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## INTERIOR MAXIMUM NORM ESTIMATES FOR SOME SIMPLE FINITE ELEMENT METHODS (\*)

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*Summary.* — The approximate solution of a simple constant coefficient second order Dirichlet problem in a plane domain  $\Omega$  by means of Galerkin's method, using continuous, piecewise linear functions on a triangulation of  $\Omega$  is considered. It is proved that if the triangulation is regular in the interior of  $\Omega$  in a certain sense, and if  $h$  is the length of the longest edge of a triangle, then the error in the interior of  $\Omega$  is bounded by  $Ch^2 |\log h|$  if the exact solution  $u$  is twice continuously differentiable and by  $Ch^2$  if the second derivatives are Hölder continuous. Similar results are obtained for continuous, piecewise bilinear functions on a division of the interior of  $\Omega$  into rectangles.

### 1. INTRODUCTION

We shall consider the approximate solution of the Dirichlet problem

$$(1.1) \quad Lu \equiv - \sum_{j,k=1}^2 a_{jk} \frac{\partial^2 u}{\partial x_j \partial x_k} = f \quad \text{in } \Omega,$$

$$(1.2) \quad u = 0 \quad \text{on } \partial\Omega,$$

using a simple finite element method based on triangulating the domain  $\Omega$ . Here  $(a_{jk})$  is a positive definite constant matrix. For this purpose, for  $h$  a small positive parameter, let  $\Omega_h \subset \Omega$  be a polygonal domain approximating  $\Omega$ . Assume that  $\Omega_h$  is the union of closed triangles having disjoint interiors, such that no vertex of any triangle lies on the interior of an edge of another triangle and such that, uniformly in  $h$ , the edges of the triangles have length bounded above and below by constant multiples of  $h$  and all the angles are bounded below. We assume throughout that the boundary  $\partial\Omega$  is sufficiently regular and the approximation of  $\Omega$  by  $\Omega_h$  is sufficiently close for the estimates to be

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quoted below the hold. Since we are aiming for interior estimates we shall not be very precise about these assumptions; in fact, we will consider the estimates (1.3), (1.4) and (1.6) below as our assumptions in this respect. In particular these will be satisfied if  $\Omega$  is convex and smooth and the boundary vertices of  $\bar{\Omega}_h$  lie on  $\partial\Omega$ .

Let  $V_h$  be the finite dimensional linear space of functions which are continuous in the whole plane, linear in each triangle and vanish outside  $\Omega_h$ . Let  $\{P_j\}_{j=1}^{N_h}$  be the interior mesh-points (vertices) of the triangulation. Then a basis of  $V_h$  is formed by the elements  $\omega_j \in V_h$  for which  $\omega_j(P_l) = \delta_{jl}$  and the representation of  $v \in V_h$  with respect to this basis takes the simple form

$$v(x) = \sum_{j=1}^{N_h} v(P_j)\omega_j(x).$$

For a given continuous function  $u$  on  $\Omega$  which vanishes on  $\partial\Omega$  we define the interpolant  $\tilde{u} \in V_h$  by

$$\tilde{u}(x) = \sum_{j=1}^{N_h} u(P_j)\omega_j(x).$$

This function satisfies, as is well-known, under suitable assumptions

$$(1.3) \quad \|\tilde{u} - u\|_{\Omega} \leq Ch^2 \|u\|_{\Omega,2} ,$$

$$(1.4) \quad |\tilde{u} - u|_{\Omega} \leq Ch^2 |u|_{\Omega,2} .$$

Here and below we denote by  $\|\cdot\|_{\Omega,k}$  (with  $k$  omitted when zero) the norm in  $W_2^k(\Omega)$  and by  $|\cdot|_{\Omega,k}$  the norm in  $C^k(\Omega)$ . This latter norm will also be used for  $k$  non-integral so that, for instance,  $|u|_{\Omega,2+\epsilon}$  is finite when  $D^\alpha u \in \text{Lip}^\epsilon(\Omega)$  for  $|\alpha| = 2$  ( $0 < \epsilon < 1$ ). In (1.3), (1.4) and in what follows  $C$  denotes a positive constant independent of  $h$  and the functions involved, but not necessarily the same at different occurrences.

