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CONVERGENCE ANALYSIS OF FINITE DIFFERENCE SCHEMES FOR SEMI-LINEAR INITIAL-VALUE PROBLEMS

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Summary. — *The approximate solution by finite differences of the initial-value problem for a semi-linear equation* $\partial u/\partial t = Pu + f(x, u)$, *with* $P = P(x, D)$ *a linear partial differential operator and* $x \in R^d$ *is considered. It is proved that under the appropriate existence, smoothness and stability assumptions relative to* L_2, *if the finite difference scheme is accurate of order* μ *then the convergence is* $O(h^{\mu})$. *The analysis is carried out in the Besov space* $B_2^{d/2,1}$ *and uses interpolation of Banach spaces.*

1. INTRODUCTION

In this paper we shall consider the approximate solution of the initial-value problem

$$\frac{\partial u}{\partial t} = P(x, D)u + f(x, u), \qquad t > 0, \quad x \in R^d,$$

$$u(x, 0) = v(x),$$

where $P(x, D)$ is a linear partial differential operator of order M and $f(x, u)$ is a sufficiently smooth function of x and u for u near the range of the solution to be approximated. For $t = nk$, with k a small positive number and n a non-negative integer, the approximation will be $G_k^n v$, where G_k is a finite difference operator of the form

$$G_k v = E_k v + k F_k v,$$

with E_k a linear operator consistent with the linear problem ($f = 0$), based on mesh-size h, with kh^{-M} = constant, and F_k chosen to accomodate the nonlinearity f.

In [1], [2], Ansorge, Hass and Geiger considered the case when the linear initial-value problem is correctly posed and the linear finite difference ope-

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rator E_k is stable, both with respect to the maximum-norm. This would include parabolic problems and scalar first order hyperbolic problems with maximum-norm stable E_k. In contrast, their theory does not cover nontrivial hyperbolic systems and equations of Schrödinger type, since these are correctly posed only in L_2 and not in L_p for $p \neq 2$. Also, even for the scalar hyperbolic case it does nor apply to operators E_k which, like for instance the Lax-Wendroff operator, are stable in L_2 but not in other L_p.

Our purpose here is therefore to treat the case when L_2 is a more natural basic space for the analysis than L_∞. It turns out, however, that L_2 itself is also not suitable if we want to make assumptions on $f(x, u)$ only near the range of the exact solution, since closeness of two functions in L_2 does not imply pointwise closeness. For this reason it is convenient to carry out the analysis in the Besov space $B = B_2^{d/2, 1}$ which is largest the L_2 based Besov space which is contained in L_∞.

Our main result is (Theorem 5.1) that if the linear initial value problem is strongly correctly posed in L_2 (for definitions, *cf.* below) and if G_k is accurate of order μ with E_k strongly L_2 stable, we have, as long as the exact solution exists (with $\| . \|$ the norm in B),

$$\| G_k^n v - u(nk) \| = O(h^\mu) \qquad \text{as} \quad h \to 0,$$

provided that v has $M + \mu$ derivatives in B. For less smooth initial data a correspondingly weaker convergence result holds (Theorem 5.2).

The proofs of our results will use concepts and techniques from the theory of interpolation spaces. For basic material needed in this paper on such spaces and in particular on Besov spaces, *see* e. g. [3], [4], [8] and [9].

We shall beging by discussing in Sections 2 and 3 the initial value problem and its approximation in an abstract Banach space setting. In Sections 4 and 5 we then specify the Banach spaces to the concrete function spaces mentioned above and show that under the appropriate hypotheses about the differential equation and the difference operator, the assumptions of the results in Sections 2 and 3 are satisfied. Sections 4 and 5 also contain some specific examples of situations covered by our theory.

The simple case of the scalar hyperbolic equation

$$\frac{\partial u}{\partial t} = \frac{\partial u}{\partial x} + u^2,$$

and a second order finite difference scheme based on the Lax-Wendroff operator was presented in [12]. The technique of working in $B_2^{d/2, 1}$ to obtain maximum-norm estimates for L_2 stable operators was employed in [13] for linear problems.

2. THE ABSTRACT INITIAL-VALUE PROBLEM

Let B be a Banach space with norm $\| . \|$ and let P be the infinitesimal generator of a strongly continuous semi-group $\{ E(t) : t \geqq 0 \}$ of bounded linear operators on B. We shall study the approximate solution in B of the initial-value problem

$$\frac{du}{dt} = Pu + Fu, \qquad u(0) = v. \tag{2.1}$$

Here v is a fixed element in B and F is a (non-linear) operator defined in some subset of B containing v.

Under the appropriate regularity assumptions on F it is clear that (2.1) has a unique local solution (*cf.* e. g. [11]); we state and prove for completeness and later reference:

LEMMA 2.1: *Assume that F is defined and Lipschitz continuous in a neighborhood V of v. Then there is a positive number T such that* (2.1) *admits a unique solution in* $[0, T]$.

Proof: We write (2.1) in the form

$$u(t) = E(t)v + \int_0^t E(t-s) F u(s)\, ds. \tag{2.2}$$

To prove existence, let δ be so small that $\{ w : \| w - v \| \leqq 2\delta \}$ is contained in V, and let T_0 be so small that $\| E(t)v - v \| \leqq \delta$ for $0 \leqq t \leqq T_0$. Then V contains the δ-neighborhood of $\{ E(t)v : 0 \leqq t \leqq T_0 \}$.

By our assumptions there are constants β_0, σ_0, γ_0 such that

$$\| E(t) w \| \leqq \beta_0 \| w \| \quad \text{for } 0 \leqq t \leqq T_0, \qquad w \in B, \tag{2.3}$$

$$\| F w_1 - F w_0 \| \leqq \sigma_0 \| w_1 - w_0 \| \qquad \text{for} \quad w_1, w_0 \in V, \tag{2.4}$$

$$\| F w \| \leqq \gamma_0 \qquad \text{for} \quad w \in V. \tag{2.5}$$

We now choose $T \leqq T_0$ such that $0 < \beta_0 \gamma_0 T \leqq \delta$ and define recursively

$$u_{n+1}(t) = E(t)v + \int_0^t E(t-s) F u_n(s)\, ds, \qquad u_0(t) = E(t)v.$$

Using (2.3)-(2.5) we find easily by induction that $u_n(t) \in V$ for $0 \leqq t \leqq T$ and that

$$\| u_{n+1}(t) - u_n(t) \| \leqq \gamma_0 \sigma_0^{-1} \frac{(\beta_0 \sigma_0 t)^{n+1}}{(n+1)!}.$$

It follows that $u_n(t)$ converges uniformly on $[0, T]$ to a function $u(t)$ with values in V which satisfies (2.2).

The uniqueness follows in a standard manner from Grönwall's inequality.

From now on we shall assume that (2.1) [or (2.2)] has a solution $u(t) = G(t)v$ for $0 \leqq t \leqq T$ (with T not necessarily small). We denote by U the range of this solution, $U = \{u(t) : 0 \leqq t \leqq T\}$ and by U_δ the closed δ-neighborhood of U. We shall consider δ fixed in the sequel and assume that F is defined on U_δ.

We shall need later to be able to solve (2.1) also with v replaced by w close to v:

LEMMA 2.2: *Assume that F is Lipschitz continuous on U_δ. Then there is a neighborhood V of v such that for $w \in V$, the differential equation in* (2.1) *has a unique solution $\tilde{u}(t) = G(t)w \in U_\delta$ for $0 \leqq t \leqq T$ with $\tilde{u}(0) = w$. Moreover, there is a positive constant ω_0 such that*

$$\|G(t)w - G(t)v\| \leqq \omega_0 \|w - v\| \quad \textit{for } w \in V, \qquad 0 \leqq t \leqq T. \tag{2.6}$$

Proof: With β_0 as in (2.3) with T_0 replaced by T and σ_0 as in (2.4) with V replaced by U_δ, we let

$$\omega_0 = \beta_0 \exp(\beta_0 \sigma_0 T) \qquad \text{and} \qquad V = \{w : \|w - v\| \leqq \delta/\omega_0\}.$$

Defining $\{\tilde{u}_n\}$ by

$$\tilde{u}_{n+1}(t) = E(t)w + \int_0^t E(t-s) F \tilde{u}_n(s)\, ds, \qquad \tilde{u}_0(t) = u(t),$$

we obtain

$$\|\tilde{u}_{n+1}(t) - u(t)\| \leqq \beta_0 \|w - v\| + \beta_0 \sigma_0 \int_0^t \|\tilde{u}_n(s) - u(s)\|\, ds.$$

It follows by induction

$$\|\tilde{u}_n(t) - u(t)\| \leqq \beta_0 \|w - v\| \exp(\beta_0 \sigma_0 t) \leqq \omega_0 \|w - v\|, \tag{2.7}$$

and, in particular, $\{u_n\} \subset U_\delta$. We also find with γ_0 a bound for F in U_δ,

$$\|\tilde{u}_{n+1}(t) - \tilde{u}_n(t)\| \leqq \beta_0 \gamma_0 \frac{(\beta_0 \sigma_0 t)^{n+1}}{(n+1)!},$$

so that $\tilde{u}_n(t)$ converges uniformly to $u(t) \in U_\delta$ which then obviously solves (2.1) with v replaced by w. The estimate (2.6) now follows immediatly from (2.7). The uniqueness is again an immediate consequence of Grönwall's inequality.

