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APPROXIMATION OF AN ELLIPTIC BOUNDARY VALUE PROBLEM WITH UNILATERAL CONSTRAINTS

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Abstract. — *In this paper we show how a method of J. Nitsche for the approximation of elliptic boundary value problems can be applied to obtain an approximation scheme and « optimal » error estimate for the approximation of a certain variational inequality.*

1. INTRODUCTION

In this paper we wish to consider the approximation of the following unilateral problem. Let $a(u, v)$ be the bilinear form

$$\sum_{i,j=1}^{2} \int_{\Omega} a_{ij}(x) u_{x_i} v_{x_j} \, dx + \int_{\Omega} c(x) uv \, dx$$

where Ω is a bounded domain in R^2, $u, v \in H^1(\Omega)$, and the a_{ij} and c satisfy

(i) $a_{ij} = a_{ji} \in C^1(\bar{\Omega})$

(ii) $\sum_{i,j=1}^{2} a_{ij}\xi_i\xi_j \geqslant \bar{\alpha} \, |\xi|^2 \quad \forall \xi \in R^2$

and some $\bar{\alpha} > 0$

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(iii) $c \in L^{\infty}(\Omega)$ and $c(x) \geqslant \lambda$ where λ is either nonnegative or negative with sufficiently small absolute value so that $a(u, v)$ is coercive on $H_0^1(\Omega)$, i.e.,

$$a(v, v) \geqslant \alpha \|v\|_1^2 \qquad \forall v \in H_0^1(\Omega)$$

for some constant $\alpha > 0$.

For $f \in L^2(\Omega)$ and $\psi \in H^2(\Omega)$ with $\psi \leqslant 0$ on $\partial\Omega$, we wish to consider the problem :

Problem P : Find $u \in \mathcal{K}$ such that

$$a(u, v - u) \geqslant (f, v - u) \qquad \forall v \in \mathcal{K}$$

where $(.\,,.)$ denotes the usual L^2 inner product and

$$\mathcal{K} = \{ v \in H_0^1(\Omega) : v \geqslant \psi \text{ a.e. in } \Omega \}.$$

This problem can also be formulated as a minimization problem, i.e.,

Problem P' : Find $u \in \mathcal{K}$ such that

$$J(u) = \inf_{v \in \mathcal{K}} J(v) \quad \text{where}$$
$$J(v) = \tfrac{1}{2}\, a(v, v) - (f, v).$$

The problem of determining error estimates for the approximation of this type of unilateral problem has previously been considered in [2] for Ω a convex domain, and in [5] using a different method, for Ω a polygonal domain. These are precisely the cases in which it is easy to construct finite dimensional subspaces of $H_0^1(\Omega)$ (i.e., satisfying zero boundary conditions) in which to look for an approximation to the solution.

In this paper we would like to show how a method of Nitsche for approximating the solution of second order elliptic Dirichlet problems using subspaces which do not satisfy zero boundary conditions can be applied to get an approximation technique and an « optimal » error estimate for this unilateral problem when Ω is not necessarily convex.

As was done in [2] and [5] we shall make use of the equivalent formulation P' of the problem to construct a finite dimensional approximate problem which can be solved by mathematical programming. Unlike those previous papers, however, we shall not simply look for the minimum of $J(v)$ over some approximation $\mathcal{K}_h$ to the convex set $\mathcal{K}$, but rather we will consider an approximate problem of the form :

Problem P'_h : Find $u_h \in \mathcal{K}_h$ such that

$$J_h(u_h) = \inf_{v_h \in \mathcal{K}_h} J_h(v_h)$$

where now $J_h(v_h)$ is a functional different from, but related to, $J(v)$.

In the next three sections we shall describe the construction of the approximating convex set $\mathcal{K}_h$ and the functional $J_h(v_h)$, and then derive an error estimate for this approximation scheme.

2. CONSTRUCTION OF $\mathcal{K}_h$

Let us begin by making the additional assumptions that $\partial\Omega \in C^2$ and that $\partial\Omega$ has only a finite number of points where the curvature changes sign. We wish to choose $\mathcal{K}_h$ to be a closed convex subset of a finite dimensional subspace V_h of $H^1(\Omega)$. We first describe the construction of V_h.

