|_{[L^2(\Omega)]^{N}}.$$

LEMMA 1 : *Problem* (18) *is equivalent to the problem :*

Find $p \in K_1 \cap M$ *minimizing* $\frac{1}{2}\|p\|^2 - (\varphi, p)$ *over* $K_1 \cap M$, *where* φ *is any solution of* $\operatorname{rot}\varphi \equiv \frac{\partial \varphi_2}{\partial x_1} - \frac{\partial \varphi_1}{\partial x_2} = -f$ (19)

$$K_1 = \{ p \in [L^2(\Omega)]^2 : |p| \leqslant 1 \text{ a.e. in } \Omega \}, \text{ and}$$
$$M = \{ p \in [L^2(\Omega)]^2 : (p, \nabla \Psi) = 0, \forall \Psi \in H^1(\Omega) \}.$$

Proof : The result follows easily by using the fact that $p \in M$ is equivalent to $p = \operatorname{rot} v$ for some $v \in H_0^1(\Omega)$ (see [13]),

$$(\varphi, \operatorname{rot} v) = (f, v), \qquad \forall v \in H_0^1(\Omega)$$
$$|\nabla v| = |\operatorname{rot} v| \qquad \text{for} \quad v \in H_0^1(\Omega)$$

(Recall that when v is a scalar, rot v is the vector deduced from the gradient by a rotation of $+\frac{\pi}{2}$)

REMARK : We note that problem (19) is in fact the original problem (see [7]).

We further note that problem (19) can be derived from the more general problem :

Find $(p, \chi) \in (K_1 \cap M) \times H^1(\Omega)$ satisfying

$$(p, q - p) - (\varphi + \nabla \chi, q - p) \geqslant 0 \qquad \forall q \in K_1. \tag{20}$$

Using a result of Brezis [2], it was proved in [15] that there exists a solution to problem (20), when f is constant.

We will assume, as in section II, that χ, which may be interpreted as a displacement, belongs to $H^2(\Omega)$ We know from [3] that $p \in [H^1(\Omega)]^2$ for $f \in L^2(\Omega)$.

Following the ideas of section III, we approximate problem (19) by the problem

Find $p_h \in K_1 \cap M_h$ minimizing

$$\frac{1}{2}\|p_h\|^2 - (\varphi, p_h) \text{ over } K_1 \cap M_h, \text{ where} \tag{21}$$

$$M_h = \{ p_h \in Y_h : (p_h, \nabla \Psi_h) = 0 \ \forall \Psi_h \in V_h \},$$

Y_h is the subspace of $[L^2(\Omega)]^2$ of piecewise constants, and

V_h is the subspace of $H^1(\Omega)$ of continuous piecewise linear functions.

THEOREM 3 : *If* $\varphi \in [H^1(\Omega)]^2$ *and* $\chi \in H^2(\Omega)$, *then we have the error estimate*

$$\| p - p_h \| \leqslant Ch \, [\|\varphi\|_1 + \|\chi\|_2],$$

where C *is a constant independent of* φ, χ *and* h. ($\|\varphi\|_1$ *is the norm of* φ in $[H^1(\Omega)]^2$ *and* $\|\chi\|_2$ *is the norm of* χ *in* $H^2(\Omega)$).

Proof : Proceeding in an identical fashion to the proof of theorem 1, we easily obtain the estimate

$$\frac{1}{2} \| p - p_h \|^2 \leqslant \frac{1}{2} \| \nabla(\chi - \chi_h) \|^2 + (\varphi, p - q_h) \qquad \forall \chi_h \in V_h,$$

where q_h has been chosen as the $[L^2(\Omega)]^2$ projection of p onto Y_h. Since

$$(\varphi_h, p - q_h) = 0 \qquad \forall \varphi_h \in Y_h,$$

we get

$$\begin{aligned}(\varphi, p - q_h) = (\varphi - \varphi_h, p - q_h) &\leqslant \|\varphi - \varphi_h\| \, \|p - q_h\| \\ &\leqslant Ch_2 \, \|\varphi\|_1 \, \|p\|_1\end{aligned}$$

(using the standard approximation properties of Y_h and the assumed regularity of p and φ). Estimating

$$\|\nabla(\chi_h - \chi)\|^2 \leqslant Ch^2 \, \|\chi\|_2^2$$

as before, we obtain the desired result.