We shall now turn to the regularity of the solutions which will be needed in analyzing the concrete finite difference schemes in Sections 4 and 5. For this purpose, let A with norm $\|\cdot\|_A$ be a densely embedded subspace of B. We shall see that under certain assumptions on $E(t)$ and F relative to A, $G(t)w$ belongs to A for w in A and close to v. These assumptions are:

(A i) For $w \in A$ we have $E(t)w \in A$ for $t \geqq 0$ and there is a constant $\beta_1 \geqq 1$ such that

$$\|E(t)w\|_A \leqq \beta_1 \|w\|_A \quad \text{for } w \in A, \qquad 0 \leqq t \leqq T.$$

(A ii) For $w \in U_\delta \cap A$ we have $Fw \in A$ and there is a constant γ_1 such that

$$\|Fw\|_A \leqq \gamma_1(\|w\|_A + 1) \qquad \text{for} \quad w \in U_\delta \cap A.$$

(A iii) For any bounded subset w of A there is a constant σ_1 such that

$$\|Fw_1 - Fw_0\|_A \leqq \sigma_1 \|w_1 - w_0\|_A \qquad \text{for} \quad w_0, w_1 \in U_\delta \cap W.$$

We then have:

LEMMA 2.3: *Assume that F is Lipschitz continuous on U_δ and that* (A i), (A ii) *and* (A iii) *hold. Then there is a neighborhood V of v in B (independent of A) such that for $w \in V \cap A$, $G(t)w$ is defined and in $U_\delta \cap A$ for $0 \leqq t \leqq T$. Moreover, there is a constant τ such that*

$$\|G(t)w\|_A \leqq \tau(\|w\|_A + 1) \quad \textit{for } w \in V \cap A, \qquad 0 \leqq t \leqq T. \tag{2.8}$$

Proof: Let V be a neighborhood of v such that (*cf.* Lemma 2.2) $w \in V$ implies that $\tilde{u}(t) = G(t)w \in U_{\delta/3}$ for $0 \leqq t \leqq T$. We shall prove that there exists a positive T_0 such that if $\tilde{u}(t) \in A$ for $0 \leqq t \leqq T_1 \leqq T$ then $\tilde{u}(t) \in A$ for $0 \leqq t \leqq \min(T_1 + T_0, T)$. This will prove that $\tilde{u}(t) \in A$ for $0 \leqq t \leqq T$.

Put $\tau = \beta_1 \exp(\beta_1 \gamma_1 T)$. Using (A i), (A ii) and Grönwall's inequality in (2.2) with v replaced by w we find at once as long as $\tilde{u}(t) \in A$, in particular for $0 \leqq t \leqq T_1$, we have

$$\|\tilde{u}(t)\|_A + 1 \leqq \beta_1 \exp(\beta_1 \gamma_1 t)(\|w\|_A + 1) \leqq \tau(\|w\|_A + 1). \tag{2.9}$$

Let now w be a fixed element in $V \cap A$ and set

$$\tilde{U} = \{\tilde{u}(t) = G(t)w : 0 \leqq t \leqq T\}.$$

Since $\tilde{U} \subset U_{\delta/3}$ and since $\tilde{U}$ is compact we may determine a positive T_0 such that $E(t)\tilde{u} \in U_{2\delta/3}$ for any $\tilde{u} \in \tilde{U}$ and $0 \leqq t \leqq T_0$, and such that in addition $\beta_0 \gamma_0 T_0 \leqq \delta/3$, where as in Lemma 2.1, β_0 and γ_0 are a stability constant for $E(t)$ in $0 \leqq t \leqq T$ and a bound for F in U_δ.

Set $w_1 = u(T_1) = G(T_1)w$ and define (*cf.* the proof of Lemma 2.1),

$$u_{n+1}(t) = E(t)w_1 + \int_0^t E(t-s)Fu_n(s)\,ds, \qquad u_0(t) = E(t)w_1.$$

We find at once recursively that $u_n(t) \in U_\delta \cap A$ for $0 \leqq t \leqq T_0$ and also that

$$\| u_{n+1}(t) \|_A + 1 \leqq \beta_1 (\| w_1 \|_A + 1) + \beta_1 \gamma_1 \int_0^t (\| u_n(s) \|_A + 1)\, ds.$$

Hence, using also (2.9) we have

$$\| u_n(t) \|_A \leqq \beta_1 \exp(\beta_1 \gamma_1 t)(\| w_1 \|_A + 1) \leqq \beta_1^2 \exp(\beta_1 \gamma_1 (T_1 + T_0))(\| w \|_A + 1).$$

In particular, $\{ u_n(t) \}$ is uniformly bounded in A for $0 \leqq t \leqq T_0$. Using (A iii) we therefore obtain (with σ_1 depending on w),

$$\| u_{n+1}(t) - u_n(t) \|_A \leqq \beta_1 \gamma_1 \sigma_1^{-1} \frac{(\beta_1 \sigma_1 t)^{n+1}}{(n+1)!} (\| w_1 \| + 1),$$

so that $u_n(t)$ converges uniformly in A on $0 \leqq t \leqq T_0$. Clearly, since the limit $\tilde{\tilde{u}}(t)$ satisfies $u(0) = \tilde{u}(T_1) = G(T_1)w$, we have

$$\tilde{\tilde{u}}(t) = G(t + T_1)w = \tilde{u}(t + T_1).$$

Together with (2.9) this completes the proof.

3. THE ABSTRACT DISCRETIZED PROBLEM

We shall now consider the approximate solution of (2.1) defined for $t = nk$ by $G_k^n v$ with k a small positive parameter and n a non-negative integer. Here G_k is an operator approximating $G(k)$ of the form

$$G_k w = E_k w + k F_k w,$$

with E_k bounded linear and F_k defined on U_δ. In applications E_k will approximate $E(k)$ and F_k will be designed to handle the nonlinear operator F in (2.1).

We shall assume below that E_k is stable in B, so that there is a $\beta \geqq 1$ such that

$$\| E_k^n w \| \leqq \beta \| w \| \quad \text{for } w \in B, \qquad nk \leqq T. \tag{3.1}$$

Further, we shall assume that for $k \leqq k_0$, F_k is Lipschitz continuous on U_δ, uniformly in k, so that

$$\| F_k w_1 - F_k w_0 \| \leqq \sigma \| w_1 - w_0 \| \quad \text{for } w_0, w_1 \in U_\delta, \qquad k \leqq k_0. \tag{3.2}$$

For $w \in U_\delta$, we define the local discretization error

$$\varepsilon_k w = k^{-1}(G_k w - G(k)w),$$

and for w in the neighborhood V of Lemma 2.2, the global discretization error

$$\tau_k(w) = k \sum_{nk \leq T} \|\varepsilon_k G(nk) w\|.$$

We shall then be able to prove:

THEOREM 3.1: *Let $u(t) = G(t) v$ be a solution of* (2.1) *for* $0 \leqq t \leqq T$ *and assume in addition to* (3.1) *and* (3.2) *that* $\lim_{k \to 0} \tau_k(v) = 0$. *Then with* $c_0 = \beta \exp(\beta \sigma T)$, *$G_k^n v$ is defined and in U_δ for $nk \leqq T$, $k \leqq k_1$ if $k_1 \leqq k_0$ is so small that $c_0 \tau_k(v) \leqq \delta$ for $k \leqq k_1$. Moreover,*

$$\|G_k^n v - G(nk) v\| \leqq c_0 \tau_k(v).$$

Proof: We shall show by induction over n that $G_k^n v$ is defined and in U_δ for $nk \leqq T$, $k \leqq k_1$ and that with $a_n = G_k^n v - G(nk) v$,

$$\|a_n\| \leqq \beta \exp(\beta\sigma nk) \tau_k(v). \tag{3.3}$$

Since this is trivially valid for $n = 0$, assume now that we have already proved the conclusion for all integers $\leqq n$ and that $(n+1) k \leqq T$. Then in particular, we have $G_k^n v \in U_\delta$ so that $G_k^{n+1} v$ is defined. We may then write, with b_n defined by the second equality,

$$a_{n+1} = E_k a_n + k(F_k G_k^n v - F_k G(nk) v) + k \varepsilon_k G(nk) v = E_k a_n + k b_n,$$

or, since $a_0 = 0$, by (3.1),

$$\|a_{n+1}\| \leqq k \sum_{j=0}^{n} \|E_k^{n-j} b_j\| \leqq \beta k \sum_{j=0}^{n} \|b_j\|.$$

Now (3.2) implies

$$\|F_k G_k^j v - F_k G(jk) v\| \leqq \sigma \|a_j\| \qquad \text{for} \quad j \leqq n,$$

so that

$$\|a_{n+1}\| \leqq \beta\sigma k \sum_{j=0}^{n} \|a_j\| + \beta \tau_k(v).$$

This clearly establishes (3.3) with n replaced by $n+1$. By the choice of k_1 we may also conclude that $G_k^{n+1} v \in U_\delta$, which completes the proof.

