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STABILITY AND DUALITY IN CONVEX MINIMIZATION PROBLEMS (1)

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Abstract. — *The purpose of this paper is to show the general relations that exist between the stability of a convex minimization problem and duality. Given an initial minimization problem, we consider a family of perturbed minimization problems (the initial problem corresponding to the perturbation zero). In this way, using the notion of conjugate functional, we define a dual problem (and also a family of perturbed dual problems).*

In the first part, the general relations between different notions of stability, the equality of the amounts of the dual and primal problems, the existence of solutions are established. Several sufficient conditions for stability are given.

In the second part, three different kinds of perturbations are studied (horizontal perturbations, vertical perturbations and mixed type perturbations). Several particular problems are considered, including spline function problems and best approximation problems.

INTRODUCTION

The purpose of this paper is to show the general relations that exist between the stability of a convex minimization problem and duality. Given an initial minimization problem, we consider a family of perturbed minimization problems (the initial problem corresponding to the perturbation zero). In this way, using the notion of conjugate functional, we define a dual problem (and also a family of perturbed dual problems). The general relations between different notions of stability, the equality of the amounts of the dual and primal problems, the existence of solutions are established. Several sufficient conditions for stability are given.

Our first motivation for working on this subject was a remarkable paper by R. T. Rockafellar [23] where the relations between the stability for a particular kind of perturbation (translation) and duality are studied (see § 2.1. below). In a first (not published) version of the present paper, we expressed

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Rockafellar's results by using the notion of inf-convolution of two convex functionals. The problem can be essentially reduced to the research of sufficient conditions for the lower-semi-continuity and sub-differentiability of the functional $h = f \nabla g$ (inf convolution of f and g). We gave several new conditions which imply these properties (see J. L. Joly [13]). Later we saw that the same type of conditions can be given for the stability (with respect to quite general perturbations) of an arbitrary convex minimization problem. These conditions are weaker than the usual ones (which use the notion of interior) and have the advantage of giving directly, when the space of the perturbations is finite dimensional, the conditions using the relative interior and the recession functional (Th. (1.7.9) and (1.8.9)).

The idea of associating a dual problem to a class of perturbations of the initial problem (in finite or infinite dimensional spaces) is due to R. T. Rockafellar [28] (we learned of the existence of this paper several months after the present paper was written).

The case when all spaces are finite dimensional is studied in great detail (using a different language : the notions of convex and concave bifunctions are introduced) in the recent and excellent book by R. T. Rockafellar [24]. Except for the theorems (1.7.9) and (1.8.9), we will not study this case specially.

In the first part we will present the theory : definition of the dual problem which is associated with the family of perturbed problems, different notions of stability, the relations between these notions, the duality and the existence of solutions, the characterization of the solutions and some sufficient conditions for stability.

In the second part, three different kinds of perturbations are studied (horizontal perturbations as in [23], vertical perturbations and mixed type perturbations). Several particular problems are considered, including spline function problems and best approximation problems.

A more detailed version of this paper has been published as a Mathematics Research Center technical summary report (1090), Madison, Wisconsin. The properties of convex functionals we will use can be found in this report or in J. J. Moreau [19].

STABILITY AND DUALITY

1.1. Definitions and notations

Let E and F be real vector spaces, in duality with respect to a bilinear form $\langle x, y \rangle$, $x \in E$, $y \in F$. We assume that E and F have been assigned locally convex Hausdorff topologies compatible with this duality.

We denote by $\bar{R}$ the set of extended real numbers

$$(\bar{R} = R \cup \{+\infty\} \cup \{-\infty\},$$

where R is the set of the reals numbers) with the natural order relation. The operation $+$ in $\overline{R}$ will have the obvious meaning of the addition with the supplementary convention :

$$(+\infty)+(-\infty)=(-\infty)+(+\infty)=(+\infty),$$

(This operation is denoted by $\dot{+}$ in [19].) We denote the effective domain of $f \in \bar{R}^E$ (f is a functional defined in E with values in $\bar{R}$) by dom (f) :

(1.1.1) $$\operatorname{dom}(f)=\{x \in E \mid f(x)<\infty\}$$

and the epigraph of f by epi (f) :

(1.1.2) $$\operatorname{epi}(f)=\{[x, \lambda] \in E \times R \mid f(x) \leqslant \lambda\}.$$

The functional $f \in \bar{R}^E$ is said to be *proper* if it does not take the value $+\infty$ identically and if it never takes the value $-\infty$.