We remark that the approximation given by (21) is not equivalent to the usual direct approximation of problem (18) by piecewise linear finite elements [10], since this would lead to an internal approximation of M, which is not the case here ($M_h \not\subset M$). For the direct approximation, non-optimal error estimates have previously been derived in [8].

5.2. Mosolov's problem [7]

This problem is usually formulated as the following :

Find $u \in H_0^1(\Omega)$ minimizing

$$\frac{1}{2} \| \nabla v \| + j(\nabla v) - (f, v) \text{ over } H_0^1(\Omega), \text{ where} \tag{22}$$

$$j(p) \equiv g \int_\Omega |p| \, dx.$$

Since $\Omega \subset \mathbf{R}^2$, we form an equivalent problem in a similar fashion to lemma 1. We get problem

Find $p \in M$ *minimizing*

$$\frac{1}{2} \|q\|^2 + j(q) - (\varphi, q) \textit{ over } M, \textit{ where } \varphi \textit{ and } M \textit{ are chosen as in section } 5 \cdot 1. \tag{23}$$

Using duality theory, we have that problem (23) is the dual of the problem

$$\sup_{\psi \in H^1(\Omega)} -\frac{1}{2}\|\{|\varphi + \nabla \Psi| - g\}^+\|^2 \quad \text{(see [17])}. \tag{24}$$

Since the problem is coercive in $H^1(\Omega)/\mathbf{R}$, we know that it has a solution $\chi \in H^1(\Omega)$. Hence (p, χ) satisfies the following extremality relation

$$(p, q - p) + j(q) - j(p) - (\varphi + \nabla\chi, q - p) \geqslant 0 \qquad \forall q \in [L^2(\Omega)]^2.$$

We will again assume that $\chi \in H^2(\Omega)$, which is a valid assumption at least for the exact solution computed by Glowinski [9]. Using our general technique once more we approximate (23) by the following problem.

Find $p_h \in M_h$ minimizing

$$\frac{1}{2}\|q_h\|^2 + j(q_h) - (\varphi, q_h) \qquad \text{over} \quad q_h \in M_h, \tag{25}$$

where M_h is defined as in section 5.1.

THEOREM 4 : *If* $\varphi \in [H^1(\Omega)]^2$ *and* $\chi \in H^2(\Omega)$, *then we have the error estimate*

$$\|p - p_h\| \leqslant Ch\,[\|\varphi\|_1 + \|\chi\|_2]$$

Proof : Proceeding in an identical fashion to the proof of theorem 3, we easily obtain the estimate

$$\frac{1}{2}\|p - p_h\|^2 \leqslant Ch^2\,[\|\varphi\|_1 + \|\chi\|_2]^2 + j(q_h) - j(p),$$

where q_h is again the $[L^2(\Omega)]^2$ projection of p onto Y_h. Hence

$$\forall T \in \mathscr{T}_h \qquad q_h|_T = \frac{1}{\operatorname{meas}(T)}\int_T p\,dx,$$

and the convexity of j implies that $j(q_h) \leqslant j(p)$. Thus, we get the desired result.

We remark that this approximation is again different from the direct approximation of (22) for which quasi-optimal error estimates have already been derived [9].

As far as we know, the approximate problem (25) has not been solved numerically. What we should suggest for such a numerical computation is to try to solve directly the approximation of the dual problem (24), when $H^1(\Omega)$ is approximated by V_h, because this problem would be the dual of (25). Furthermore, it is a problem of unconstrained minimization of a differentiable (but not strictly convex) function.

ACKNOWLEDGEMENTS

We wish to thank Professors Nedelec and Raviart for several helpful discussions.

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