Let h, $0 < h < 1$ be a parameter. Under the assumptions we have made about $\partial\Omega$, there will exist an h_0 such that for each value of $h \leqslant h_0$, we can find a polygon Ω_h with all its vertices lying on $\partial\Omega$, each side of the polygon less than or equal to h, and having the set of its vertex points including each point of $\partial\Omega$ where the curvature changes sign. Now subdivide Ω_h into triangles T_q, $q = 1, 2, \ldots, N_h$ such that the following two conditions are satisfied.

(1) The ratio of any two triangle sides in the triangulation is bounded by a constant b independent of h.

(2) All the angles in the triangulation are greater or equal to some angle ϕ independent of h.

Define $V_h(\Omega \cup \Omega_h) = \{ v_h : v_h$ is linear in each triangle T_q, $q = 1, \ldots, N_h$, continuous on $\Omega \cup \Omega_h$, and equal to zero in $(\Omega \cup \Omega_h) - \Omega_h \}$. Also define $V_h(\Omega)$ as the restriction to Ω of functions in $V_h(\Omega \cup \Omega_h)$. Clearly, $V_h(\Omega)$ is a finite dimensional subspace of $H^1(\Omega)$. We then define $\mathcal{K}_h = \{ v_h \in V_h(\Omega) : v_h \geqslant \psi$ at every vertex of each triangle T_q, $q = 1, \ldots, N_h \}$. Clearly $\mathcal{K}_h$ is a closed convex subset of $V_h(\Omega)$. Furthermore it consists of only a finite number of (linear) constraints, a fact which is critical to being able to solve the approximate problem P_h'. Since we will need this fact later, we also remark that under the conditions we have assumed on Ω_h and $\partial\Omega$, the elements of $V_h(\Omega)$ satisfy the inverse condition

$$\sum_{j=1}^{2} \int_{\partial\Omega} \left| \frac{\partial v_h}{\partial x_j} \right|^2 d\sigma \leqslant C_1 h^{-1} \|v_h\|_1^2 \tag{3}$$

for some constant C_1 independent of h. Here $\|v_h\|_1$ denotes the Sobolev $H^1(\Omega)$ norm

$$\left[\sum_{j=1}^{2} \int_{\Omega} \left| \frac{\partial v_h}{\partial x_j} \right|^2 dx + \int_{\Omega} v_h^2 \, dx \right]^{\frac{1}{2}}.$$

We now turn our attention to the description of the functional $J_h(v_h)$.

3. CONSTRUCTION OF $J_h(v_h)$

We begin by looking at the following bilinear form, which we consider initially on $H^2(\Omega) \times H^2(\Omega)$. Let

$N_\gamma(u, v) = a(u, v) - \langle u, Bv \rangle - \langle Bu, v \rangle + \gamma h^{-1} \langle u, v \rangle$ where $\langle .,. \rangle$

denotes the L^2 inner product on $\partial\Omega$, γ is a positive constant and $Bu = \sum_{i,j=1}^{2} a_{ij} u_{x_i} \nu_j$ where ν is the outward normal to $\partial\Omega$. If we define $Au = -\sum_{i,j=1}^{2} (a_{ij} u_{x_i})_{x_j} + cu$, then we easily see that for $u \in H^2(\Omega) \cap H_0^1(\Omega)$, and $v \in H^1(\Omega)$, $N_\gamma(u, v)$ also makes sense and

$$N_\gamma(u, v) = (Au, v). \tag{4}$$

We also observe that $N_\gamma(u, v)$ makes sense on $V_h(\Omega) \times V_h(\Omega)$, even though $V_h(\Omega) \not\subset H^2(\Omega)$.

For each value of $h \leqslant h_0$, we will also need to use the norm

$$\|\varphi\|_N = \left[\sum_{j=1}^{2} \left\| \frac{\partial \varphi}{\partial x_j} \right\|_0^2 + h \left| \frac{\partial \varphi}{\partial x_j} \right|_0^2 + h^{-1} |\varphi|_0^2 \right]^{\frac{1}{2}}$$

where $\|\cdot\|_0$ denotes the $L^2(\Omega)$ norm and $|\cdot|_0$ denotes the $L^2(\partial\Omega)$ norm. This norm is well defined for $\varphi \in H^2(\Omega) \cup V_h(\Omega)$. Finally, we will be making use of the following three properties of $\|\cdot\|_N$ and $N_\gamma(u, v)$. The proofs of these properties in a slightly more special case than considered here can be found in Nitsche [7], although the results follow basically from the Schwartz inequality and the inverse condition (3).