In applications, if the initial-values are known to have a certain regularity, it is often possible to derive precise estimates for the discretization errors and hence of the right hand side in (3.3) in terms of k. In the following theorem we shall state such an estimate in which for later use (the proof of Theorem 3.3) we consider initial-values also in a neighborhood of v. We shall assume then that there exists a neighborhood V_0 contained in the neighborhood V of Lemma 2.2, such that if A is a densely embedded subspace of B, then there are positive numbers c_1 and ν such that

$$\| \varepsilon_k G(t) w \| \leq c_1 k^\nu (\| w \|_A + 1) \quad \text{for } w \in V_0 \cap A, \qquad k \leq k_0, \quad t+k \leq T. \quad (3.4)$$

Recall from Lemma 2.3 that under the assumptions (A i), (A ii), (A iii), if $w \in A$ and is close to v we have $G(t) w \in A$ for $0 \leq t \leq T$ and the estimate (2.8) holds. This will be used in the verification of (3.4) for the concrete difference schemes in Sections 4 and 5.

THEOREM 3.2: *Assume that F is Lipschitz continuous on U_δ and that G_k satisfies* (3.1), (3.2) and (3.4). *Then there is a neighborhood V of v in B, a positive constant c_2 and for each $w \in V \cap A$ a positive k_2 such that for $k \leq k_2$ and $t = nk \leq T$, $G(t) w$ and $G_k^n w$ are defined and in U_δ and*

$$\| G_k^n w - G(nk) w \| \leq c_2 k^\nu (\| w \|_A + 1). \quad (3.5)$$

For k_2 we may take any number with $k_2 \leq k_0$ and $c_2 k_2^\nu (\| w \|_A + 1) \leq \delta/2$.

Proof: By Lemma 2.2 we may choose $V \subset V_0$ such that for $w \in V$, $G(t) w$ is defined and in $U_{\delta/2}$ for $0 \leq t \leq T$. Setting $\tilde{U} = \{ G(t) w : 0 \leq t \leq T \}$ we find for all such w that $\tilde{U}_{\delta/2} \subset U_\delta$. Moreover, for $w \in V$ we have by (3.4),

$$c_0 \tau_k(w) \leq c_0 c_1 T k^\nu (\| w \|_A + 1) = c_2 k^\nu (\| w \|_A + 1).$$

It hence follows from Theorem 3.1 that $G_k^n w$ is defined and in $\tilde{U}_{\delta/2} \subset U_\delta$ for $k \leq k_2$, $nk \leq T$ and that (3.5) holds, which proves the theorem.

In particular, if $v \in A$ we may apply Theorem 3.2 to $w = v$ and obtain then a $0(k^\nu)$ global error estimate for small k.

We shall conclude this section by deriving a convergence estimate for initial data in a space which is intermediate between B and A. In order to define such spaces we introduce for any pair of Banach spaces B_0 and B_1 with $B_1 \subset B_0$ the functional

$$K(t, v; B_0, B_1) = \inf_{w \in B_1} (\| v - w \|_{B_0} + t \| w \|_{B_1}) \quad \text{for } w \in B_0, \qquad t > 0.$$

For $0 < \theta < 1$ the Banach space $(B_0, B_1)_{\theta, \infty}$ is then defined by the norm

$$\| v \|_{(B_0, B_1)_{\theta, \infty}} = \sup_{t > 0} t^{-\theta} K(t, v; B_0, B_1).$$

We shall prove:

THEOREM 3.3: *Under the assumptions of Theorem 3.2, let $v \in A_\theta = (B, A)_{\theta, \infty}$ for some θ with $0 < \theta < 1$. Then there are constants c_3 and k_3 such that $G_k^n v$ is defined and in U_δ for $nk \leq K$, $k \leq k_3$ and*

$$\| G_k^n v - G(nk) v \| \leq c_3 k^{\theta\nu}. \quad (3.6)$$

As a preliminary step we prove:

LEMMA 3.1: *Under the assumptions* (3.1) *and* (3.2) *on* G_k, *let* $k \leqq k_0$, $nk \leqq T$ and $\omega = \beta \exp(\beta\sigma T)$. *Then if* $G_k^j v$, $G_k^j w \in U_\delta$ *for* $j < n$ *we have*

$$\| G_k^n v - G_k^n w \| \leqq \omega \| v - w \|.$$

Proof: We shall prove by induction over j that with $d_j = G_k^j v - G_k^j w$,

$$\| d_j \| \leqq \beta \exp(\beta\sigma jk) \| v - w \| \qquad \text{for} \quad j \leqq n. \tag{3.7}$$

Since this estimate clearly holds for $j = 0$, assume it has been proved for $j < m < n$. We have with e_j defined by the second equality,

$$d_{j+1} = E_k d_j + k(F_k G_k^j v - F_k G_k^j w) = E_k d_j + k e_j,$$

and hence

$$d_m = E_k^m (v - w) + k \sum_{j=0}^{m-1} E_k^{m-1-j} e_j.$$

It follows that

$$\| d_m \| \leqq \beta \| v - w \| + k \beta\sigma \sum_{j=0}^{m-1} \| d_j \|,$$

from which (3.7) now easily follows for $j = m$ by the induction assumption. This proves the lemma.

Proof of Theorem 3.3: Let ω_0, ω and c_2 be as in Lemmas 2.2 and 3.1 and Theorem 3.2 and let V be the intersection of the neighborhoods in Lemma 2.2 and Theorem 3.2. We shall then prove the theorem with

$$c_3 = 2 \max(\omega_0 + \omega, c_2)(\| v \|_{A_0} + 1) \qquad \text{and} \qquad k_3 \leqq k_0$$

such that

$$\{ w : \| w - v \| \leqq 2 k_3^{\theta\nu} \| v \|_{A_\theta} \} \subset V \qquad \text{and} \qquad c_3 k_3^{\theta\nu} \leqq \delta/2.$$

The result clearly holds for $n = 0$. Assume it has already been established for integers less than n. In particular, then $G_k^{n-1} v \in U_\delta$ so that $G_k^n v$ is defined. By definition, we may choose $w \in A$ (depending on k) such that

$$\| v - w \| + k^\nu \| w \|_A \leqq 2 K(k^\nu, v; B, A) \leqq 2 k^{\theta\nu} \| v \|_{A_\theta}. \tag{3.8}$$

Then, for $k \leqq k_3$ we have $w \in V$ and $c_2 k^\nu (\| w \|_A + 1) \leqq \delta/2$. We conclude by Theorem 3.2 that $G(nk) w$ and $G_k^n w$ are defined and in U_δ and (3.5) holds. This yields, using also Lemmas 2.2 and 3.1,

$$\begin{aligned} &\| G_k^n v - G(nk) v \| \\ &\leqq \| G_k^n v - G_k^n w \| + \| G(nk) v - G(nk) w \| + \| G_k^n w - G(nk) w \| \\ &\leqq (\omega_0 + \omega) \| v - w \| + c_2 k^\nu (\| w \|_A + 1). \end{aligned}$$

Using now (3.8) and the definition of c_3, we get

$$\|G_k^n v - G(nk)v\| \leqq \max(\omega_0+\omega, c_2)(\|v-w\|+k^\nu\|w\|_A+k^\nu) \leqq c_3 k^{\theta_\nu},$$

which completes the proof.

4. THE CONCRETE INITIAL-VALUE PROLLEM

From now our Banach spaces will consist of functions on R^d with values in $R^{d'}$. More precisely, B will consist of such functions in the Besov space $B_2^{d/2,1}$. For arbitrary positive s, $B_2^{s,1}$ may be defined, with any $N > s$, by

$$\|w\|_{B_2^{s,1}} = \int_0^\infty t^{-s-1}\omega_2^N(w, t)\,dt, \tag{4.1}$$

where with $\Delta_h w(x) = w(x+h) - w(x)$,

$$\omega_2^N(w, t) = \sup_{|h| \leqq t} \|\Delta_h^N w\|_{L_2}.$$

In addition to $B = B_2^{d/2,1}$ (in which the norm will still be denoted $\|.\|$) we shall use $B^m = B_2^{m+d/2,1}$ with m a non-negative integer. A norm in B^m, equivalent to the one defined by (4.1) is then

$$\|w\|_m = \sum_{|\alpha| \leqq m} \|D^\alpha w\|.$$

We shall later also have reason to use the space $B_2^{s,\infty}$, defined with $N > s$ by

$$\|w\|_{B_2^{s,\infty}} = \sup_{t>0} t^{-s}\omega_2^N(w, t). \tag{4.2}$$

Our choice of the space $B_2^{d/2,1}$ is motivated by the fact that this is the largest Besov space $B_2^{s,q}$ based on L_2 which is contained in L_∞; we have the sharp Sobolev type inequality (*cf.* e. g. [3], Theorem 2.4):

$$\|w\|_{L_\infty} \leqq \varkappa \|w\| \quad \text{for all } w \in B. \tag{4.3}$$