Introducing the bilinear form

$$A(v, w) = \int_{\Omega} \sum_{j,k=1}^2 a_{jk} \frac{\partial v}{\partial x_j} \frac{\partial w}{\partial x_k} dx,$$

the finite element problem may be formulated as follows : Find  $v \in V_h$  such that

$$A(v, w) = (f, w) \equiv \int_{\Omega} f(x)w(x) dx, \text{ for all } w \in V_h .$$

An equivalent formulation is : Find  $\{v(P_j)\}_1^{N_h}$  such that

$$(1.5) \quad \sum_{j=1}^{N_h} v(P_j)A(\omega_j, \omega_l) = (f, \omega_l) \quad , \quad l = 1, \dots, N_h .$$

It is well-known that this problem admits a unique solution  $v \in V_h$ , and it has been proved (cf. [2], [3]) that under suitable assumptions, if  $u$  is the exact solution of (1.1), (1.2), then  $v$  is an approximation of  $u$  of the same order as the interpolant in the sense that

$$(1.6) \quad \|v - u\|_{\Omega} \leq Ch^2 \|u\|_{\Omega,2} .$$

For the maximum norm of the error it was also proved that

$$|v - u|_{\Omega} \leq Ch \|u\|_{\Omega,2} .$$

However, although this latter estimate is optimal in the sense that its order cannot be improved without changing the norm on the right hand side, one could hope to replace  $h$  by  $h^2$  under more stringent regularity assumptions. We shall in fact be able to prove such second order estimates in the interior of the domain  $\Omega$  when the triangulation is regular in the interior of  $\Omega$  in the sense that there are three different directions in the plane such that for any  $\Omega^0$  with  $\Omega^0 \subset\subset \Omega$  (i.e.  $\bar{\Omega}^0 \subset \Omega$ ) and for  $h$  sufficiently small, the triangles intersecting  $\Omega^0$  are defined by equidistant lines parallel to the given directions.

The following is our main result.

**Theorem 1.** Assume that the triangulation is regular in the interior. Then for  $\Omega^0 \subset\subset \Omega$  there is a constant  $C$  and for any  $\varepsilon > 0$  a constant  $C_{\varepsilon}$  such that

$$|v - u|_{\Omega^0} \leq \begin{cases} Ch^2 \log \frac{1}{h} |u|_{\Omega,2} , \\ C_{\varepsilon} h^2 |u|_{\Omega,2+\varepsilon} . \end{cases}$$

The proof will depend on interpreting (1.5) as a finite difference equation. For this purpose we collect some results from finite difference theory in Section 2 and complete the proof of Theorem 1 in Section 3. In Section 4 we discuss briefly the case of regular quadrilateral elements in the interior of  $\Omega$ . A weaker form of Theorem 1 was given in [5].

## 2. ELLIPTIC FINITE DIFFERENCE OPERATORS

For real-valued functions defined on the square mesh  $hZ^2$  we denote  $u_{\alpha} = u(\alpha h)$ , and introduce the translation operator  $T^{\beta}$  defined by  $T^{\beta}u_{\alpha} = u_{\alpha+\beta}$ . We shall consider second order finite difference operators of the form

$$L_h = h^{-2} \sum_{\beta} b_{\beta} T^{\beta} ,$$

where only a finite number of the constant real coefficients  $b_{\beta}$  are non-zero. It is well-known (cf. [6]) that such a finite difference operator  $L_h$  is consistent with

the differential operator  $L = \sum_{|\gamma|=2} a_\gamma D^\gamma$  with  $D^\gamma = (\partial/\partial x_1)^{\gamma_1}(\partial/\partial x_2)^{\gamma_2}$  if and only if it can be written

$$(2.1) \quad L_h = \sum_{\beta, \gamma} c_{\beta\gamma} T^\beta \partial^\gamma \quad \text{with} \quad \sum_{\beta} c_{\beta\gamma} = a_\gamma,$$

where  $\partial^\gamma = \partial_1^{\gamma_1} \partial_2^{\gamma_2}$  and  $\partial_l = h^{-1}(T^{e_l} - I)$  for  $l = 1, 2$ , with  $e_1 = (1, 0)$  and  $e_2 = (0, 1)$ .

For mesh-functions we define, with  $k$  a non-negative integer, the following norms

$$\|v\|_{h, \Omega, k} = \sum_{|\gamma| \leq k} \|\partial^\gamma v\|_{h, \Omega} \quad \text{with} \quad \|v\|_{h, \Omega} = \left( h^2 \sum_{\alpha h \in \Omega} v^2 \right)^{1/2},$$

$$|v|_{h, \Omega, k} = \sum_{|\gamma| \leq k} |\partial^\gamma v|_{h, \Omega} \quad \text{with} \quad |v|_{h, \Omega} = \max_{\alpha h \in \Omega} |v_\alpha|,$$

and for  $0 < \varepsilon < 1$ ,

$$|v|_{h, \Omega, \varepsilon} = |v|_{h, \Omega} + \max_{\substack{\alpha \neq \beta \\ \alpha h, \beta h \in \Omega}} \frac{|v_\alpha - v_\beta|}{|(\alpha - \beta)h|^\varepsilon}.$$