(5) There exists constants C_2 and γ_0, independent of h, such that for all $\gamma \geqslant \gamma_0$

$$\|\varphi_h\|_N^2 \leqslant C_2 N_\gamma(\varphi_h, \varphi_h) \qquad \forall \varphi_h \in V_h(\Omega).$$

(6) There exists a constant C_3 independent of h such that

$$|N_\gamma(u, v)| \leqslant C_3 \|u\|_N \|v\|_N \qquad \forall u, v \in H^2(\Omega) \cup V_h(\Omega).$$

(7) There exists a constant C_4 independent of h such that

$$\|\varphi\|_1 \leqslant C_4 \|\varphi\|_N \qquad \forall \varphi \in H^2(\Omega) \cup V_h(\Omega).$$

We can now define

$$J_h(v_h) = \tfrac{1}{2} N_\gamma(v_h, v_h) - (f, v_h) \qquad (\gamma \text{ taken} \geqslant \gamma_0).$$

From (5) it follows that the minimization problem P_h' will have a unique solution. We also remark, since we will need it later to derive an error estimate, that Problem P_h' is equivalent to :

Problem P_h : Find $u_h \in \mathcal{K}_h$ such that

$$N_\gamma(u_h, v_h - u_h) \geqslant (f, v_h - u_h) \qquad \forall v_h \in \mathcal{K}_h.$$

4. DERIVATION OF THE ERROR ESTIMATE

It now remains to derive an error estimate for this approximation procedure. The theorem we will prove is :

Theorem 1 : Let u and u_h be the respective solutions of Problems P and P'_h with $\mathcal{K}$ and $\mathcal{K}_h$ defined as above. Then there exists a constant C independent of u and depending only on the data Ω, f, ψ, a_{ij} and c such that

$$\|u - u_h\|_1 \leqslant Ch.$$

To prove this theorem we will need to make use of the following approximation results which involve technical modifications of results that can be found in Nitsche [6] and [7].

Lemma 1 : Let $V_h(\Omega)$ be as described in section 2. If $\psi \in H^2(\Omega)$ with $\psi \leqslant 0$ on $\partial\Omega$, then

$$\|\sup(0, \psi)\|_{L^2(\Omega \cup \Omega_h - \Omega_h)} \leqslant Ch^2 \|\psi\|_2 \tag{8}$$

and there exists an element S_h^ψ in $V_h(\Omega)$ which interpolates ψ at all vertices of the triangles T_q, $q = 1, \ldots, N_h$ such that

$$\|\psi - S_h^\psi\|_{L^2(\Omega \cap \Omega_h)} \leqslant Ch^2 \|\psi\|_2 \tag{9}$$

for some constant C independent of u and h.

Lemma 2 : If $u \in H^2(\Omega) \cap H_0^1(\Omega)$, there exists an element V_h in $V_h(\Omega)$ which interpolates u at all vertices of the triangles T_q, $q = 1, \ldots, N_h$ such that

$$\|u - v_h\|_k \leqslant Ch^{2-k} \|u\|_2 \tag{10}$$

and

$$|u - v_h|_k \leqslant Ch^{3/2-k} \|u\|_2 \tag{11}$$

$k = 0,1$ and C a constant independent of u and h.

In the above

$$|\varphi|_1^2 = \sum_{j=1}^{2} \int_{\partial\Omega} \left|\frac{\partial\varphi}{\partial x_j}\right|^2 d\sigma, \quad \text{and} \quad \|\varphi\|_2^2 = \sum_{i,j=1}^{2} \int_\Omega \left|\frac{\partial^2\varphi}{\partial x_i \partial x_j}\right|^2 dx + \|\varphi\|_1^2.$$

We will also need to make use of a regularity result for the solution u and another result which characterizes the solution u. Brezis and Stampacchia. have shown in [1] that under the conditions we have imposed, the solution u of Problem P belongs to $H^2(\Omega)$. Furthermore, $\|Au\|_0 \leqslant \|\sup(f, A\psi)\|_0$, and since $\|u\|_2 \leqslant C \|Au\|_0$, we also have an estimate for $\|u\|_2$ depending only on the data of the problem.