As a result of this, we have:

LEMMA 4.1: *For any m (and $d' = 1$ sothat point wise multiplication is defined), B^m is a Banach algebra, and for given l and m there is a constant C such that*

$$\|w_0 w_1 \ldots w_l\|_m \leqq C \sum_{\Sigma_j m_j = m} \|w_0\|_{m_0}\|w_1\|_{m_1} \ldots \|w_l\|_{m_l}. \tag{4.4}$$

Proof: It is enough to prove (4.4) for $l = 1$ and using Leibniz' formula we may restrict ourselves to the case $m = 0$. Let $N > d$ and recall the discrete Leibniz formula

$$\Delta_h^N(w_0 w_1)(x) = \sum_{j=0}^N \binom{N}{j} \Delta_h^{N-j} w_0(x+jh)\,\Delta_h^j w_1(x).$$

Using (4.3) we find at once that the terms with $j \leqq d/2$ (and hence $N-j > d/2$) are bounded in L_2 by

$$C \| \Delta_h^{N-j} w_0 \|_{L_2} \| \Delta_h^j w_1 \|_{L_\infty} \leqq C \| \Delta_h^{N-j} w_0 \|_{L_2} \| w_1 \|_{L_\infty} \leqq C \omega_2^{N-j}(w_0, h) \| w_1 \|,$$

and similarly, the terms with $j > d/2$ are bounded by $C \omega_2^j (w_1, h) \| w_0 \|$. Hence

$$\omega_2^N (w_0 w_1, t) \leqq C \{ \| w_1 \| \sum_{j \leqq d/2} \omega_2^{N-j}(w_0, t) + \| w_0 \| \sum_{j > d/2} \omega_2^j (w_1, t) \},$$

and the result now follows by the definition of B.

We shall now consider the concrete initial-value problem

$$\left. \begin{aligned} \frac{\partial u}{\partial t} &= P(x,D)\, u + f(x, u), \\ u(x, 0) &= v(x), \end{aligned} \right\} \qquad (4.5)$$

where $P = P(x, D)$ is a linear differential operator of order M, the $d' \times d'$ matrix coefficients of which have bounded continuous derivatives of all orders, and where f is a given function on $R^d \times R^{d'}$. We shall assume that (4.5) admits a solution $u = u(x, t)$ in B for $0 \leqq t \leqq T$, and we shall be concerned with proving in this concrete situation the conclusions of Lemmas 2.2 and 2.3 (with a suitable A). Setting

$$F u(x) = f(x, u(x)), \qquad (4.6)$$

our efforts will mainly be devotes to the verification of the assumptions on the operator F made in Section 2.

We shall need to assume below that f satisfies the following regularity assumptions:

(fi) $D_x^\alpha D_\xi^\beta f(x, \xi)$ are bounded continuous on $R^d \times R^{d'}$ for all α, β;

($f$ii) $D_x^\alpha D_\xi^\beta f(x, \xi)$ are bounded in $L_2(R^d)$, uniformly for $\xi \in R^{d'}$, when $\alpha \neq 0$.

Since we shall be interested in the behavior of F only in a neighborhood U_δ in B of the given solution u, it is in fact sufficient to assume f defined and satisfying the regularity conditions on $R^d \times \Omega$ where Ω is some neighborhood in $R^{d'}$ of the closure of $\{ u(x, t) : x \in R^d, 0 \leqq t \leqq T \}$. For, if $\| w - v(t) \| \leqq \delta$ for some t, we conclude by (4.3) that $| w(x) - u(x, t) | \leqq \varkappa\delta$ so that for δ small, $w(x)$ is in Ω for all $x \in R^d$. On the other hand, a function f satisfying the regularity assumptions on $R^d \times \Omega$ may be extended to $R^d \times R^{d'}$ without loss of these properties. Notive that ($f$ii) is always satisfied if f is independent of x, or more generally, if f is independent of x outside some compact set in R^d. In each individual result below, only a finite number of the derivatives of f will enter; for convenience we refrain from keeping track of the exact number.

In addition to (f i), (f ii) we shall demand that

(f iii) $$f(x, 0) \in L_2(R^d).$$

For f independent of x, this requirement reduces to $f(0) = 0$. Notice that since the functions in B are small for large $|x|$ we have $0 \in U_\delta$ for any $\delta > 0$.

We now turn to the technical work. We shall first prove in Lemma 4.3 below that the condition (A ii) of Section 2 is satisfied with $A = B^m$. As a preliminary step we prove an estimate for Fw in the Sobolev space W_2^N (*cf.* [10]). Recall that the norm in W_2^N is defined by

$$\| w \|_{W_2^N} = \sum_{|\alpha| \leqq N} \| D^\alpha w \|_{L_2}.$$

LEMMA 4.2: *Let F be defined by* (4.6) *with f satisfying* (f i), (f ii). *Then for any positive N there is a constant C such that for* $|\alpha| = N$, $w \in W_2^N$,

$$\| D^\alpha(F w) \|_{L_2} \leqq C(\| w \|_{L_\infty} + 1)^{N-1} (\| w \|_{W_2^N} + 1).$$

Proof: The derivatives of order N of $f(x. w(x))$ are linear combinations of erms of the forms

$$D_x^\alpha f(x, w) \quad \text{with} \quad |\alpha| = N, \tag{4.7}$$

and

$$(D_x^\alpha D_\xi^\beta f)(x, w) \prod_l D_x^{\gamma_l} w_{\varkappa_l}, \tag{4.8}$$

where $w = (w_1, \ldots, w_{d'})$ and

$$\left.\begin{array}{l} |\alpha| < N, \quad 1 \leqq |\beta| \leqq N - |\alpha|, \quad l = 1, \ldots, |\beta|, \\ \sum_l |\gamma_l| = N - |\alpha|, \\ \gamma_l \neq 0, \quad 1 \leqq \varkappa_l \leqq d'. \end{array}\right\} \tag{4.9}$$

The terms of the form (4.7) are clearly bounded in L_2 by (f ii). In order to estimate a term of the form (4.8) we shall apply the inequality (*see* [10]):

$$\| D^\gamma w \|_{L_q} \leqq C \| w \|_{L_\infty}^{1-\theta} \| w \|_{W_2^M}^\theta, \quad \text{where} \quad 0 < \theta = \frac{|\gamma|}{M} \leqq 1, \quad q = \frac{2}{\theta},$$

with

$$M = N - |\alpha|, \quad \gamma = \gamma_l, \quad \theta = \theta_l = \frac{|\gamma_l|}{N - |\alpha|}, \quad q = q_l = \frac{2}{\theta_l}.$$

Noticing that $\sum q_l^{-1} = 2^{-1}$ we hence obtain, using first Hölder's inequality and $(f\text{i})$, that (4.8) is majorized in L_2-norm by

$$C\prod_l \| D^{\gamma_l} w\|_{L_{q_l}} \leqq C\|w\|_{L_\infty}^{\Sigma(1-\theta_l)}\|w\|_{W_2^M}^{\Sigma\theta_l} = C\|w\|_{L_\infty}^{|\beta|-1}\|w\|_{W_2^{N-\alpha}}.$$

In view of (4.9) this proves the Lemma.

LEMMA 4.3: *Assume that f satisfies $(f\text{i})$, $(f\text{ii})$ and $(f\text{iii})$ and let W be a bounded set in B. Then for any nonnegative m there is a constant C such that*

$$\|Fw\|_m \leqq C(\|w\|_m+1) \qquad \text{for} \quad w\in W\cap B^m.$$

Proof: Let $N > m+d/2$ and set $K(t, w; L_2, W_2^N)$. Then the norm in B^m is equivalent to

$$\int_0^1 t^{-1-(m+d/2)/N} K(t, w)\,dt + \|w\|_{L_2},$$

and it is therefore sufficient to prove

$$\|Fw\|_{L_2} \leqq C(\|w\|_{L_2}+1) \qquad \text{for} \quad w\in W \tag{4.10}$$

and

$$K(t, Fw) \leqq C(K(t, w)+t) \quad \text{for } w\in W\cap B^m, \qquad 0\leqq t\leqq 1. \tag{4.11}$$

By $(f\text{iii})$, (4.10) follows immediately from

$$|Fw(x)| \leqq |Fw(x)-FO(x)|+|f(x, 0)| \leqq C|w(x)|+|f(x, 0)|.$$

In order to prove (4.11) we recall that it is known (*cf.* [10]) that there is a constant C independent of f and w such that if we take

$$\tilde{w}(x) = \mathscr{F}^{-1}(\exp(-t|\xi|^N)\hat{w}), \qquad \text{where} \quad \hat{w} = \mathscr{F}w,$$

then for $0 \leqq t \leqq 1$,

$$\|w-\tilde{w}\|_{L_2}+t\|\tilde{w}\|_{W_2^N} \leqq CK(t, w). \tag{4.12}$$

Since $\exp(-|\xi|^N)$ is a multiplier on $\mathscr{F} L_1$ (*cf.* e. g. [7]) we have in addition

$$\|\tilde{w}\|_{L_\infty} \leqq C\|w\|_{L_\infty} \leqq C\|w\| \leqq C \qquad \text{for} \quad w\in W.$$

It follows by (4.10) and Lemma 4.2 that

$$\|F\tilde{w}\|_{W_2^N} \leqq C(\|\tilde{w}\|_{W_2^N}+1).$$

Hence, using $(f\text{i})$ and (4.12) we obtain

$$\begin{aligned} K(t, Fw) &\leqq \|Fw-F\tilde{w}\|_{L_2}+t\|F\tilde{w}\|_{W_2^N} \\ &\leqq C\{\|w-\tilde{w}\|_{L_2}+t(\|\tilde{w}\|_{W_2^N}+1)\} \leqq C(K(t, w)+t), \end{aligned}$$

which completes the proof.

août 1976.