If C is a subset of E, the indicator functional χ_C of C is defined by

(1.1.3) $$\chi_C(x)=\begin{cases} 0 \text{ if } x \in C \\ +\infty \text{ if } x \notin C. \end{cases}$$

We denote by conv (E) the set of all functionals $f \in \bar{R}^E$ which are convex (i.e. which have a convex epigraph).

The l.s.c. (lower semi-continuous) hull $\bar{f}$ of a functional $f \in \bar{R}^E$ is the greatest l.s.c. functional which is a minorant of f :

(1.1.4) $$\bar{f}(x)=\liminf_{x' \rightarrow x} f(x')$$

(The epigraph of $\bar{f}$ is the closure of the epigraph of f.)

We shall denote by $\Gamma(E)$ the set of all functionals $f \in$ conv (E) which are the supremum of a family of continuous affine functionals :

(1.1.5) $$f(x)=\sup_{i \in I}(\langle x, y_i\rangle-r_i),$$

where $y_i \in F$ and $r_i \in R$. It will be convenient also to denote by $\Gamma_0(E)$ the set of all $f \in \Gamma(E)$ which do not take the values $+\infty$ or $-\infty$ identically. One can prove that $\Gamma_0(E)$ is exactly the set of all proper l.s.c. convex functionals.

The conjugate functional f^* of a given functional $f \in \bar{R}^E$ is defined by

(1.1.6) $$f^*(y)=\sup_{x \in E}(\langle x, y\rangle-f(x)).$$

It is an element of $\Gamma(F)$. By the same construction we obtain the conjugate functional f^{**} of f^*. It is an element of $\Gamma(E)$.

The Γ-hull f^Γ of a functional $f \in \bar{R}^E$ is the greatest minorant of f which belongs to $\Gamma(E)$. One can prove that $f^\Gamma = f^{**}$ (and consequently, $f = f^{**}$ iff f belongs to $\Gamma(E)$).

A functional $f \in \operatorname{conv}(E)$ is said to be *sub-differentiable* at $x_0 \in E$ if $f(x_0)$ is finite and if there exists $y_0 \in F$ such that :

$$f(x) \geqslant f(x_0) + \langle x - x_0, y_0 \rangle, \text{ for all } x \in E.$$

(i.e. there exists a continuous affine minorant of f which takes at x_0 the same value). Such an element y_0 is called a sub-gradient of f at x_0. We shall call subdifferential of f at x_0 the (eventually empty) set $\partial f(x_0)$ of all sub-gradients of f at x_0. It is a closed convex set of F. As the Γ-hull $f^\Gamma = f^{**}$ of $f \in \operatorname{conv}(E)$ is the supremum of all continuous affine minorants of f, we have :

(1.1.7) $$\partial f(x_0) \neq \varnothing \quad \text{implies} \quad f(x_0) = f^{**}(x_0).$$

Since f and f^{**} have the same continuous affine minorants we also have :

(1.1.8) $$f(x_0) = f^{**}(x_0) \quad \text{implies} \quad \partial f(x_0) = \partial f^{**}(x_0).$$

The following inequality always holds for $f \in \operatorname{conv}(E)$:

$$f(x) + f^*(y) \geqslant \langle x, y \rangle, \text{ for all } x \in E, y \in F$$

and we have the following characterization of a sub-gradient of f at x_0 :

$$y_0 \in \partial f(x_0) \quad \text{iff} \quad f(x_0) + f^*(y_0) = \langle x_0, y_0 \rangle.$$

This gives the following result in the case where $f \in \Gamma_0(E)$, ([19], 10-a) :
(1.1.9) If $f \in \Gamma_0(E)$, then the three following statements are equivalent :

(i) $y_0 \in \partial f(x_0)$,

(ii) $x_0 \in \partial f^*(y_0)$,

(iii) $f(x_0) + f^*(y_0) = \langle x_0, y_0 \rangle$.

1.2. The minimization problem

Let X and Y be two locally convex Hausdorff linear topological spaces in duality with respect to the bilinear form $\langle x, y \rangle$, $x \in X$, $y \in Y$.

Consider the following minimization problem :

(P) $$\alpha = \operatorname*{Inf}_{x \in X} f(x)$$

with $f \in \Gamma(X)$. We will denote by A the (eventually empty) set of the solutions. If α is finite, we have (cf. [19], 10-b) :

$$A = \{\bar{x} \in X \mid \alpha = f(\bar{x})\} = \partial f^*(0). \tag{1.2.1}$$

The space X will be called the space of the *variables* for the problem (P).

