REVUE FRANÇAISE D'AUTOMATIQUE, INFORMATIQUE, RECHERCHE OPÉRATIONNELLE. ANALYSE NUMÉRIQUE

JIM JR. DOUGLAS
TODD DUPONT
MARY FANETT WHEELER

An L^∞ estimate and a superconvergence result for a Galerkin method for elliptic equations based on tensor products of piecewise polynomials

Revue française d'automatique, informatique, recherche opérationnelle. Analyse numérique, tome 8, nº R2 (1974), p. 61-66

<http://www.numdam.org/item?id=M2AN_1974__8_2_61_0>

R.A.I.R.O.
(8e année, août 1974, R-2, p. 61 à 66)

AN L^∞ ESTIMATE AND A SUPERCONVERGENCE RESULT FOR A GALERKIN METHOD FOR ELLIPTIC EQUATIONS BASED ON TENSOR PRODUCTS OF PIECEWISE POLYNOMIALS

by Jim DOUGLAS, Jr. (*), Todd DUPONT (*)
and Mary Fanett WHEELER (**)

Abstract. — *In the case of a rectangular domain uniform error estimates of optimal order are proved for Galerkin approximate solutions of a Dirichlet problem with variable coefficients. For the case of Laplace's equation and a special choice of the Galerkin space, convergence is shown to be faster at the knots than is possible globally.*

1. INTRODUCTION

Let $\Omega = (0, 1) \times (0, 1)$ and suppose that $a(x, y)$ is a $C^2(\bar{\Omega})$ function that is positive on $\bar{\Omega}$. Take u to be the solution of the Dirichlet problem

$$(1) \qquad \begin{aligned} -\nabla \cdot (a \nabla u) &= f \text{ on } \Omega, \\ u &= 0 \text{ on } \partial\Omega. \end{aligned}$$

Assume that u is sufficiently smooth. For a partition $\delta = \{ x_i \}^N_{i=0}$, $0 = x_0 < x_1 < ... < x_N = 1$, let $I_i = [x_{i-1}, x_i]$, $h_i = x_i - x_{i-1}$, $h = \max h_i$ and $\tau = \max (h/h_i)$. Fix an integer $r \geqslant 3$, and for a non-negative integer $k < r$ define

$$(2) \qquad \mathcal{M}^0_k(r, \delta) = \{ V \in C^k([0, 1]) \; ; \; V \in P_r(I_i), i = 1, ..., N \; ; \; V(0) = V(1) = 0 \},$$

where $P_r(E)$ is the set of functions whose restrictions to E are polynomials of degree less than $r + 1$. Let

$$(3) \qquad \mathcal{M} = \mathcal{M}^0_k(r, \delta) \otimes \mathcal{M}^0_k(r, \delta),$$

(*) University of Chicago, Chicago.
(**) Rice University, Houston.

and define the Galerkin approximation $U \in \mathcal{M}$ to u by

$$(a \nabla U, \nabla V) = (f, V), \qquad V \in \mathcal{M}, \tag{4}$$

where $(\cdot, \cdot)$ is the $L^2(\Omega)$ inner product.

We shall show in Section 2 that in the case $a \equiv 1$ the results of [6] and [4] can be combined to show that

$$\|U - u\|_{L^\infty(\Omega)} \leqslant Ch^{r+1}, \tag{5}$$

where C depends on u and a bound for τ. We shall also show that (5) holds in the case that $a(x, y)$ is not necessarily constant and $k = 0$. In the case that $a \equiv 1$ and $k = 0$ we can show that (5) can be generalized to domains which are unions of rectangles with sides parallel to the coordinate axes. The error estimate (5) is of optimal order in the sense that the exponent of h cannot be increased for general u; the smoothness we require on the solution u is, however, not minimal.

In Section 3, we show that when $k = 0$ and $a \equiv 1$,

$$\max_{(x,y)\in\delta\times\delta} |(U - u)(x, y)| \leqslant Ch^{r+2}, \tag{6}$$

where C depends on u and a bound for τ. This is a super-convergence result in the sense that the rate of convergence proved at the knots is greater than is possible globally.

For $1 \leqslant p \leqslant \infty$ and s a nonnegative integer define $W^{p,s}$ to be the class of functions in $L^p(\Omega)$ whose distribution derivatives through order s are also in $L^p(\Omega)$, and let

$$\|V\|_{W^{p,s}} = \sum_{|\alpha| \leqslant s} \|D^\alpha V\|_{L^p(\Omega)} \quad , \qquad V \in W^{p,s}.$$

In the special case $p = 2$ we denote the norm on $W^{2,s}$ as $\| \ \|_s$, and we define for $s > 0$ and $\varphi \in L^2(\Omega)$ the norm

$$\|\varphi\|_{-s} = \sup \{ (\varphi, \Psi) : \Psi \in W^{2,s}, \|\Psi\|_s = 1 \} .$$

2. $L^\infty(\Omega)$ BOUNDS

The principal results of this section are summarized in the following two theorems.