For mesh-functions with finite support we also use the inner product

$$(v, w)_h = h^2 \sum_{\alpha} v_\alpha w_\alpha.$$

The adjoint  $L_h^*$  of  $L_h$  is defined by

$$L_h^* u = h^{-2} \sum_{\beta} b_\beta T^{-\beta} u,$$

and satisfies for  $v, w$  with finite support,

$$(L_h v, w)_h = (v, L_h^* w)_h.$$

The operator  $L_h$  is said to be elliptic if with  $\langle \beta, \theta \rangle = \beta_1 \theta_1 + \beta_2 \theta_2$ ,

$$p(\theta) = \sum_{\beta} b_\beta e^{i\langle \beta, \theta \rangle} \neq 0 \quad \text{for} \quad 0 \neq \theta \in Q = \{ \theta; |\theta_j| \leq \pi, j = 1, 2 \}.$$

For such operators we shall need the fundamental solution given by the following lemma.

**Lemma 2.1.** Let  $L_h$  be elliptic. Then there is a mesh-function  $g_\alpha$  defined on  $hZ^2$  such that

$$L_h g_\alpha = h^{-2} \delta_{\alpha, 0},$$

and for any  $\gamma$  there is a constant  $C$  such that

$$|\partial^\gamma g_\alpha| \leq C(h|\alpha| + h)^{-|\gamma|}.$$

*Proof.* See [4] where  $g_\alpha$  is given in the form

$$g_\alpha = (2\pi)^{-2} \int_Q \frac{e^{i\langle \alpha, \theta \rangle} - 1}{p(\theta)} d\theta.$$

We shall now use the fundamental solution to derive an estimate for the maximum norm of a mesh-function  $v$  in the interior of  $\Omega$  in terms of  $L_h v$  and the discrete  $L_2$ -norm of  $v$ .

**Lemma 2.2.** Let  $\Omega^1 \subset\subset \Omega^2 \subset\subset \Omega$  and let  $\chi \in C_0^\infty(\Omega^2)$  with  $\chi \equiv 1$  in a neighborhood of  $\Omega^1$ . Then there is a constant  $C$  such that for all mesh-functions  $v$  on  $hZ^2$ ,

$$|v|_{h, \Omega^1} \leq C \left\{ \max_{\alpha h \in \Omega^1} |(L_h v, \chi T^{-\alpha} g)_h|_{h, \Omega^1} + \|v\|_{h, \Omega^2} \right\},$$

where  $g$  is the fundamental solution from Lemma 2.1 for the elliptic operator  $L_h^*$ .

*Proof.* It follows from Lemma 2.1 that there is a constant  $C$  such that for  $\alpha h \in \Omega^1$ ,

$$|L_h^*(\chi T^{-\alpha} g)_\beta - h^{-2} \delta_{\alpha, \beta}| \leq C.$$

Consequently,

$$(2.2) \quad |(v, L_h^*(\chi T^{-\alpha} g))_h - v_\alpha| \leq Ch^2 \sum_{\beta h \in \Omega^2} |v_\beta| \leq C \|v\|_{h, \Omega^2}.$$

Since  $\chi$  has compact support in  $\Omega^2$  we have

$$(2.3) \quad (v, L_h^*(\chi T^{-\alpha} g))_h = (L_h v, \chi T^{-\alpha} g)_h.$$

Together (2.2) and (2.3) imply

$$|v_\alpha| \leq C \left\{ |(L_h v, \chi T^{-\alpha} g)_h| + \|v\|_{h, \Omega^2} \right\},$$

which proves the lemma.

In the application of this estimate below we shall have occasion to use the following lemma :

**Lemma 2.3.** Let  $M_h$  be a second order finite difference operator of the form (2.1). Then with the notation of Lemma 2.2 there is a constant  $C$  and for each  $\varepsilon \in (0, 1)$  a constant  $C_\varepsilon$  such that for any mesh-function  $w$  on  $hZ^2$ ,

$$|(M_h w, \chi T^{-\alpha} g)_h|_{h, \Omega^2} \leq \begin{cases} C \log \frac{1}{h} |w|_{h, \Omega^2}, \\ C_\varepsilon |w|_{h, \Omega^2, \varepsilon}. \end{cases}$$

*Proof.* Since  $\chi \in C_0^\infty(\Omega^2)$  we have for small  $h$  and  $\alpha h \in \Omega^1$ ,

$$|(M_h w, \chi T^{-\alpha} g)_h| = |(w, M_h(\chi T^{-\alpha} g))_h| \leq C |w|_{h, \Omega^2} h^2 \max_{|\gamma| \leq 2} \sum_{\beta \in \Omega^2} |\partial^\gamma g_{\beta-\alpha}|.$$

By Lemma 2.1 we have for  $\alpha h \in \Omega_1$ ,  $|\gamma| \leq 2$ , with  $d$  the diameter of  $\Omega$ ,

$$h^2 \sum_{\beta \in \Omega^2} |\partial^\gamma g_{\beta-\alpha}| \leq C \sum_{h|\beta| \leq d} (|\beta| + 1)^{-2} \leq C \log \frac{1}{h},$$

which proves the first inequality.