Now, let U and V be two locally convex Hausdorff linear topological spaces in duality with respect to the bilinear form (u, v), $u \in U$, $v \in V$. The space U will be the space of the perturbations for the problem (P). We assume that we have a convex functional $\varphi \in \Gamma_0(X \times U)$ such that

$$f(x) = \varphi(x, 0) \quad , \quad \text{for all } x \in X. \tag{1.2.2}$$

For the perturbation $u \in U$, we consider the perturbed problem :

$$h(u) = \operatorname*{Inf}_{x \in X} \varphi(x, u). \tag{P_u}$$

The initial problem (P) corresponds to the value $u = 0$:

$$\alpha = h(0).$$

Note that the functional h belongs to $\operatorname{conv}(U)$ but not in general to $\Gamma_0(U)$.

Using the definition of the Γ-hull and the properties of the conjugate functional, we have :

$$-\alpha = -h(0) \leqslant -h^{**}(0) = \operatorname*{Inf}_{v \in V} h^*(v). \tag{1.2.3}$$

The spaces $X \times U$ and $Y \times V$ are two locally convex Hausdorff linear topological spaces in duality with respect to the bilinear form :

$$\langle\langle [x, u], [y, v] \rangle\rangle = \langle x, y \rangle + (u, v).$$

Let $\psi \in \Gamma_0(Y \times V)$ be the conjugate functional of φ :

$$\psi(y, v) = \operatorname*{Sup}_{\substack{x \in X \\ u \in U}} (\langle x, y \rangle + (u, v) - \varphi(x, u)).$$

Then we obtain for h^* the following formulae :

$$\begin{aligned} h^*(v) &= \sup_{u \in U} ((u, v) - \operatorname*{Inf}_{x \in X} \varphi(x, u)) \\ &= \sup_{\substack{x \in X \\ u \in U}} (\langle x, 0 \rangle + (u, v) - \varphi(x, u)) \end{aligned}$$

i.e. finally :

$$h^*(v) = \psi(0, v). \tag{1.2.4}$$

We shall put :

(1.2.5) $$g(v) = \psi(0, v).$$

The inequality (1.2.3) leads us to consider the following minimization problem :

(Q) $$\beta = \operatorname*{Inf}_{v \in V} g(v).$$

The problem (Q) will be called the dual problem of (P) with respect to the amily of perturbed problems (P) (which is defined by φ).

We will denote by B the (possibly empty) set of the solutions of (Q) ; If β is finite, we have :

(1.2.6) $$B = \{ \bar{v} \in V \mid \beta = g(\bar{v}) \} = \partial g^*(0).$$

For this problem (Q), the space V is the space of the variables and the space Y will be the space of the perturbations. For the perturbation $y \in Y$, we consider the *perturbed dual problem* :

(Q_y) $$k(y) = \operatorname*{Inf}_{v \in V} \psi(y, v)$$

We have : $$\beta = k(0).$$

Note that the functional k belongs to conv (Y) but not in general to $\Gamma_0(Y)$.

By (1.2.3) the following inequality :

(1.2.7) $$-\beta \leqslant \alpha$$

is always true. We will give in the next paragraphs some conditions which imply the equality $-\beta = \alpha$. It is important to note that this construction is completely symmetrical with respect to (P) and (Q). We have clearly :

(1.2.8) $$-\beta = -k(0) \leqslant -k^{**}(0) = \operatorname*{Inf}_{x \in X} k^*(x)$$

and by a direct calculation we obtain :

(1.2.9) $$k^*(x) = \varphi(x, 0) = f(x)$$

hence :

$$-\beta \leqslant \operatorname*{Inf}_{x \in X} \varphi(x, 0) = \operatorname*{Inf}_{x \in X} f(x) = \alpha.$$

Thus, applying the same transformation to the problem (Q) we obtain the problem (P) : *the problem (P) is the dual of (Q) with respect to the family of perturbed problems* (Q_y) (which is defined by ψ).