We shall now prove in Lemma 4.5 below the Lipschitz continuity properties of the operator F required in the theory of Setcion 2. Again we start with a technical Lemma:

LEMMA 4.4: *Assume that f satisfies $(f\text{i})$ and $(f\text{ii})$ and let W be a bounded set in B. Then for any non-negative m there is a constant C such that*

$$\| F w_0 . w_1 \|_m \leqq C \sum_{m_0+m_1=m} (\| w_0 \|_{m_0}+1) \| w_1 \|_{m_1}$$

for

$$w_0 \in W \cap B^m, \qquad w_1 \in B^m.$$

Proof: The function $f(x, \xi)-f(x, 0)$ satisfies $(f\text{i})$, $(f\text{ii})$ and $(f\text{iii})$ and hence by Lemmas 4.1 and 4.3,

$$\begin{aligned} \| (f(., w_0)-f(., 0)) w_1 \|_m &\leqq C \sum_{m_0+m_1=m} \| f(., w_0)-f(., 0) \|_{m_0} \| w_1 \|_{m_1} \\ &\leqq C \sum_{m_0+m_1=m} (\| w_0 \|_{m_0}+1) \| w_1 \|_{m_1}. \end{aligned}$$

Since obviously in view of $(f\text{i})$,

$$\| f(., 0) w_1 \|_m \leqq C \| w_1 \|_m ,$$

the result follows

LEMMA 4.5: *Assume that f satisfies $(f\text{i})$, $(f\text{ii})$ and $(f\text{iii})$, and let m be non-negative. Then for any bounded set W in B^m there is a constant C such that*

$$\| F w_1 - F w_0 \|_m \leqq C \| w_1 - w_0 \|_m \qquad \text{for} \quad w_0, w_1 \in W. \tag{4.13}$$

Proof: Since f is defined everywhere on $R^d \times R^{d'}$ it is no restriction to assume that W is convex. With $w_0, w_1 \in W$ we then have $w_s = w_0 + s(w_1 - w_0) \in W$ for $0 \leqq s \leqq 1$ and we may write

$$\begin{aligned} F w_1(x) - F w_0(x) &= \int_0^1 \frac{d}{ds} f(x, w_s(x))\, ds \\ &= \int_0^1 \langle \operatorname{grad}_\xi f(x, w_s(x)), w_1(x) - w_0(x) \rangle\, ds. \end{aligned} \tag{4.14}$$

Applying Lemma 4.4 to $\operatorname{grad}_\xi f$ we obtain since W is bounded in B^m,

$$\| \langle \operatorname{grad}_\xi f(., w_s), w_1 - w_0 \rangle \|_m \leqq C(\| w_s \|_m + 1) \| w_1 - w_0 \|_m \leqq C \| w_1 - w_0 \|_m ,$$

which together with (4.14) proves (4.13) and hence the Lemma.

Notice that Lemma 4.5 contains both the Lipschitz continuity of F on U_δ required in Lemma 2.2 ($m = 0$) and the condition (A iii) ($m > 0$).

We have thus proved that the assumptions on F in Lemmas 2.2 and 2.3 are satisfied for the concrete initial-value problem (4.5). For the purpose of satisfying also the assumption (A i) on $E(t)$ we now demand that the initial-value problem for the linear equation,

$$\frac{\partial u}{\partial t} = P(x, D)u, \tag{4.15}$$

be strongly correctly posed in L_2, that is that for each N there is a constant C such that

$$\| E(t) w \|_{W_2^N} \leqq C \| w \|_{W_2^N} \quad \text{for} \quad 0 \leqq t \leqq T.$$

Choosing $N = 0$ and some $N > m + d/2$, it follows at once by interpolation that

$$\| E(t) w \|_m \leqq C \| w \|_m \quad \text{for} \quad 0 \leqq t \leqq T,$$

which is (A i) for $A = B^m$. We may hence conclude from Lemmas 2.2 and 2.3:

LEMMA 4.6· *Assume that the initial-value problem* (4.15) *is strongly correctly posed in* L_2 *and that* f *satisfies* (f i), (f ii) *and* (f iii). *Then there is a neighborhood* V *of* v *in.* B *such that for* $w \in V$, (4.5) *has a unique solution* $G(t)\, w \in U_\delta$ *for* $0 \leqq t \leqq T$. *If in addition* $w \in B^m$ *then* $G(t)\, w \in B^m$ *for* $0 \leqq t \leqq T$. *Moreover there are constants* ω_0 *and* τ_m *such that*

$$\| G(t) w - G(t) v \| \leqq \omega_0 \| w - v \| \quad \text{for} \quad w \in V,$$

$$\| G(t) w \|_m \leqq \tau_m (\| w \|_m + 1) \quad \text{for} \quad W \in V \cap B^m$$

For solutions which are smooth with respect to x we may use the differential equation to determine corresponding differentiability properties with respect to t. In Section 5 we shall need the following bounds for derivatives in t (*cf.* Segal [11]):

LEMMA 4.7: *Under the assumptions of Lemma* 4.6, *let* j *be a non-negative integer. Then there is a constant* C *such that for* $w \in V \cap B^{jM}$ (*with* V *as in Lemma* 4.6), $G(t)\, w$ *is* j *times continuously differentiable with respect to* t *in* $[0, T]$, *and*

$$\| D_t^l G(t) w \| \leqq C(\| w \|_{jM} + 1) \quad \text{for} \quad l \leqq j.$$

Proof: We shall prove by induction over j that for any non-negative j and m, under the appropriate smoothness assumption,

$$\| D_t^j G(t) w \|_m \leqq C(\| G(t) w \|_{m+jM} + 1). \tag{4.16}$$

Taking $m = 0$ the desired result then follows by Lemma 4.6.

Clearly (4.16) holds for $j = 0$. In order to carry out the step from j to $j+1$, we first notice that by our assumptions on the M^{th} order differential operator $P = P(x, D)$, we have for any $N > 0$,

$$\| P w \|_{W_2^N} \leqq C \| w \|_{W_2^{N+M}},$$

and hence by interpolation

$$\| P w \|_m \leqq C \| w \|_{m+M}. \tag{4.17}$$

Differentiating the differential equation for $\tilde{u}(t) = G(t) w$ we obtain

$$D_t^{j+1} \tilde{u}(t) = P D_t^j \tilde{u}(t) + D_t^j F \tilde{u}(t).$$

By (4.17) and the induction assumption we have

$$\| P D_t^j \tilde{u}(t) \|_m \leqq C \| D_t^j \tilde{u} \|_{m+M} \leqq C(\| \tilde{u}(t) \|_{m+(j+1)M} + 1).$$

It will therefore be sufficient to prove

$$\| D_t^j F \tilde{u}(t) \|_m \leqq C(\| \tilde{u}(t) \|_{m+jM} + 1). \tag{4.18}$$

For this purpose we notice that $D_j^t f(x, u(x, t))$ is a linear combination of expressions of the form

$$D_\xi^\beta f(x, \tilde{u}(x, t)) \prod_{l=1}^{b} D_t^{j_l} \tilde{u}_{\varkappa_l}(x, t),$$

with $1 \leqq |\beta| = b \leqq j$, $\Sigma_l j_l = j$, $j_l \neq 0$, $1 \leqq \varkappa_l \neq d'$. By Lemmas 4.4 (applied to $D_\xi^\beta f$, noticing that $\tilde{u}$ is in U_δ and hence bounded in B) and 4.1, the B^m norm of each term can be estimated by a multiple of

$$\sum_{m_0, \ldots, m_b} (\| \tilde{u}(t) \|_{m_0} + 1) \prod_{l=1}^{b} \| D_t^{j_l} \tilde{u}(t) \|_{m_l} \quad \text{with} \quad \sum_{l=0}^{b} m_l = m. \tag{4.19}$$

Hence by the induction hypothesis, each term in (4.19) can be estimated by a multiple of

$$\prod_{l=0}^{b} (\| \tilde{u}(t) \|_{m_l + j_l M} + 1), \quad \text{where} \quad j_0 = 0.$$

We now use the convexity inequality

$$\| u \|_\mu \leqq C \| u \|^{1-\theta} \| u \|_\nu^\theta, \quad 0 < \theta = \frac{\mu}{\nu} \leqq 1, \tag{4.20}$$

and obtain since $\tilde{u}$ is bounded in B,

$$\| \tilde{u}(t) \|_{m_l + j_l M} \leqq C \| \tilde{u}(t) \|_{m+jM}^{\theta_l} \quad \text{with} \quad \theta_l = \frac{m_1 + j_l M}{m + j M}.$$

Using also the fact that $1+x^{\theta} \leqq 2(1+x)^{\theta}$ for $x \geqq 0, 0 \leqq \theta < 1$ and $\Sigma_l \theta_l = 1$, we conclude

$$\| D_t^j f(., \tilde{u}(., t)) \|_m \leqq C \prod_{l=0}^{b} (\| \tilde{u}(t) \|_{m+jM}^{\theta_l} + 1) \leqq C(\| \tilde{u}(t) \|_{m+jM} + 1),$$

which concludes the proof of (4.18) and thus of (4.16).