Since  $M_h$  annihilates constants, we also find

$$(M_h w, \chi T^{-\alpha} g)_h = (M_h(w - w_\alpha), \chi T^{-\alpha} g)_h = (w - w_\alpha, M_h^*(\chi T^{-\alpha} g))_h,$$

so that for  $\alpha h \in \Omega^1$ ,

$$\begin{aligned} |(M_h w, \chi T^{-\alpha} g)_h| &\leq C h^2 \sum_{\beta \in \Omega^2} \frac{|w_\beta - w_\alpha|}{(|\beta - \alpha| h + h)^2} \\ &\leq C |w|_{h, \Omega^2, \varepsilon} h^2 \sum_{h|\beta| \leq d} (|\beta| h + h)^{-(2-\varepsilon)} \leq C_\varepsilon |w|_{h, \Omega^2, \varepsilon}, \end{aligned}$$

which proves the second inequality.

### 3. PROOF OF THEOREM 1

Let  $\Omega^0 \subset \subset \Omega^1 \subset \subset \Omega^2 \subset \subset \Omega$ . In considering the regular triangulation in the interior of  $\Omega$  it is no restriction of generality, since  $L$  has arbitrary coefficients, to assume that the three families of straight lines defining the triangles are  $x_1 = nh$ ,  $x_2 = nh$ ,  $x_1 + x_2 = nh$  with  $n = 0, \pm 1, \dots$ . In this case the mesh-points of  $\Omega^2$ , for small  $h$ , are of the form  $\alpha h$  with  $\alpha \in Z^2$  and we may denote the corresponding basis functions by  $\omega_\alpha$  and set  $u_\alpha = u(\alpha h)$ . The basis of our analysis is then the following representation of the Galerkin equation (1.5) corresponding to the point  $P = \alpha h$  as a finite difference equation. Here we use in addition to the forward difference quotients  $\partial_j$  also the backward difference quotients,

$$\bar{\partial}_j v_\alpha = h^{-1}(v_\alpha - v_{\alpha - e_j}).$$

**Lemma 3.1.** For  $\alpha h \in \Omega^2$  the Galerkin equation (1.5) may be written

$$(3.1) \quad L_h v_\alpha = h^{-2}(f, \omega_\alpha),$$

where  $L_h$  is the elliptic finite difference operator

$$L_h = - \sum_{j,k=1}^2 a_{jk} \bar{\partial}_j \partial_k.$$

*Proof.* It suffices to show (3.1) for  $\alpha = 0$ . Let then  $P_0 = (0,0)$  and let  $\{P_j\}_{j=1}^6$  be its neighbors,  $P_1 = (h, 0)$ ,  $P_2 = (0, h)$ ,  $P_3 = (-h, h)$ ,  $P_4 = (-h, 0)$ ,  $P_5 = (0, -h)$ ,  $P_6 = (h, -h)$ .

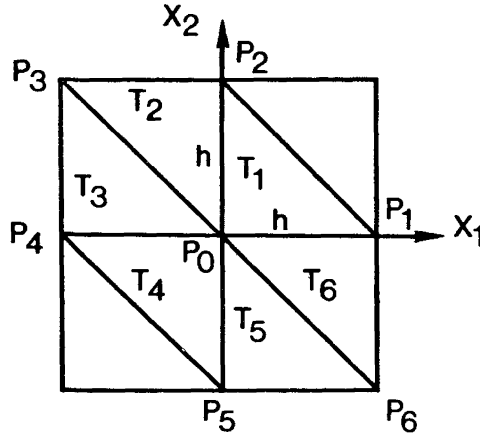


Figure 1

Simple calculations give then for the corresponding basis functions

$$\begin{aligned}
 A(\omega_0, \omega_0) &= 2(a_{11} + a_{12} + a_{22}), \\
 A(\omega_1, \omega_0) &= A(\omega_4, \omega_0) = -(a_{11} + a_{12}), \\
 A(\omega_2, \omega_0) &= A(\omega_5, \omega_0) = -(a_{12} + a_{22}), \\
 A(\omega_3, \omega_0) &= A(\omega_6, \omega_0) = a_{12}.
 \end{aligned}$$