REMARKS

1° Without assumption, all possible cases can happen in the inequality (1.2.7) (including the cases where $\alpha = +\infty$ or $-\infty$ and $\beta = +\infty$ or $-\infty$). The condition « $\varphi(x, 0)$ not identically $+\infty$ » (i.e. $f \in \Gamma_0(X)$) implies $\alpha < \infty$ and, in the same way, the condition « $\psi(0, v)$ not identically $+\infty$ » (i.e. $g \in \Gamma_0(V)$) implies $\beta < \infty$. Thus, these two conditions together imply that both α and β are finite. The condition « α finite » implies that k has at least one continuous affine minorant (and of course that $\beta > -\infty$).

2° Usually, the dual of a minimization problem is written in the form of a maximization problem. As a matter of fact, instead of (Q), we could consider the following problem :

$$(\tilde{Q}) \qquad \tilde{\beta} = \operatorname*{Sup}_{v \in V} \tilde{g}(v)$$

with $\tilde{g} = -g$ and $\tilde{\beta} = -\beta$. This would lead to the inequality $\tilde{\beta} \leqslant \alpha$. But this way, we would have had to consider concave functionals, conjugate of concave functionals, etc..., and the presentation would have become slightly more complicated. Another advantage of our presentation is that the dual problem (Q) has exactly the same form as the primal problem (P).

1.3. Stability of the minimization problem

We shall give in this paragraph some relations between the stability of the minimization problem and the fact that the equality $-\beta = \alpha$ holds in the inequality (1.2.7). First we introduce two notions of stability for a minimization problem :

(1.3.1) *Definition*

The problem (P) will be said *stable* if $h(0)$ is finite and if h is continuous at $0 \in U$.

It is convenient to introduce another notion of stability which is weaker than the preceding one :

(1.3.2) *Definition*

The problem (P) will be said *inf-stable* if $h(0)$ is finite and if h is l.s.c. at $0 \in U$, i.e. :

$$\bar{h}(0) = \liminf_{u \to 0} h(u) = h(0).$$

We could introduce the notion of sup-stability (replace l.s.c. by u.s.c. in the definition), but in fact, as the functional h is convex, the problem (P) would be sup-stable iff it is stable. Thus the only two notions we will use are *stability* and *inf-stability*. The inf-stability of (P) (or of (Q)) is related to the duality in the following manner :

(1.3.3) **Theorem**

The following three statements are equivalent :

(i) the problem (P) is inf-stable,

(ii) the problem (Q) is inf-stable,

(iii) $-\beta = \alpha =$ a finite number.

Proof :

As the condition (iii) is symmetrical with respect to (P) and (Q), we have only to prove that (i) and (iii) are equivalent. By (1.2.3), (1.2.4) and (1.2.5) we have :

$$-\alpha = -h(0) \leqslant -h^{**}(0) = \beta. \tag{1.3.4}$$

In the same way, by (1.2.8) and (1.2.9), we have :

$$-\beta = -k(0) \leqslant -k^{**}(0) = \alpha. \tag{1.3.5}$$

Thus the condition $-\alpha = \beta$ is equivalent to the condition $h(0) = h^{**}(0)$ (or to the condition $k(0) = k^{**}(0)$). Suppose that (P) is inf-stable ; we have $\bar{h}(0) = h(0)$, a finite number. The l.s.c. convex functional $\bar{h}$, which is finite at 0, cannot take the value $-\infty$ ([19], 2. f) ; hence $\bar{h} \in \Gamma_0(U)$ and we have $\bar{h} = h^{\Gamma} = h^{**}$. Thus we have $h(0) = h^{**}(0)$, a finite number.

Conversely, suppose that $h(0) = h^{**}(0) =$ a finite number. The inequality $h^{\Gamma}(0) \leqslant \bar{h}(0) \leqslant h(0)$ implies then that $\bar{h}(0) = h(0) =$ a finite number; hence (P) is inf-stable.

(1.3.6) REMARK

If the problems (P) and (Q) are inf-stable, by (1.2.1), (1.2.6) and (1.1.8), the sets of their solutions can be written :

$$A = \partial k(0) \qquad \text{and} \qquad B = \partial h(0).$$

We will give later conditions which imply the stability or the inf-stability of the problem (P) (or (Q)).