Theorem 1. Suppose that $a \equiv 1$ and $\tau_0 \geqslant 1$. There exists a constant C such that if u satisfies (1), U satisfies (4), and $\tau \leqslant \tau_0$, then

$$\|U - u\|_{L^\infty(\Omega)} \leqslant C[\|u\|_{r+2} + \|u\|_{W^{\infty,r+1}}]h^{r+1}. \tag{7}$$

Theorem 2. Suppose that $a \in C^2(\bar{\Omega})$, $\tau_0 \geqslant 1$, and $k = 0$. There is a constant C depending on τ_0 and $\|a\|_{W^{\infty,2}}$ such that, if u and U satisfy (1) and (4), respectively, and $\tau \leqslant \tau_0$,

$$\|U - u\|_{L^\infty(\Omega)} \leqslant C\,[\|u\|_{r+2} + \|u\|_{W^{\infty,r+1}}]h^{r+1}. \tag{8}$$

These theorems will be proved by noting that in each case U is close in $L^\infty(\Omega)$ to a function $W \in \mathcal{M}$ which we shall construct using one-dimensional projections. Let $W = P \otimes Pu$, where P is the one-dimensional $H_0^1((0, 1))$ projection into $\mathcal{M}_k^0(r, \delta)$. The function W can be viewed as being constructed by first projecting $u(\cdot, y)$ into $\mathcal{M}_k^0(r, \delta)$ for each y and then projecting this function into $\mathcal{M}_k^0(r, \delta)$ for each x, or by first projecting for all fixed x and then projecting for all fixed y. In particular W satisfies

$$\begin{aligned} (W_x, V_x) &= ((I \otimes P)(u_x), V_x), \qquad V \in \mathcal{M}, \\ (W_y, V_y) &= ((P \otimes I)(u_y), V_y), \qquad V \in \mathcal{M}. \end{aligned} \tag{9}$$

First we note that the function W is close to u. In [4] it is shown that for $r + 1 \geqslant m \geqslant 1$ and $V \in W^{\infty,m}((0, 1))$, $V(0) = V(1) = 0$,

$$\|(I - P)V\|_{L^\infty((0,1))} \leqslant Ch^m \|V^{(m)}\|_{L^\infty((0,1))}, \tag{10}$$

where C depends on τ_0. Thus,

$$\begin{aligned} \|(I \otimes I - P \otimes P)u\|_{L^\infty(\Omega)} &\leqslant \|(I - P) \otimes Pu\|_{L^\infty(\Omega)} + \|P \otimes (I - P)u\|_{L^\infty(\Omega)} \\ &\quad + \|(I - P) \otimes (I - P)u\|_{L^\infty(\Omega)} \\ &\leqslant \|(I - P) \otimes Iu\|_{L^\infty(\Omega)} + \|I \otimes (I - P)u\|_{L^\infty(\Omega)} \\ &\quad + 3\,\|(I - P) \otimes (I - P)u\|_{L^\infty(\Omega)} \\ &\leqslant C\left[\left\|\left(\frac{\partial}{\partial x}\right)^{r+1} u\right\|_{L^\infty(\Omega)} + \left\|\left(\frac{\partial}{\partial y}\right)^{r+1} u\right\|_{L^\infty(\Omega)} \right. \\ &\quad \left. + \left\|\frac{\partial}{\partial x}\left(\frac{\partial}{\partial y}\right)^{r} u\right\|_{L^\infty(\Omega)}\right] h^{r+1} \\ &\leqslant C\,\|u\|_{W^{\infty,r+1}}\, h^{r+1}. \end{aligned} \tag{11}$$

Hence, in order to prove Theorems 1 and 2, it suffices to show in each case that $\|U - W\|_{L^\infty(\Omega)}$ is bounded by the right-hand side of (7) or (8). We can get the uniform closeness of U and W from (2.8) of [6], with say $\alpha = 3/2$. However, this imposes more smoothness than we need, and since we shall need a slightly different argument to get the super-convergence results of the next section, we shall instead derive our $L^\infty(\Omega)$ estimate from an $L^2(\Omega)$ estimate.