This proves (3.1). We obtain for the characteristic polynomial of  $L_h$ ,

$$\begin{aligned}
 p(\theta) &= - \sum_{j,k=1}^2 a_{jk}(1 - e^{-i\theta_j})(e^{i\theta_k} - 1) \\
 &= \sum_{j,k=1}^2 a_{jk}(1 - \cos \theta_j)(1 - \cos \theta_k) + \sum_{j,k=1}^2 a_{jk} \sin \theta_j \sin \theta_k,
 \end{aligned}$$

so that since  $(a_{jk})$  is positive definite the operator  $L_h$  is an elliptic finite difference operator.

Our purpose is now to rewrite the right hand side of the difference equation (3.1) for  $f = Lu$  as a combination of difference operators applied to



certain averages of  $u$  along the edges of the triangulation. We define

$$U_\alpha^l = h^{-1} \int_0^h u(\alpha h + te_l) dt \quad , \quad l = 1, 2,$$

$$U_\alpha^3 = h^{-1} \int_0^h u(\alpha_1 h + t, \alpha_2 h + h - t) dt,$$

and obtain (cf. [1]) the following :

**Lemma 3.2.** With the above notation we have for  $\alpha h \in \Omega^2$ ,

$$3.2) \quad h^{-2}(Lu, \omega_\alpha) = -a_{11}[\bar{\partial}_1 \bar{\partial}_2 U_\alpha^3 + \partial_1(\bar{\partial}_1 - \bar{\partial}_2)U_\alpha^2] \\ - 2a_{12} \bar{\partial}_1 \bar{\partial}_2 U_\alpha^3 - a_{22}[\bar{\partial}_1 \bar{\partial}_2 U_\alpha^3 + \partial_2(\bar{\partial}_2 - \bar{\partial}_1)U_\alpha^1].$$

*Proof.* It is clearly sufficient to consider  $\alpha = 0$ . We obtain then, since  $\omega_0$  has its support in  $\bigcup_{i=1}^6 T_i$  (see fig. 1) and has constant gradient in  $T_i$ ,

$$\left( \frac{\partial^2 u}{\partial x_j \partial x_k}, \omega_0 \right) = - \left( \frac{\partial u}{\partial x_k}, \frac{\partial \omega_0}{\partial x_j} \right) = - \sum_{i=1}^6 \frac{\partial \omega_0}{\partial x_j} \Big|_{T_i} \int_{T_i} \frac{\partial u}{\partial x_k} dx \\ = - \sum_{i=1}^6 \frac{\partial \omega_0}{\partial x_j} \Big|_{T_i} \oint_{T_i} \nu_k u ds,$$

where  $\nu = (\nu_1, \nu_2)$  denotes the exterior normal. For  $j, k = 1$  we have hence, using the values of  $\frac{\partial \omega_0}{\partial x_1}$  in  $T_i$ ,

$$h^{-2} \left( \frac{\partial^2 u}{\partial x_1^2}, \omega_0 \right) = h^{-2} [U_0^3 - U_{-e_2}^3 - U_{-e_1}^3 + U_{-e_1 - e_2}^3 \\ + U_{e_1 - e_2}^2 - U_{e_2}^2 - U_0^2 + U_{e_1}^2] = \bar{\partial}_1 \bar{\partial}_2 U_0^3 + \partial_1(\bar{\partial}_1 - \bar{\partial}_2)U_0^2,$$

which is the coefficient of  $-a_{11}$  in (3.2). The remaining combinations of  $j$  and  $k$  are treated similarly.

We define now the following discrete averages,

$$u_\alpha^l = \frac{1}{2}(u_\alpha + u_{\alpha+e_l}), \quad l = 1, 2, \quad u_\alpha^3 = \frac{1}{2}(u_{\alpha+e_1} + u_{\alpha+e_2}),$$

and find easily, analogously to (3.2),

$$L_h u_\alpha = -a_{11}[\bar{\partial}_1 \bar{\partial}_2 u_\alpha^3 + \partial_1(\bar{\partial}_1 - \bar{\partial}_2)u_\alpha^2] \\ - 2a_{12} \bar{\partial}_1 \bar{\partial}_2 u_\alpha^3 - a_{22}[\bar{\partial}_1 \bar{\partial}_2 u_\alpha^3 + \partial_2(\bar{\partial}_2 - \bar{\partial}_1)u_\alpha^1].$$