When applying the general convergence results of Section 3 to difference schemes for specific initial-value problems, it will be necessary to convert known results on these problems to the present framework. A result in this direction is the following lemma which shows that is the linear problem (4.15) is strongly correctly posed and F has certain local boundedness and Lipschitz continuity properties, then the solution with initial data in W_2^N remains if W_2^N (and hence in B if $N > d/2$) for as long as it is bounded in L_∞. Recall that the step from B to B^m was already taken in Lemma 2.3.

LEMMA 4.8: *Let $N > d/2$, $T > 0$, ans assume that there is a constant $\beta \geqq 1$ such that*

$$\| E(t) w \|_{W_2^N} \leqq \beta \| w \|_{W_2^N} \quad \text{for} \quad 0 \leqq t \leqq T, \tag{4.21}$$

and that for arbitrary positive σ and γ, there are $c_1(\sigma)$ and $c_2(\gamma)$ such that

$$\| F w_1 - F w_0 \|_{W_2^N} \leqq c_1(\sigma) \| w_1 - w_0 \|_{W_2^N} \quad \text{for } \| w_j \|_{W_2^N} \leqq \sigma, \qquad j = 0, 1,$$

$$\| F w \|_{W_2^N} \leqq c_2(\gamma)(\| w \|_{W_2^N} + 1) \quad \text{for } w \in W_2^N, \qquad \| w \|_{L_\infty} \leqq \gamma. \tag{4.22}$$

Let $v \in W_2^N$ and assume that there is a classical solution $u(t)$ with $\| u(t) \|_{L_\infty} \leqq \gamma_0$ on $[0, T]$. Then $u(t) \in W_2^N$ on $[0, T]$.

Proof: We notice first that for as long a subinterval of $[0, T]$ as $u(t) = G(t) v$ is in W_2^N we have

$$\| G(t) v \|_{W_2^N} \leqq \sigma_0 = -1 + (\beta \| v \|_{W_2^N} + 1) \exp(\beta c_2(\gamma_0) T). \tag{4.23}$$

For, by the integral form (2.2) of the initial-value problem we obtain using (4.21) and (4.22) and the boundedness of $u(t)$,

$$\| G(t) v \|_{W_2^N} \leqq \beta \| v \|_{W_2^N} + \beta c_2(\gamma_0) \int_0^t (\| G(s) v \|_{W_2^N} + 1) \, ds$$

from which (4.23) follows at once. We find easily, as in the proof of Lemma 2.1, that with σ_0 given by (4.23), there is a T_0 such that the initial-value problem has a solution $G(t) w$ in W_2^N on $[0, T_0]$ for all w with $\| w \|_{W_2^N} \leqq \sigma_0$. Since $v \|_{W_2^N} \leqq \sigma_0$ it follows that $G(t) v$ is in W_2^N for $0 \leqq t \leqq T_0$ and by (4.23) we have $\| G(T_0) v \|_{W_2^N} \leqq \sigma_0$. Hence $G(T_0) v$ may be taken as new initial-values so that we can conclude $G(t) v \in W_2^N$ on $[0, 2 T_0]$ and by continuation of this procedure on $[0, T]$.

We conclude this section by some examples.

Example 1: Consider for $d = 1$ the equation

$$\frac{\partial u}{\partial t} = \frac{\partial u}{\partial x} + \rho u^{r+1}, \qquad \rho \text{ Const.} \neq 0, \quad r = 1, 2, \ldots \tag{4.24}$$

The exact solution of the initial-value problem is then

$$u(x, t) = v(x+t)\left[1 - r\rho t v(x+t)^r\right]^{-1/r}.$$

For $\rho < 0$ and r even the solution is bounded for all positive t and hence T is arbitrary. For $\rho > 0$ or $\rho > 0$ and r odd we may choose any T such that $T < T_0 (r \rho \| v \|_{L_\infty}^r)^{-1}$. In both cases we conclude that for $v \in W_2^1$ we have that $u(x, t)$ belongs to W_2^1 (and hence to B) for $t \in [0, T]$. (It is in fact easy to see that $u \in B^m$ if $v \in B^m$).

Example 2: Consider for d arbitrary the symmetric hyperbolic system ($u \in R^{d'}$):

$$\frac{\partial u}{\partial t} = \sum_{j=1}^{d} A_j(x) \frac{\partial u}{\partial x_j} + f(x, u), \qquad A_j(x)^* = A_j(x). \tag{4.25}$$

The linear problem is strongly correctly posed in L_2 and hence, under the appropriate regularity conditions on f, the corresponding initial-value problem admits a solution in $[0, T]$ for some $T > 0$ by Lemma 2.1, which by Lemma 2.3 is smooth provided the initial data are smooth.

In some cases the solution continues to exist for all positive t. For example, let $d = 3$ and consider the scalar second order problem

$$\frac{\partial^2 y}{\partial t^2} = \sum_{j=1}^{3} \frac{\partial^2 y}{\partial x_j^2} - y^3, \tag{4.26}$$

$$y(x, 0) = \varphi, \qquad \frac{\partial y}{\partial t}(x, 0) = \psi.$$

Introducing y, $\partial y/\partial t$ and $\partial y/\partial x_j$, $j = 1, 2, 3$, as new variables, this problem can be reduced in a standard manner to an initial-value problem for a symmetric hyperbolic system (with $d' = 5$). It was proved by Jörgens ([5], *cf.* Satz 2 and Hilfssatz 1) for a class of equations containing (4.26) that for any positive T and sufficiently smooth φ and ψ there exists a classical solution of (4.26) in $[0, T]$ with y, $\partial y/\partial t$, $\partial y/\partial x_j$ uniformly bounded. This implies the existence of a bounded classical solution $u(t)$ of the corresponding symmetric hyperbolic system. Since the assumptions of Lemma 4.8 are satisfied for this system (with $N = 2$) we conclude that if the initial data $v = (\varphi, \psi, \text{grad } \varphi)$ of this solution are in W_2^2, then $u(t)$ also belongs to W_2^2 (and in particular to B) on $[0, T]$.

Example 3: Consider the Schrödinger type equation

$$\frac{\partial u}{\partial t} = i \sum_{j=1}^{d} \frac{\partial^2 u}{\partial x_j^2} + f(x, u). \tag{4.27}$$

Taking real and imaginary parts we obtain a system (with $d' = 2$) for which the linear initial-value problem is strongly correctly posed in L_2. The semi-linear system therefore has a solution in B at least locally in t. Again, it may be that the solution continues to exist for all positive t. Such a case is the equation ($d = 1$):

$$\frac{\partial u}{\partial t} = i \frac{\partial^2 u}{\partial x^2} - |u|^2 u.$$

Here it is easy to see (*cf.* [6], Chapter 1, Section 10) that for arbitrary $T > 0$ the solution is bounded in W_2^1 and hence in B on $[0, T]$, provided this is the case initially.

5. THE CONCRETE DISCRETE PROBLEM

We shall now consider the approximation of the concrete initial-value problem discussed in Section 4 by means of a finite difference operator of the form

$$G_k w = E_k w + k F_k w.$$

Here we shall assume that E_k is a linear explicit finite difference operator,

$$E_k w(x) = \sum_{\alpha} a_\alpha(x, h) w(x - \alpha h) \qquad \text{with} \quad \frac{k}{h^M} = \lambda = \text{Const.},$$

which is strongly stable in L_2 so that for each T and N,

$$\| E_k^n w \|_{W_2^N} \leqq C \| w \|_{W_2^N} \qquad \text{for} \quad nk \leqq T. \tag{5.1}$$

The finitely many coefficient matrices $a_\alpha(x, h)$ are assumed to have bounded continuous derivatives of all orders for $(x, h) \in R^d \times (0, 1]$ say. Further, $F_k v$ will be an expression of the form

$$F_k w(x) = \psi(x, h, w(x - \alpha_1 h), \ldots, w(x - \alpha_J h)),$$

for some finite set $\{\alpha_j\} \subset Z^d$. We shall assume that the function $\psi = \psi(x, h, \xi_{(1)}, \ldots, \xi_{(J)})$ is defined on $R^d \times (0, 1] \times R^{d'} \times \ldots \times R^{d'}$ with values in $R^{d'}$ and that

(ψ i) $D_x D_h^\beta D_{\xi_{(1)}}^{\gamma_1} \ldots D_{\xi_{(J)}}^{\gamma_J} \psi$ are bounded in all variables:

(ψ ii) for $\alpha \neq o$ these derivatives are bounded in $L_2(R^d)$, uniformly in $h, \xi_{(1)}, \ldots, \xi_{(J)}$. Notice that as before, if we work in U_δ, the behavior of ψ for large values of the $\xi_{(j)}$ is immaterial.