1.4. Differential stability of the minimization problem

We denote by $h_0(u)$ the one-sided directional derivative of h at 0 with respect to a direction u :

$$h_0(u) = h'(0, u) = \lim_{\substack{\lambda \to 0 \\ \lambda > 0}} \frac{h(\lambda u) - h(0)}{\lambda}$$

(1.4.1) *Definition*

The problem (P) will be said *dif-stable* if $h(0)$ is finite and if h is Gateaux-differentiable at $0 \in U$, i.e. if there exists $v_0 \in V$ such that :

$$h_0(u) = (u, v_0), \text{ for all } u \in U.$$

As for the stability, we shall introduce another notion which is weaker than the preceding one :

(1.4.2) *Definition*

The problem (P) will be said *inf-dif-stable* if $h(0)$ is finite and if there exists $v_0 \in V$ such that :

$$h_0(u) \geqslant (u, v_0), \text{ for all } u \in U.$$

We could introduce the notion of sup-dif-stability (replace $\geqslant$ by $\leqslant$ in the definition). But as the functional h_0 is convex and $h_0(0) = 0$, the problem (P) would be sup-dif-stable iff it is dif-stable. Thus we have only two notions : dif-stability and inf-dif-stability.

(1.4.3) **Proposition**

The problem (P) is inf-dif-stable iff $\partial h(0)$ is non empty.

Proof :

Suppose $\partial h(0)$ is non empty (this implies that $h(0)$ is finite). We put $h_0(u) = h'(0, u)$. We have (see [19], 10-f) :

(1.4.4) $$h_0^* = \chi_{\partial h(0)}, \text{ hence :}$$

(1.4.5) $$h_0^{**}(u) = \sup_{v \in \partial h(0)} (u, v).$$

Thus, if $v_0 \in \partial h(0)$, we have :

$$h_0(u) \geqslant h_0^{**}(u) \geqslant (u, v_0), \quad \text{for all } u \in U.$$

Conversely, suppose that (P) is inf-dif-stable : As $h_0 \in \operatorname{conv}(U)$ has a continuous affine minorant, h_0^* does not take the value $+\infty$ identically and by (1.4.4), $\partial h(0)$ is non-empty.

(1.4.6) **Proposition** [23]

The problem (P) is inf-dif-stable iff $h(0)$ is finite and there exists a neighborhood $\mathcal{U}$ of $0 \in U$ and a number $M \in R$ such that :

$$h_0(u) \geqslant M, \text{ for all } u \in \mathcal{U}.$$

Instead of the words «inf-stable» and «inf-dif-stable» R. T. Rockafellar [23], in the case where the perturbations are translations, uses « normal » and « stably set » (note that we always suppose in our definitions that $h(0)$ is finite). In fact, in order to define a stably set problem this author takes the property of the proposition (1.4.6) and then proves that it is equivalent to « $\partial h(0)$ non empty »; (this equivalence is true for every convex functional h and thus the proposition is quite independant of the perturbations).

REMARKS

(1.4.7) If the problem (P) is stable, then it is also inf-dif-stable.

(If the functional $h \in \text{conv}(U)$ is finite and continuous at $0 \in U$, then the sub-differential $\partial h(0)$ is non-empty; See [19], 10.c.)

(1.4.8) If the problem (P) is stable and if $\partial h(0)$ consists of just one element, then the problem (P) is dif-stable. (If $h \in \text{conv}(U)$ is finite continuous at $0 \in U$, then the formula (1.4.5) becomes (See [19], 10.f) :

$$h_0(u) = \max_{v \in \partial h(0)} (u, v)$$

which gives exactly (1.4.1) in the case where $\partial h(0)$ consists of just one element).

1.5. Stability and duality

The theorem (1.3.3) shows that the inf-stability of (P) (or of (Q)) is equivalent to the fact that the equality $-\beta = \alpha$ holds (with a finite amount). The next theorem gives the relation between inf-stability and inf-dif-stability. By (1.2.1) and (1.2.9), if α is finite, the set of solutions of (P) can be written :

$$A = \partial f^*(0) = \partial k^{**}(0). \tag{1.5.1}$$

In the same way, by (1.2.6), (1.2.4) and (1.2.5), if β is finite the set of solutions of (Q) can be written :

$$B = \partial g^*(0) = \partial h^{**}(0). \tag{1.5.2}$$

(1.5.3) **Theorem**

The following two statements are equivalent :

(i) The problem (P) is inf-dif-stable,

(ii) The problem (P) is inf-stable and the problem (Q) has solutions.