Hence, setting  $w_\alpha^l = u_\alpha^l - U_\alpha^l$ ,  $l=1, 2, 3$ , we find

$$(3.3) \quad L_h u_\alpha - h^{-2}(Lu, \omega_\alpha) = - \sum_{l=1}^3 M_h^l w_\alpha^l,$$

where

$$M_h^1 = a_{22} \partial_2(\bar{\partial}_2 - \bar{\partial}_1), \quad M_h^2 = a_{11} \partial_1(\bar{\partial}_1 - \bar{\partial}_2), \quad M_h^3 = (a_{11} + 2a_{12} + a_{22}) \bar{\partial}_1 \bar{\partial}_2.$$

For the  $w^l$  we have the following estimates :

**Lemma 3.3** There is a constant  $C$  such that for  $0 \leq \epsilon < 1$  and  $l = 1, 2, 3$ ,

$$|w^l|_{h, \Omega^2, \epsilon} \leq Ch^2 |u|_{\Omega, 2+\epsilon}.$$

*Proof.* We have for instance

$$\begin{aligned} w_\alpha^1 &= \frac{1}{2}(u(\alpha h) + u(\alpha h + e_1 h)) - h^{-1} \int_0^h u(\alpha h + te_1) dt \\ &= \frac{1}{2h} \int_0^h t(h-t) \frac{\partial^2 u}{\partial x_1^2}(\alpha h + te_1) dt, \end{aligned}$$

from which the estimate immediately follows for  $l = 1$ . The cases  $l = 2, 3$  are similar.

We now complete the proof of Theorem 1. Since  $L_h$  is elliptic we may apply Lemma 2.2 to the restriction of  $u - v$  to the mesh-points of  $\Omega^2$ . We obtain by Lemma 3.1 and (3.3),

$$L_h(u - v)_\alpha = L_h u_\alpha - L_h v_\alpha = L_h u_\alpha - h^{-2}(Lu, \omega_\alpha) = - \sum_{l=1}^3 M_h^l w_\alpha^l,$$

and hence applying Lemmas 2.3 and 3.3,

$$(3.4) \quad \begin{aligned} |(L_h(u - v), \chi T^{-\alpha} g)|_{h, \Omega^1} &\leq \sum_{l=1}^3 |(M_h^l w^l, \chi T^{-\alpha} g)|_{h, \Omega^1} \\ &\leq \begin{cases} Ch^2 \log \frac{1}{h} |u|_{\Omega, 2}, \\ C_\epsilon h^2 |u|_{\Omega, 2+\epsilon} \text{ for } 0 < \epsilon < 1. \end{cases} \end{aligned}$$

We further obtain, using (1.3) and (1.6),

$$(3.5) \quad \begin{aligned} \|u - v\|_{h, \Omega^2} = \|\tilde{u} - v\|_{h, \Omega^2} &\leq C \|\tilde{u} - v\|_\Omega \leq C(\|v - u\|_\Omega + \|\tilde{u} - u\|_\Omega) \\ &\leq Ch^2 \|u\|_{\Omega, 2} \leq Ch^2 |u|_{\Omega, 2}. \end{aligned}$$

Together, Lemma 2.2, (3.4) and (3.5) now prove that

$$|\tilde{u} - v|_{\Omega^0} \leq |u - v|_{h, \Omega^1} \leq \begin{cases} Ch^2 \log \frac{1}{h} |u|_{\Omega, 2} , \\ C_\varepsilon h^2 |u|_{\Omega, 2+\varepsilon} \text{ for } 0 < \varepsilon < 1. \end{cases}$$

Using also (1.4) this completes the proof of Theorem 1.

#### 4. QUADRILATERAL ELEMENTS

In this section we shall consider the case in which the elements in the interior of the domain are squares with sides of length  $h$  and the approximating functions are continuous in the union of the squares and bilinear in each square. Such a division of  $\Omega$  in the interior may then be completed by means of triangles to a polygonal domain  $\Omega_h \subset \Omega$  and we shall denote by  $V_h$  the finite dimensional linear space of functions which are continuous in the whole plane, bilinear in the squares, linear in the triangles and which vanish outside  $\Omega_h$ . We may again in the present space  $V_h$  define basis functions  $\omega_j$  with  $\omega_j(P_l) = \delta_{jl}$  where  $\{P_l\}_1^{N_h}$  are the interior mesh-points and we assume that the approximation of  $\Omega$  by  $\Omega_h$  is such that the interpolant

$$\tilde{u}(x) = \sum_{j=1}^{N_h} u(P_j) \omega_j(x)$$

satisfies (1.3) and (1.4) as before. The finite element problem (1.5) still has a unique solution  $v \in V_h$  and the error estimate (1.6) holds.