It is also known that a.e. in Ω either

$$Au = f \quad \text{and} \quad u > \psi$$

or

$$Au \geqslant f \qquad \text{and} \qquad u = \psi$$

It follows easily from these last results that

$(Au - f, v - u) \geqslant 0 \quad \forall v \in \mathcal{K}^1 \quad$ where $\quad \mathcal{K}^1 = \{ v \in H^1(\Omega) : v \geqslant \psi \text{ a.e. in } \Omega \}$,

and hence since $u \in H^2(\Omega) \cap H_0^1(\Omega)$ that

$$N_\gamma(u, v - u) \geqslant (f, v - u) \qquad \forall v \in \mathcal{K}^1. \tag{12}$$

Using (12) and the properties (5), (6), and (7), we are now able to prove the following lemma which relates the error $\|u - u_h\|_1$ to the degree of approximation of elements in $\mathcal{K}^1$ by elements in $\mathcal{K}_h$ and vice-versa. After proving some easy approximation results based on lemmas 1 and 2, we will then be able to apply this lemma to prove Theorem 1.

Lemma 3 :

$$\begin{aligned}(13)\quad & \|u - u_h\|_1 \leqslant \|u - v_h\|_1 \\ & + C_4 \{ (C_2C_3)^2 \|v_h - u\|_N^2 + 2C_2 \|Au - f\|_0 [\|v_h - u\|_0 + \|v - u_h\|_0] \}^{\frac{1}{2}} \\ & \qquad \forall v \in \mathcal{K}^1, \quad v_h \in \mathcal{K}_h.\end{aligned}$$

Proof : By the triangle inequality

$$\|u - u_h\|_1 \leqslant \|u - v_h\|_1 + \|v_h - u_h\|_1 \qquad \forall v_h \in \mathcal{K}_h$$

By properties (5) and (7)

$$\|v_h - u_h\|_1^2 \leqslant C_4^2 \|v_h - u_h\|_N^2 \leqslant C_4^2 C_2 N_\gamma(v_h - u_h, v_h - u_h).$$

Now $N_\gamma(v_h - u_h, v_h - u_h) \leqslant N_\gamma(v_h, v_h - u_h) - (f, v_h - u_h)$

(since u_h solves Problem P_h)

$$\begin{aligned} &= N_\gamma(v_h - u, v_h - u_h) + N_\gamma(u, v_h - u) + N_\gamma(u, u - v) \\ &\quad + N_\gamma(u, v - u_h) - (f, v_h - u) - (f, u - v) - (f, u - v_h) \\ &\leqslant N_\gamma(v_h - u, v_h - u_h) + N_\gamma(u, v_h - u) + N_\gamma(u, v - u_h) \\ &\quad - (f, v_h - u) - (f, v - u_h) \qquad \forall v \in \mathcal{K}^1 \quad \text{by (12).}\end{aligned}$$

Since $N_\gamma(u, \varphi) = (Au, \varphi)\ \forall \varphi \in H^1(\Omega)$ by (4),

$$\begin{aligned}\|v_h - u_h\|_N^2 &\leqslant C_2 N_\gamma(v_h - u, v_h - u_h) + C_2(Au - f, v_h - u) + C_2(Au - f, v - u_h) \\ &\leqslant C_2C_3 \|v_h - u\|_N \|v_h - u_h\|_N + C_2 \|Au - f\|_0 \|v_h - u\|_0 \\ &\quad + C_2 \|Au - f\|_0 \|v - u_h\|_0 \quad \text{by (6)}\end{aligned}$$

and the Schwartz inequality.

Since $C_2C_3 \|v_h - u\|_N \|v_h - u_h\|_N \leqslant \dfrac{\|v_h - u_h\|_N^2}{2} + \dfrac{(C_2C_3)^2}{2} \|v_h - u\|_N^2$, we obtain

$$\|v_h - u_h\|_N^2 \leqslant (C_2C_3)^2 \|v_h - u\|_N^2 + 2C_2 \|Au - f\|_0 [\|v_h - u\|_0 + \|v - u_h\|_0].$$

The result follows easily.

To apply lemma 3, we need estimates for the quantities $\|u - v_h\|_N$, $\|u - v_h\|_1$, $\|u - v_h\|_0$ and $\|u_h - v\|_0$ for some $v_h \in \mathcal{K}_n$ and some $v \in \mathcal{K}^1$. These estimates are similar to those derived in [2] and will follow basically from lemmas 1 and 2.