Proof :

Suppose that (P) is inf-dif-stable, i.e. by proposition (1.4.3) that $\partial h(0) \neq \varnothing$. By (1.1.7), we have $h(0) = h^{**}(0)$, hence $-\beta = \alpha$ (see the proof of theorem (1.3.3)). Further, by (1.1.8), $\partial h(0) = \partial h^{**}(0)$, hence $B = \partial h^{**}(0)$ is non-empty. *Conversely*, suppose (P) to be inf-stable (i.e. $h(0) = h^{**}(0) =$ a finite number) and $B = \partial h^{**}(0)$ to be non-empty. By (1.1.8), $\partial h(0)$ is equal to $\partial h^{**}(0)$, hence is non-empty, and (P) is inf-dif-stable.

(1.5.4) **Corollary**

The following three statements are equivalent :

(i) The problems (P) and (Q) are inf-stable and have solutions,

(ii) The problems (P) and (Q) are inf-dif-stable,

(iii) The problem (P) is inf-dif-stable and has solutions.

Proof :

It is a direct consequence of theorems (1.3.3) and (1.5.3).

(1.5.5) REMARK

The notion of stability (for the problem (P)) depends on the topology which has been assigned to the space U. But the two notions of inf-stability and inf-dif-stability do not depend on this topology. They only depend on the duality between U and V : If (P) is inf-stable (inf-dif-stable) then it has still this property for every other locally convex topology on U which is compatible with the duality (for example, the weak topology $\sigma(U, V)$ or the Mackey topology $\tau(u, v)$) (It is clear that the equality $-\beta = \alpha$ and the existence of solutions for the problem (Q) do not depend on the topology).

1.6. Characterization of the solutions

The sets of solutions A and B of the problems (P) and (Q) are given by the formulae (1.5.1) and (1.5.2) which are true without any assumption.

With the assumption that (P) is inf-dif-stable we will obtain a characterization of the solutions which is very important for the applications. But first we give a sufficient condition for which we need no assumption.

(1.6.1) **Theorem**

If $\bar{x} \in X$ and $\bar{v} \in V$ satisfy one of the three (equivalent) conditions :

(i) $$\varphi(\bar{x}, 0) + \psi(0, \bar{v}) = 0,$$

(ii) $$[\bar{x}, 0] \in \partial\psi(0, \bar{v}),$$

(iii) $$[0, \bar{v}] \in \partial\varphi(\bar{x}, 0),$$

then $\bar{x}$ is solution of (P) and $\bar{v}$ is solution of (Q).

Proof :

Suppose that $\bar{x} \in X$ and $\bar{v} \in V$ satisfy (i).

We have

$$\alpha \leqslant \varphi(x, 0) \quad , \quad \text{for all } x \in X$$

and

$$\beta \leqslant \psi(0, v) \quad , \quad \text{for all } v \in V.$$

Further we have $0 \leqslant \alpha + \beta \leqslant \varphi(x, 0) + \psi(0, v)$, for all $x \in X$ and all $v \in V$.

If $\bar{x} \in X$ and $\bar{v} \in V$ satisfy :

$$0 = \varphi(\bar{x}, 0) + \psi(0, \bar{v}),$$

they necessarily satisfy :

$$\alpha = \varphi(\bar{x}, 0), \ \beta = \psi(0, \bar{v}) \ \text{ with } 0 = \alpha + \beta,$$

i.e. $\bar{x}$ is a solution of (P), $\bar{v}$ a solution of (Q) and the equality $-\beta = \alpha$ holds. The three conditions are equivalent by (1.1.9).

(1.6.2) **Theorem**

If the problem (P) is inf-dif-stable, then $\bar{x} \in X$ is a solution of (P) iff there exists $\bar{v} \in V$ satisfying one of the three (equivalent) conditions of theorem (1.6.1) ; such an element $\bar{v}$ is then necessarily a solution of the problem (Q).

Proof :

We know (th. (1.6.1)) that the condition is sufficient.

Conversely, suppose that $\bar{x} \in X$ is a solution of (P) ; we have $\alpha = \varphi(\bar{x}, 0)$. As the problem (P) is inf-dif-stable, by theorem (1.5.3), we have $\alpha + \beta = 0$ and the problem (Q) has solutions. Let $\bar{v}$ be a solution of (Q) ; we have $\beta = \psi(0, \bar{v})$. Thus we have $\varphi(\bar{x}, 0) + \psi(0, \bar{v}) = \alpha + \beta = 0$.

We can obtain equivalent characterisation theorems by using a notion of generalized Lagrangian.

(1.6.3) *Definition*

We shall call *Lagrangian of* (P) the following functional defined on $X \times V$ with values in $\bar{R}$.

$$l(x, v) = \operatorname*{Sup}_{u \in U} ((u, v) - \varphi(x, u)).$$

For a fixed x, the functional $v \rightarrow l(x, v)$ is convex and belongs to $\Gamma(V)$.