We shall prove the following analogue of Theorem 1.

**Theorem 2.** Under the present assumptions and for  $\Omega^0 \subset \subset \Omega$  there is a constant  $C$  and for any  $\varepsilon \in (0, 1)$  a constant  $C_\varepsilon$  such that

$$|v - u|_{\Omega^0} \leq \begin{cases} Ch^2 \log \frac{1}{h} |u|_{\Omega, 2} , \\ C_\varepsilon h^2 |u|_{\Omega, 2+\varepsilon} . \end{cases}$$

Let  $\Omega^0 \subset \subset \Omega^1 \subset \subset \Omega^2 \subset \subset \Omega$  and assume that the mesh-points of  $\Omega^2$  for small  $h$  are in  $\Omega \cap hZ^2$ . We now denote the basis function corresponding to the point  $h\alpha \in \Omega^2$  by  $\omega_\alpha$ . This time we may represent the interior Galerkin equations as follows.

**Lemma 4.1.** For  $\alpha h \in \Omega^2$  the Galerkin equation (1.5) may be written

$$(4.1) \quad L_h v_\alpha = h^{-2}(f, \omega_\alpha),$$

where  $L_h$  is the elliptic finite difference operator

$$L_h = -a_{11} \frac{1}{6} (T^{\epsilon_2} + 4 + T^{-\epsilon_2}) \partial_1 \bar{\partial}_1 - a_{22} \frac{1}{6} (T^{\epsilon_1} + 4 + T^{-\epsilon_1}) \partial_2 \bar{\partial}_2 - \frac{1}{2} a_{12} (\partial_1 + \bar{\partial}_1) (\partial_2 + \bar{\partial}_2).$$

*Proof.* As above it suffices to consider  $\alpha = 0$  and we introduce  $P_0 = (0, 0)$  and its neighbors as in fig. 2 :

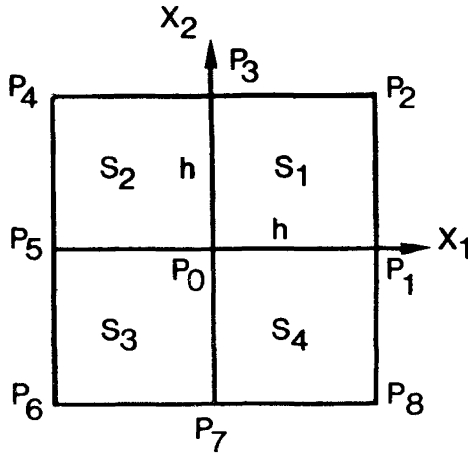


Figure 2

We obtain now

$$A(\omega_0, \omega_0) = \frac{4}{3} (a_{11} + a_{22}),$$

$$A(\omega_1, \omega_0) = A(\omega_5, \omega_0) = -\frac{2}{3} a_{11} + \frac{1}{3} a_{22} ,$$

$$A(\omega_3, \omega_0) = A(\omega_7, \omega_0) = \frac{1}{3} a_{11} - \frac{2}{3} a_{22} ,$$

$$A(\omega_2, \omega_0) = A(\omega_6, \omega_0) = -\frac{1}{6} a_{11} - \frac{1}{6} a_{22} - \frac{1}{2} a_{12} ,$$

$$A(\omega_4, \omega_0) = A(\omega_8, \omega_0) = -\frac{1}{6} a_{11} - \frac{1}{6} a_{22} + \frac{1}{2} a_{12} ,$$

which proves (4.1). The characteristic polynomial of  $L_h$  is now

$$p(\theta) = \frac{1}{3} a_{11}(4 + 2 \cos \theta_2)(1 - \cos \theta_1) \\ + \frac{1}{3} a_{22}(4 + 2 \cos \theta_1)(1 - \cos \theta_2) + 2a_{12} \sin \theta_1 \sin \theta_2 ,$$

or with  $s_j = \sin \frac{1}{2} \theta_j$ ,  $c_j = \cos \frac{1}{2} \theta_j$ ,

$$(4.2) \quad \frac{1}{4} p(\theta) = a_{11} \left( 1 - \frac{2}{3} s_2^2 \right) s_1^2 + 2a_{12} s_1 s_2 c_1 c_2 + a_{22} \left( 1 - \frac{2}{3} s_1^2 \right) s_2^2 .$$

Since the matrix  $(a_{jk})$  is positive definite we have

$$a_{11} c_2^2 s_1^2 + 2a_{12} s_1 s_2 c_1 c_2 + a_{22} c_1^2 s_2^2 \geq 0,$$

so that

$$\frac{1}{4} p(\theta) \geq \frac{1}{3} (a_{11} + a_{22}) s_1^2 s_2^2 \geq 0.$$

Hence for  $p$  to vanish we must have  $s_1 = 0$  or  $s_2 = 0$  and we then see from (4.2) that  $s_1 = s_2 = 0$  which proves the ellipticity.