Lemma 4 : Let v_h be the element of $V_h(\Omega)$ described in lemma 2 when u is the solution of Problem P. Then $v_n \in \mathcal{K}_n$ and

$$\|u - v_h\|_0 \leqslant Ch^2 \|u\|_2 \tag{14}$$

$$\|u - v_h\|_1 \leqslant Ch \|u\|_2 \tag{15}$$

$$\|u - v_h\|_N \leqslant Ch \|u\|_2 \tag{16}$$

for some constant C independent of h and u.

Proof : By the regularity result for the solution u and the Sobolev imbedding theorem, both u and ψ are continuous. Since $u \in \mathcal{K}$, $u(x) \geqslant \psi(x)\ \forall x \in \bar{\Omega}$. Hence by the definition of v_h, if w is a vertex of one of the triangles T_q, $v_h(w) = u(w) \geqslant \psi(w)$ which implies $v_h \in \mathcal{K}_n$. Estimates (14) and (15) follow directly from (10). (16) follows from the definition of $\|\cdot\|_N$ and estimates (10) and (11), i.e.,

$$\begin{aligned}\|u - v_h\|_N^2 = \sum_{j=1}^{2} \left[\left\| \frac{\partial(u - v_h)}{\partial x_j} \right\|_0^2 + h \left| \frac{\partial(u - v_h)}{\partial x_j} \right|_0^2 \right] + h^{-1} |u - v_h|_0^2 \\ \leqslant \|u - v_h\|_1^2 + h\,|u - v_h|_1^2 + h^{-1}\,|u - v_h|_0^2 \\ \leqslant [Ch \|u\|_2]^2 + h[Ch^{\frac{1}{2}} \|u\|_2]^2 + h^{-1}[Ch^{\frac{3}{2}} \|u\|_2]^2 \\ \leqslant Ch^2 \|u\|_2^2.\end{aligned}$$

Lemma 5 : Let u_h be the solution of Problem P_h' and let $v = \sup \{ u_h, \psi \}$. Then $v \in \mathcal{K}^1$ and $\|u_h - v\|_0 \leqslant Ch^2$ for some constant C independent of h and u_h.

Proof : Since $u_h \in V_h(\Omega) \subset H^1(\Omega)$ and $\psi \in H^2(\Omega)$, clearly $v \in \mathcal{K}^1$. Now

$$\begin{aligned}\|u_h - v\|_0^2 &= \|u_h - v\|_{L^2(\Omega \cup \Omega_h - \Omega_h)}^2 + \|u_h - v\|_{L^2(\Omega_h \cap \Omega)} \\ &= \|u_h - v\|_{L^2(\Omega_h \cap \Omega)} + \|\sup \{ 0, \psi \}\|_{L^2(\Omega \cup \Omega_h - \Omega_h)}.\end{aligned}$$

Let S_h^ψ be the element of $V_h(\Omega)$ described in lemma 1.

Then, using the fact that $u_h \in \mathcal{K}_h$ and the piecewise linearity of u_h and S_h^ψ, one can show (see lemma 4 of [2]) that $\|u_h - v\|_{L^2(\Omega_h \cap \Omega)} \leqslant \|\psi - S_h^\psi\|_{L^2(\Omega_h \cap \Omega)}$. The lemma follows by applying (8) and (9).

We now note that Theorem 1 follows immediately from applying lemmas 4 and 5 and the a priori estimates for u and Au stated previously, to the approximation result (13).

5. CONCLUSIONS

We remark that since the best approximation to elements in $H^2(\Omega)$ by elements in $V_h(\Omega)$ is of order h in the $H^1(\Omega)$ norm, our estimate is optimal in the sense that it duplicates up to a multiplicative constant the best approximation properties of the subspace. We further note that by a counter-example of Lewy and Stampacchia [3], u does not belong to $C^2(\Omega)$ and hence no higher order accuracy can be achieved by using better subspaces, e.g. higher order splines, since this would require additional regularity of the solution.

Although we have looked at this problem for a special type of non-convex Ω, the result will continue to hold on domains in which (3) is satisfied as long as $\mathcal{K}_h$ can be chosen with the following three properties.

a) $\mathcal{K}_h$ should consist of only a finite number of constraints (so that one can solve Problem P_h').

b) Some approximation to the solution u satisfying conditions (10) and (11) lies in $\mathcal{K}_h$.

c) $\mathcal{K}_h$ sufficiently confines the approximate solution u_h so that u_h can be approximated by an element of $\mathcal{K}^1$ to order h^2 in the $L^2(\Omega)$ norm.

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