If we denote by φ_x the functional defined by : $\varphi_x(u) = \varphi(x, u)$, for all $u \in U$, we have in fact :

$$l(x, v) = \varphi_x^*(v).$$

For a fixed v, the functional $x \to -l(x, v)$ is convex but in general, it does not belong to $\Gamma_0(X)$. We have obviously :

$$\psi(y, v) = \sup_{x \in X} (\langle x, y \rangle + l(x, v)). \tag{1.6.4}$$

Hence, for $y = 0$:

$$\beta = \operatorname*{Inf}_{v \in V} \operatorname*{Sup}_{x \in X} l(x, v). \tag{1.6.5}$$

On the other hand, as $l(x, v) = \varphi_x^*(v)$ and $\varphi_x \in \Gamma(U)$,

$$\varphi(x, u) = \varphi_x(u) = \varphi_x^{**}(u) = \operatorname*{Sup}_{v \in V} ((u, v) - l(x, v)). \tag{1.6.6}$$

Hence, for $u = 0$:

$$\alpha = \operatorname*{Inf}_{x \in X} \operatorname*{Sup}_{v \in V} (-l(x, v)),$$

or equivalently,

$$-\alpha = \operatorname*{Sup}_{x \in X} \operatorname*{Inf}_{v \in V} l(x, v). \tag{1.6.7}$$

Then the equality $-\alpha = \beta$ corresponds to the equality :

$$\operatorname*{Inf}_{v \in V} \operatorname*{Sup}_{x \in X} l(x, v) = \operatorname*{Sup}_{x \in X} \operatorname*{Inf}_{v \in V} l(x, v)$$

We have the following characterization theorems, which are equivalent to the theorems (1.6.1) and (1.6.2) :

(1.6.8) **Theorem**

If $\bar{x} \in X$ and $\bar{v} \in V$ satisfy :

$l(x, \bar{v}) \leqslant l(\bar{x}, \bar{v}) \leqslant l(\bar{x}, v)$, for all $x \in X$ and all $v \in V$, then x is a solution of (P) and $\bar{v}$ is a solution of (Q).

Proof :

The condition of the theorem is equivalent to :

$$\operatorname*{Sup}_{x \in X} l(x, \bar{v}) = l(\bar{x}, \bar{v}) = \operatorname*{Inf}_{v \in V} l(\bar{x}, v).$$

Now, by (1.6.4) and (1.6.6) we have :

$$\operatorname*{Sup}_{x\in X} l(x, \bar{v}) = \psi(0, \bar{v}),$$

$$\operatorname*{Inf}_{v\in V} l(\bar{x}, v) = -\varphi(\bar{x}, 0).$$

Thus the condition of the theorem implies :

$\varphi(\bar{x}, 0) + \psi(0, \bar{v}) = 0$, which is the condition (i) of theorem (1.6.1).

(1.6.9) **Theorem**

If the problem (P) is inf-dif-stable, then $\bar{x} \in X$ is a solution of (P) iff there exists $\bar{v} \in V$ such that :

$l(x, \bar{v}) \leqslant l(\bar{x}, \bar{v}) \leqslant l(\bar{x}, v)$, for all $x \in X$ and all $v \in V$, (such an element $\bar{v}$ is necessarily a solution of (Q)).

Proof :

See the proofs of theorems (1.6.2) and (1.6.8).

REMARK :

The family of perturbed problems (P_u) is completely defined by the functional φ. This functional defines a unique Lagrangian l. Conversely, using (1.6.6), to a given Lagrangian l corresponds a unique functional φ, i.e. a unique family of perturbed problems (P_u).

1.7. Conditions for the stability of a minimization problem

In this paragraph we will give several conditions which imply the inf-dif-stability (and sometimes the stability) of the problem (P).

(1.7.1) **Theorem**

If $h(0)$ is finite and if there exists $x_0 \in X$ such that the functional $\varphi_{x_0} : u \in U \to \varphi(x_0, u)$ is finite and continuous at $0 \in U$, then the problem (P) is stable (hence also inf-dif-stable).