For the purpose of giving an analogue of Lemma 3.2 we define

$$U_\alpha^l = h^{-2} \int_{-h}^h u(\alpha h + se_l)(h - |s|) ds, \quad l = 1, 2,$$

$$U_\alpha^3 = h^{-2} \int_0^h \int_0^h u(\alpha h + x) dx.$$

We then have the following :

**Lemma 4.2.** With the above notation we have for  $\alpha h \in \Omega^2$ ,

$$h^{-2}(Lu, \omega_\alpha) = -a_{11} \partial_1 \bar{\partial}_1 U_\alpha^2 - a_{22} \partial_2 \bar{\partial}_2 U_\alpha^1 - 2a_{12} \bar{\partial}_1 \bar{\partial}_2 U_\alpha^3 .$$

*Proof.* It is again sufficient to consider  $\alpha = 0$ . Applying the notation of fig. 2 we have

$$\left( \frac{\partial^2 u}{\partial x_j \partial x_k}, \omega_0 \right) = - \left( \frac{\partial u}{\partial x_k}, \frac{\partial \omega_0}{\partial x_j} \right) = - \sum_{l=1}^4 \int_{S_l} \frac{\partial u}{\partial x_k} \frac{\partial \omega_0}{\partial x_j} dx.$$

For  $j = k = 1$  we obtain, using Green's formula

$$\left( \frac{\partial^2 u}{\partial x_1^2}, \omega_0 \right) = h^{-2} \int_{S_1 \cup S_4} \frac{\partial u}{\partial x_1} (h - |x_2|) dx - h^{-2} \int_{S_2 \cup S_3} \frac{\partial u}{\partial x_1} (h - |x_2|) dx \\ = U_{e_1}^2 - 2U_0^2 + U_{-e_1}^2 = h^2 \partial_1 \bar{\partial}_1 U_0^2 ,$$

and similarly for  $\left(\frac{\partial^2 u}{\partial x_2^2}, \omega_0\right)$ . Finally,

$$\left(\frac{\partial^2 u}{\partial x_1 \partial x_2}, \omega_0\right) = h^{-2} \int_{S_1 \cup S_2} \frac{\partial u}{\partial x_1} (h - |x_1|) dx - h^{-2} \int_{S_3 \cup S_4} \frac{\partial u}{\partial x_1} (h - |x_1|) dx.$$

For the integral over  $S_1$  we obtain

$$\begin{aligned} h^{-2} \int_{S_1} \frac{\partial u}{\partial x_1} (h - |x_1|) dx &= h^{-2} \int_{S_1} \left[ \frac{\partial}{\partial x_1} ((h - x_1)u) + u \right] dx \\ &= -h^{-1} \int_0^h u(0, x_2) dx_2 + U_0^3. \end{aligned}$$

Adding the analogous expressions for the remaining  $S_i$  we obtain

$$\left(\frac{\partial^2 u}{\partial x_1 \partial x_2}, \omega_0\right) = h^2 \bar{\partial}_1 \bar{\partial}_2 U_0^3,$$

which completes the proof.

Defining this time

$$\begin{aligned} u_\alpha^l &= \frac{1}{6} (u_{\alpha-e_1} + 4u_\alpha + u_{\alpha+e_1}), \quad l = 1, 2, \\ u_\alpha^3 &= \frac{1}{4} (u_\alpha + u_{\alpha+e_1} + u_{\alpha+e_2} + u_{\alpha+e_1+e_2}), \end{aligned}$$

and again  $w_\alpha^l = u_\alpha^l - U_\alpha^l$ ,  $l = 1, 2, 3$ , we find now

$$L_h u_\alpha - h^{-2} (Lu, \omega_\alpha) = -a_{11} \partial_1 \bar{\partial}_1 w_\alpha^2 - a_{22} \partial_2 \bar{\partial}_2 w_\alpha^1 - 2a_{12} \bar{\partial}_1 \bar{\partial}_2 w_\alpha^3.$$

For the  $w_\alpha^l$  the obvious analogue of Lemma 3.3 holds. In fact, the  $w_\alpha^l$  depend linearly on  $u$  and vanish for  $u$  linear so that the result follows using the integral representation with second derivatives of the remainder in Taylor's formula.

The proof of Theorem 2 can now be completed analogously to Theorem 1.

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