Our first purpose is to show that under the above assumptions, the operator G_k satisfies the assumptions (3.1) and (3.2) made on G_k in Section 3 with respect to the concrete space $B = B_2^{d/2,1}$:

LEMMA 5.1: *Assume that E_k is strongly stable in L_2 and that ψ satisfies (ψ i) and (ψ ii). Then E_k is stable in $B = B_2^{d/2,1}$ and F_k is Lipschitz continuous in U_δ, uniformly for small k.*

Proof: The stability of E_k in B follows at once by linear interpolation from (5.1) with $N = 0$ and $N = [d/2]+1$, say. In order to prove the Lipschitz continuity of F_k, it suffices, in the same way as in the proof of Lemma 4.5, to show that for all first order derivatives $\psi_{(j),k} = \partial\psi/\partial\xi_{(j),k}$ of ψ, we have

$$\| \psi_{(j),k}(., h, w_0(.-\alpha_1 h), \ldots, w_0(.-\alpha_J h)).w_1 \| \leqq C(1+\|w_0\|)\|w_1\|.$$

The proof of this fact follows as in Lemmas 4.2 through 4.4 (with $m = 0$) from (ψ i) and (ψ ii).

In order to apply the general convergence results of Section 3, it remains to discuss the discretization errors. We say that G_k is accurate of order μ if for smooth solutions $\tilde{u}$ of the differential equation (4.5).

$$\tilde{u}(t+k) - G_k\tilde{u}(t) = k\,O(h^\mu) \qquad \text{as} \quad h \to 0. \tag{5.2}$$

This relation is required to hold only formally, at each $x \in R^d$, so that as the left side is developed in a Taylor series with respect to h (recalling that $k = \lambda h^M$) and using the differential equation to replace derivatives in t by derivatives in x, then the appropriate number of terms cancel. In order to obtain an estimate for the local truncation error in B we shall then need to estimate the remainder term in this Taylor series.

We first consider the remainder in the Taylor expansion of $\tilde{u}(t+k) = G(t+k)w$,

$$R_m\tilde{u}(t) = \tilde{u}(t+k) - \sum_{j=0}^{m} \frac{k^j}{j!} D_t^j\tilde{u}(t) = \int_0^k \frac{(k-s)^m}{m!} D_t^{m+1} u(t+s)\,ds.$$

Using Lemma 4.7 we obtain

$$\|R_m\tilde{u}(t)\| \leqq \int_0^k \frac{(k-s)^m}{m!} \|D_t^{m+1}\tilde{u}(t+s)\|\,ds \leqq C k^{m+1}(\|w\|_{(m+1)M}+1).$$

With $m = \mu$ this estimate is of the order of the right side in (5.2) if $M = 1$ In order to obtain the appropriate estimate for $M > 1$ we notice that since

$$R_m\tilde{u}(t) = R_{m-1}\tilde{u}(t) - \frac{k^m}{m!} D_t^m\tilde{u}(t),$$

we also have

$$\|R_m \tilde{u}(t)\| \leqq C k^m \sup_{0 \leqq t \leqq X} \|D_t^m \tilde{u}(t)\| \leqq C k^m (\|w\|_{mM}+1).$$

The following Lemma will now provide an estimate for

$$w \in B^p \quad \text{with} \quad mM < p \leqq (m+1)M.$$

In the proof we shall again apply some interpolation theory. In particular, in addition to the interpolation space $(B_0, B_1)_{\theta,\infty}$ defined in Section 3, we shall use the space $(B_0, B_1)_{\theta,1}$ with norm

$$\int_0^\infty t^{-\theta-1} K(t, w; B_0, B_1)\, dt.$$

LEMMA 5.2: *Assume that the linear problem* (4.15) *is strongly correctly posed in* L_2 *and that* (fi), ($f$ii) *and* ($f$iii) *hold. Let* V *be the neighborhood in Lemmas* 4.6 *and* 4.7. *Then there is a neighborhood* V_0 *of* v *in* B, *with* $V_0 \subset V$ *and such that if* $p = mM+q$, *with* m *a non-negative integer and* $0 < q < M$, *then there is a constant* C *such that*

$$\|R_m G(t) w\| \leqq C k^{p/M} (\|w\|_p + 1) \quad \textit{for} \quad w \in V_0 \cap B^p \quad \textit{and} \quad t+k \leqq T.$$

Proof: Let $C^j(B)$ denote the space of j times continuously differentiable functions $U = U(t)$ on $[0, T]$ with values in B and set

$$\|U\|_{C^j(B)} = \max_{l \leqq j} \sup_{[0,T]} \|U^{(l)}(t)\|.$$

With this notation, the remainder R_m is a linear operator on $C^m(B)$ and we have for $t+k \leqq T$,

$$\|R_m U(t)\| \leqq C \min(k^m \|U\|_{C^m(B)}, k^{m+1} \|U\|_{C^{m+1}(B)}).$$

We hence obtain by linear interpolation theory, with

$$C^{m,\theta}(B) = (C^m(B), C^{m+1}(B))_{\theta,\infty},$$

$$\|R_m U(t)\| \leqq C k^{p/M} \|U\|^{C^{m,\theta}}{}_{(B)}, \qquad \theta = \frac{q}{M}.$$

Let ε be so small that $V^{(2\varepsilon)} = \{w : \|w-v\| < 2\varepsilon\}$ is contained in V. The Lemma will follow (with $V_0 = V^{(\varepsilon)}$) if we can prove that for $w \in V^{(\varepsilon)} \cap B^p$,

$$\|G(t) w\|^{C^{m,\theta}}{}_{(B)} \leqq C(\|w\|_p + 1). \tag{5.3}$$

Now recall the inequality

$$\|U\|_{C^m(B)} \leqq C \|U\|_{C^0(B)}^{1/(m+1)} \|U\|_{C^{m+1}(B)}^{m/(m+1)},$$

which is equivalent to the inclusion $(C^0(B), C^{m+1}(B))_{m/(m+1),1} \subset C^m(B)$. On the other hand, the reiteration theorem of interpolation theory states that with $\eta = (m+\theta)/(m+1)$,

$$C^{m,\theta}(B) \supset ((C^0(B), C^{m+1}(B))_{m/(m+1),1}, C^{m+1}(B))_{\theta,\infty} = (C^0(B), C^{m+1}(B))_{\eta,\infty}.$$

In view of the definition of this latter interpolation space, it suffices, in order to prove (5.3), to show that

$$K(s, G(t)w;\, C^0(B), C^{m+1}(B)) \leqq C s^{\eta}(\|w\|_p+1). \tag{5.4}$$

For this purpose, let $v \in V^{(\varepsilon)} \cap B^p$ and choose $w_1 \in B^{(m+1)M}$ so that

$$\|w-w_1\|+s\|w_1\|_{(m+1)M} \leqq 2K(s, w;\, B, B^{(m+1)M}) \leqq C_0 s^{\eta}\|w\|_p.$$

Then for $s \leqq s_0$ with s_0 such that $C_0 s_0^{\eta}(\|w\|_p+1) = \varepsilon$ we have $w_1 \in V^{(2\varepsilon)} \subset V$ so that by Lemmas 4.6 and 4.7,

$$\|G(t)w_1 - G(t)w\| \leqq C\|w_1-w\|,$$

$$\|D_t^j G(t)w_1\| \leqq C(\|w_1\|_{(m+1)M}+1), \qquad j \leqq m+1.$$

For these s we therefore obtain

$$\begin{aligned} &K(s, G(t)w;\, C^0(B), C^{m+1}(B)) \\ &\leqq \|G(t)w - G(t)w_1\|_{C^0(B)} + s\|G(t)w_1\|_{C^{m+1}(B)} \\ &\leqq C\|w-w_1\| + Cs(\|w_1\|_{(m+1)M}+1) \leqq C s^{\eta}(\|w\|_p+1). \end{aligned}$$

On the other hand, for $s \geqq s_0$,

$$K(s, G(t)w;\, C^0(B), C^{m+1}(B)) \leqq \|G(t)w\|_{C^0(B)} \leqq C \leqq C s^{\eta}(\|w\|_p+1),$$

which completes the proof of (5.4).