Proof of Theorem 1. We see from (2.1) of [6] that, with $\mathcal{V} = U - W$,

$$(\nabla \mathcal{V}, \nabla V) = (\nabla(u - W), \nabla V) = -(\eta, V), \qquad V \in \mathcal{M}, \tag{12}$$

where

$$\eta = I \otimes (I - P)u_{xx} + (I - P) \otimes Iu_{yy}. \tag{13}$$

Thus we see that

$$\|\mathcal{V}\|_1 \leqslant C\|\eta\|_{-1} \tag{14}$$

and that

$$|(\nabla \mathcal{V}, \nabla V)| \leqslant C\|\eta\|_{-2} \|V\|_2, \qquad V \in \tilde{\mathcal{M}}, \tag{15}$$

where $\tilde{\mathcal{M}} = \mathcal{M} \cap W^{2,2}$. It follows easily from (14) and the well-known properties of $I - P$ (see [1] or [5]) that

$$\|\mathcal{V}\|_1 \leqslant Ch^{r+1}(\|u_{xx}\|_r + \|u_{yy}\|_r) \leqslant Ch^{r+1} \|u\|_{r+2}. \tag{16}$$

Next, take φ such that

$$-\Delta\varphi = \mathcal{V} \text{ in } \Omega, \qquad \varphi = 0 \text{ on } \partial\Omega. \tag{17}$$

Then, we see that for appropriate $V \in \tilde{\mathcal{M}}$,

$$\begin{aligned} \|\mathcal{V}\|_0^2 &\leqslant (\nabla\mathcal{V}, \nabla(\varphi - V)) + C\|\eta\|_{-2} \|V\|_2 \\ &\leqslant C\|\mathcal{V}\|_1 h \|\varphi\|_2 + C\|\eta\|_{-2} \|\varphi\|_2, \end{aligned} \tag{18}$$

where we used the fact that for $\varphi \in W^{2,2}$

$$\inf\{\|V - \varphi\|_0 + h\|V - \varphi\|_1 + h^2\|V - \varphi\|_2 : V \in \tilde{\mathcal{M}}\} \leqslant Ch^2\|\varphi\|_2; \tag{19}$$

note that either $\tilde{\mathcal{M}} = \mathcal{M}$ or $\tilde{\mathcal{M}} \supset \mathcal{M}_1^0(r, \delta) \otimes \mathcal{M}_1^0(r, \delta)$. We see from (18) and elliptic regularity that

$$\|\mathcal{V}\|_0 \leqslant C(h\|\mathcal{V}\|_1 + \|\eta\|_{-2}) \leqslant Ch^{r+2}\|u\|_{r+2}. \tag{20}$$

It then follows from (20) and homogeneity that

$$\|\mathcal{V}\|_{L^\infty(\Omega)} \leqslant \frac{C}{\min h_i}\|\mathcal{V}\|_0 \leqslant C\tau_0 h^{r+1}\|u\|_{r+2}. \tag{21}$$

Combining (21) with (11) gives the conclusion.

Proof of Theorem 2. Again let $\mathcal{V} = U - W$. It follows from (2.14) of [6] that

$$\|\mathcal{V}\|_1 \leqslant C\|u\|_{r+2}h^{r+1}. \tag{22}$$

It is easily seen from the proof of Lemma 2.7 of [3] that, since $r \geqslant 3$, $(I - P)z$ is orthogonal to linear polynomials on each subinterval I_j. We note that, if $V \in \mathcal{N} = \mathcal{M}_0^0(1, \delta) \otimes \mathcal{M}_0^0(1, \delta)$, then with

$$\tilde{a}(x, y) = a(x_i, x_j) + a_x(x_i, x_j)(x - x_i) + a_y(x_i, x_j)(y - y_j) \text{ on } I_i \times I_j,$$

$$\begin{aligned} \int_{I_i \times I_j} \tilde{a}(u_x - W_x) V_x \, dx\, dy &= \int_{I_i \times I_j} [I \otimes (I - P)(u_x)] \tilde{a} V_x \, dx\, dy \\ &\leqslant C \| I \otimes (I - P)(u_x) \|_{L^2(I_i \times I_j)} h^2 \| (\tilde{a} V_x)_{yy} \|_{L^2(I_i \times I_j)} \\ &\leqslant C h^2 \| I \otimes (I - P)(u_x) \|_{L^2(I_i \times I_j)} \| V \|_{W^{2,2}(I_i \times I_j)}. \end{aligned} \tag{23}$$

In deriving this relation we used the orthogonality mentioned above twice, the fact that $\tilde{a} V_x$ is linear in y for each x, and the Peano Kernel theorem.