Returning to the discretization error we shall now prove:

LEMMA 5.3: *Under the assumptions of Lemma 5.2 about the initial-value problem, let G_k be accurate of order μ with ψ satisfying (ψ i), (ψ ii). Then with $p = M+\mu$ and V_0 the neighborhood of Lemma 5.2 there is a constant C such that for small k, $t+k \leqq T$,*

$$\|\varepsilon_k G(t)w\| \leqq C h^{\mu}(\|w\|_p+1) \qquad \text{for} \quad w \in V_0 \cap B^p.$$

Proof: We write

$$p = M+\mu = mM+q \qquad \text{with} \quad 0 < q \leqq M.$$

Setting $\tilde{u}(t) = G(t)w$ we have

$$G(k)\tilde{u}(t) = \sum_{j=0}^{m} \frac{k^j}{j!} D_t^j \tilde{u}(t) + \tilde{R}_m,$$

where by Lemma 5.2,

$$\|\tilde{R}_m\| \leqq C k h^{\mu}(\|w\|_{M+\mu}+1). \tag{5.5}$$

Letting further $a(h)$ be defined by

$$a(h) = E_k \tilde{u} = \sum_{\alpha} a_{\alpha}(x, h)\tilde{u}(x-\alpha h) \qquad \text{with} \quad \tilde{u} = \tilde{u}(t),$$

we have

$$E_k \tilde{u}(x) = \sum_{j=0}^{p-1} \frac{h^j}{j!} a^{(j)}(0) + Q_p,$$

where

$$Q_p = \int_0^h \frac{(h-s)^{p-1}}{(p-1)!} a^{(p)}(s)\, ds.$$

Since obviously by our assumptions on the coefficients of E_k,

$$\|a^{(p)}(s)\| \leqq C\|\tilde{u}\|_p,$$

we find by Lemma 4.6,

$$\|Q_p\| \leqq C h^p \|\tilde{u}\|_p = C k h^{\mu} \|\tilde{u}\|_{M+\mu} \leqq C k h^{\mu}(\|w\|_{M+\mu}+1). \tag{5.6}$$

Setting also

$$\psi(h) = F_k \tilde{u}(x) = \psi(x, h, \tilde{u}(x-\alpha_1 h), \ldots, \tilde{u}(x-\alpha_J h)),$$

we have

$$F_k \tilde{u}(x) = \sum_{j=0}^{\mu-1} \frac{h^j}{j!} \psi^{(j)}(0) + \tilde{Q}_{\mu},$$

with

$$\tilde{Q}_{\mu} = \int_0^h \frac{(h-s)^{\mu-1}}{(\mu-1)!} \psi^{(\mu)}(s)\, ds.$$

Using the properties (ψ i), (ψ ii) we now find by the same line of reasoning as in Section 4 (*cf.* in particular the expression for the derivatives in the proofs of Lemmas 4.2 and 4.7) that for small s,

$$\|\psi^{(\mu)}(s)\| \leqq C\Big(\prod_{\Sigma m_j \leqq \mu} \|\tilde{u}\|_{m_j}+1\Big).$$

Using the convexity inequality (4.20) and Lemma 4.6, recalling that $\tilde{u}$ is bounded in B, we conclude for $0 \leqq s \leqq h$,

$$\|\psi^{(\mu)}(s)\| \leqq C(\|\tilde{u}\|_{\mu}+1) \leqq C(\|w\|_{\mu}+1),$$

so that

$$\|\tilde{Q}_{\mu}\| \leqq C h^{\mu}(\|w\|_{\mu}+1). \tag{5.7}$$

By the accuracy assumption, the polynomials in h will have to cancel in such a fashion that

$$k\,\varepsilon_k G(t)\,w = E_k \tilde{u} + k F_k \tilde{u} - G(k)\tilde{u} = Q_p + k\,\tilde{Q}_\mu - \tilde{R}_m .$$

The Lemma therefore follows at once from (5.5), (5.6) and (5.7).

As an immediate consequence of Lemmas 5.1 and 5.3 we now find that the assumptions on G_k of Theorem 3.2 are satisfied. Using also Lemma 4.6 we may hence conclude:

THEOREM 5.1: *Assume that the initial-value problem* (4.5) *has a solution* $u(t) = G(t)\,v$ *in* B *for* $0 \leqq t \leqq T$, *that the linear problem* (4.15) *is strongly correctly posed in* L_2, *and that* f *satisfies* (f i), (f ii), (f iii). *Let* $G_k = E_k + kF_k$ *be a finite difference approximation to* (4.5) *which is accurate of order* μ, *with* E_k *strongly stable in* L_2 *and* F_k *satisfying* (ψ i), (ψ ii). *Then if* $v \in B^{M+\mu} = B_2^{M+\mu+d/2,\,1}$ *there are constants* C *and* k_0 *such that*

$$\|G_k^n v - G(nk)\,v\| \leqq C\,h^\mu \qquad \text{for} \quad k \leqq k_0, \quad nk \leqq T.$$

In order to state a result for less regular initial data, we recall that the Besov space $B_2^{s,\infty}$ defined in (4.2) can also be described as an interpolation space, namely

$$B_2^{s,\infty} = (B_2^{s_0,1}, B_2^{s_1,1})_{\theta,\infty}, \quad \text{where } \theta = \frac{s-s_0}{s_1-s_0}, \qquad s_0 < s < s_1.$$

With $s_0 = d/2$, $s_1 = M+\mu+d/2$, Theorem 3.3 therefore at once yields the following interpolated result:

THEOREM 5.2: *Under the assumptions of Theorem* 5.1, *let* $0 < s < M+\mu$. *Then if* $v \in B_2^{s+d/2,\infty}$ *there are constants* C *and* k_0 *such that*

$$\|G_k^n v - G(nk)\,v\| \leqq C\,h^{s\mu/(M+\mu)} \qquad \text{for} \quad k \leqq k_0, \quad nk \leqq T.$$

Given a finite difference operator E_k of accuracy μ for the linear problem it is easy to determine F_k in such a way that G_k is also accurate of order μ. For, if $Mm < M+\mu \leqq M(m+1)$, we obtain using the differential equation,

$$\begin{aligned} u(t+k) &= \sum_{j=0}^{m} \frac{k^j}{j!} D_t^j u(t) + O(k^{m+1}) \\ &= \sum_{j=0}^{m} \frac{k^j}{j!} P^j u(t) + k\,\mathrm{N}_k^{(m)} u + O(k^{m+1}) \qquad \text{as} \quad k \to 0, \end{aligned}$$

where $N_k^{(m)}$ is a non-linear function of u and certain of its derivatives. We have for instance, with $f = f(x, u(x))$,

$$\mathrm{N}_k^{(1)} u = f, \qquad \mathrm{N}_k^{(2)} u = f + \frac{1}{2} k (P f + \mathrm{grad}_u f\,(P u + f)).$$

Since by assumption E_k is accurate of order μ, we have

$$\sum_{j=0}^{m} \frac{k^j}{j!} P^j u(t) = E_k u(t) + k\, O(h^\mu) \qquad \text{as} \quad h \to 0,$$

and we hence find that in order to achieve (5.2) we can construct F_k by replacing derivatives by difference quotients in $N_k^{(m)} u$ in such a way that formally,

$$F_k u(t) = \mathrm{N}_k^{(m)} u(t) + O(h^\mu) \qquad \text{as} \quad h \to 0.$$

For example, for the symmetric hyperbolic system (4.25), the linear, first order accurate, strongly L_2 stable Friedrichs operator

$$E_k u(x) = \frac{1}{2} \sum_{j=1}^{d} \left\{ \left(\frac{1}{d} I + \lambda A_j(x) \right) u(x + h e_j) + \left(\frac{1}{d} I - \lambda A_j(x) \right) u(x - h e_j) \right\}$$
$$\text{with} \quad \lambda < (d \max_j \| A_j \|_{L_\infty})^{-1},$$

we have $M = \mu = m = 1$ so that we may take $F_k u = N_k^{(1)} u = f(x, u)$.

Consider also the scalar eqtation (4.24) and let E_k be the Lax-Wendroff operator

$$E_k u(x) = \frac{1}{2} (\lambda^2 + \lambda) u(x+h) + (1 - \lambda^2) u(x) + \frac{1}{2} (\lambda^2 - \lambda) u(x-h),$$
$$\text{with} \quad \lambda \leqq 1.$$

Here $M = 1$, $\mu = m = 2$, $P = \partial/\partial x$, $f(x, u) = \rho\, u^{r+1}$ and we find,

$$\mathrm{N}_k^{(2)} u = \rho \left\{ u^{r+1} + k \frac{\partial}{\partial x} u^{r+1} + \frac{1}{2} k\, \rho (r+1) u^{2r+1} \right\}.$$

To preserve second accuracy we now only have to approximate $\partial u^{r+1}/\partial x$ for first order accuracy, because of the factor k in front of this term. The special case $\rho = r = 1$ was treated in detail in [12].

For the Schrödinger equation (4.21), we have $M = 2$ and so in order to retain second order accuracy with a second order E_k, we may always choose $F_k u = N_k^{(1)} u = f(x, u)$.

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