Using a similar result for the corresponding term with y-derivatives, we see that for $V \in \mathcal{N}$

$$\begin{aligned} (a \nabla \mathcal{V}, \nabla V) &= ([(a - \tilde{a}) + \tilde{a}] \nabla (u - W), \nabla V) \\ &\leqslant C h^2 \| u - W \|_1 \| V \|_1 + C h^2 (\| I \otimes (I - P)(u_x) \| \\ &\qquad + \| (I - P) \otimes I(u_y) \|) \, ||| V ||| \\ &\leqslant C h^{r+2} \| u \|_{r+1} \, ||| V |||, \end{aligned} \tag{24}$$

where

$$||| V |||^2 = \sum_{ij} \| V \|^2_{W^{2,2}(I_i \times I_j)} .$$

The bound for $\| u - W \|_1$ is derived by a computation that parallels (11) exactly. Let φ satisfy

$$- \nabla \cdot (a \nabla \varphi) = \mathcal{V} \text{ in } \Omega, \quad \varphi = 0 \text{ on } \partial\Omega. \tag{25}$$

Then for appropriate $V \in \mathcal{N}$

$$\begin{aligned} \| \mathcal{V} \|_0^2 &= (a \nabla \mathcal{V}, \nabla(\varphi - V)) + (a \nabla \mathcal{V}, \nabla V) \\ &\leqslant C \| \mathcal{V} \|_1 h \| \mathcal{V} \|_0 + C h^{r+2} \| u \|_{r+1} \| \mathcal{V} \|_0 \\ &\leqslant C h^{r+2} \| u \|_{r+2} \| \mathcal{V} \|_0. \end{aligned} \tag{26}$$

This gives the needed bound on $\| \mathcal{V} \|_{L^\infty(\Omega)}$ and thus the conclusion.

Notice that, in the case $k = 0$, Pz interpolates z at the knots x_i [2]; thus $P \otimes Pu$ can be determined in a completely local fashion on each $I_i \times I_j$. We can use this to get $L^\infty(\Omega)$ convergence rates even in the case in which Ω is a finite union of rectangles with sides parallel to the coordinate axes. It is easily checked that the $W^{2,1}$ bounds for $\mathcal{V}$ hold even for these domains. Hence, in the case $a \equiv 1$ we can use the analogue of (14) to see that

$$\|\mathcal{V}\|_1 = C_\alpha h^{r+1+\alpha} \|u\|_{r+2+\alpha} \quad , \quad 0 \leqslant \alpha \leqslant 1. \tag{27}$$

Thus, for any $\alpha \in (0, 1)$ we see that

$$\|\mathcal{V}\|_{L^\infty(\Omega)} \leqslant C_\alpha h^{r+1} \|u\|_{r+2+\alpha}. \tag{28}$$

This combined with the analogue of (11) on each $I_i \times I_j$ implies that $\|u - U\|_{L^\infty(\Omega)}$ goes to zero as h^{r+1}, provided $u \in W^{2,r+2+\alpha}$ for some $\alpha > 0$.

3. A SUPER-CONVERGENCE RESULT

In the case $k = 0$ the functions W and u are equal at the knots (x_i, x_j); this, combined with the proof of Theorem 1, allows us to show the following theorem.

Theorem 3. Suppose that $k = 0$, $a \equiv 1$ and $\tau_0 \geqslant 1$. Then there is a constant C such that for $\tau \leqslant \tau_0$

$$\max_{ij} |(U - u)(x_i, x_j)| \leqslant Ch^{r+2} \|u\|_{r+3}. \tag{29}$$

Proof. It follows from (2.8) of [6] that

$$\|\mathcal{V}\|_1 \leqslant Ch^{r+2} \|u\|_{r+3}. \tag{30}$$

Thus from (20) we see that

$$\|\mathcal{V}\|_0 \leqslant Ch^{r+3} \|u\|_{r+3}. \tag{31}$$

This implies that

$$\max_{ij} |(U - u)(x_i, x_j)| = \max_{ij} |(U - W)(x_i, x_j)| \leqslant Ch^{r+2} \|u\|_{r+3}.$$

REFERENCES

[1] J. H. BRAMBLE and J. E. OSBORN, *Rate of convergence estimates for nonselfadjoint eigenvalue approximations*, Math. Comp., *27* (1973), 525-549.

[2] J. DOUGLAS, Jr., and T. DUPONT, *Galerkin approximations for the two point boundary problem using continuous, piecewise-polynomial spaces, Numer. Math.*, 22 (1974), 99-109.

[3] J. DOUGLAS, Jr., and T. DUPONT, *Superconvergence for Galerkin methods for the two point boundary problem via local projections, Numer. Math.*, 21 (1973), 270-278.

[4] J. DOUGLAS, Jr., T. DUPONT and L. WAHLBIN, *Optimal L_∞ error estimates for Galerkin approximations to solutions of two point boundary problems, to appear.*

[5] J. DOUGLAS, Jr., T. DUPONT and M. F. WHEELER, *A quasi-projection approximation applied to Galerkin procedures for parabolic and hyperbolic equations, to appear.*

[6] J. DOUGLAS, Jr., T. DUPONT and M. F. WHEELER, *A Galerkin procedure for approximating the flux on the boundary for elliptic and parabolic boundary value problems*, this Journal, 47-59.