Proof :

As the functional φ_{x_0} is finite and continuous at $0 \in U$, there exist $M \in R$ and a neighborhood $\mathcal{U}$ of $0 \in U$ such that :

$$\varphi_{x_0}(u) = \varphi(x_0, u) \leqslant M \quad , \quad \text{for all } u \in \mathcal{U}.$$

Thus, we have :

$$h(u) = \operatorname*{Inf}_{x\in X} \varphi(x, u) \leqslant \varphi(x_0, u) \leqslant M, \text{ for all } u \in \mathcal{U}.$$

As $h \in \operatorname{conv}(U)$ is bounded on a neighborhood of 0, h is continuous at every point of the interior of its effective domain (See [19], 5*a*), in particular at 0.

REMARK :

As the notion of inf-dif-stability only depends on the duality (cf. (1.5.5)), if $h(0)$ is finite and if there exists $x_0 \in X$ such that the functional

$$\varphi_{x_0} : u \in U \rightarrow \varphi(x_0, u)$$

is finite and $\tau(U, V)$-continuous at $0 \in U$, then the problem (P) is still inf-dif-stable (but of course stable for this $\tau(U, V)$-topology only).

In order to obtain weaker sufficient conditions for stability of (P) we will need the following two definitions :

(1.7.2) *Definition*

A functional $h \in \operatorname{conv}(U)$ is said to be quasi-continuous at u_0 if its effective domain contains u_0 and spans a non-empty closed flat L_h of finite co-dimension, and if the restriction of h to L_h is continuous at u_0.

(1.7.3) *Definition*

A functional $h \in \operatorname{conv}(U)$ is said to be quasi-continuous if its effective domain spans a non-empty closed flat L_h of finite co-dimension and if the restriction of h to L_h is continuous at every point of the (non-empty) relative interior of dom (h) in L_h.

We shall prove several intermediate propositions before we state the main theorem (1.7.7) :

(1.7.4) **Proposition**

If there exists $x_0 \in X$ such that the convex functional :

$$\varphi_{x_0} : u \in U \rightarrow \varphi(x_0, u)$$

is quasi-continuous, then h is also quasi-continuous.

Proof :

Let L_0 and L be respectively the flats spanned by dom (φ_{x_0}) and dom (h). By the definition of h, we have dom $(h) \supset$ dom (φ_{x_0}), epi $(h) \supset$ epi (φ_{x_0}) and $L \supset L_0$ (hence L is finite-co-dimensional). A convex functional is continuous at every point of the (non-empty) interior of its effective domain iff its epigraph has a non-empty interior. Thus to prove that h is quasi-continuous, we have only to prove that epi (h) has a non-empty interior in $L \times R$. The set epi (φ_{x_0})

has a non-empty interior in L_0. If $L = L_0$, the property is obvious. Suppose now that $L \underset{\neq}{\supset} L_0$. There exists $z_1 \in U \times R$ such that $z_1 \in \text{epi}\,(h)$ and $z_1 \notin \text{epi}\,(\varphi_{x_0})$. The set co $(z_1 \cup \text{epi}\,(\varphi_{x_0}))$ is contained in epi (h) and has a non-empty interior in the flat V_1 spanned by epi (φ_{x_0}) and z_1, which is contained in L. Thus, epi $(h) \cap V_1$ has a non-empty interior in V_1. We can do the same construction until we have $V_k = L$ (since L_0 is a finite co-dimensional flat).

(1.7.5) *Definition*

Two convex subsets C_1 and C_2 are said to be *united* if they cannot be properly separated, i.e. if all closed hyperplane which separate C_1 and C_2 contain both of them. In the same way, the convex functionals f_1 and f_2 are said to be *united* if dom (f_1) and dom (f_2) are united.

(1.7.6) *Proposition*

If h is quasi-continuous and if $\{0\}$ and dom (h) are united, then h is quasi-continuous at $0 \in U$.

Proof :

Let us denote by L the flat which is spanned by dom (h). First we remark that $0 \in L$ (If we had $0 \notin L$ we could strictly separate $\{0\}$ and L, i.e. find a closed hyperplane which separates $\{0\}$ and L but does not intersect them and this would contradict the assumption « $\{0\}$ and dom (h) are united »). Now, we shall prove that $0 \in U$ belongs to the relative interior Ω of dom (h) in the flat L (i.e. that the restriction of h to L is continuous at 0).

Suppose that $0 \notin \Omega$. Then, there exists a closed hyperplane H in L which separates 0 and Ω and we have $\Omega \not\subset H$. Let N be a supplementary (necessarily finite-dimensional) linear space of L in U. The flat $H + N$ is a closed hyperplane in U and separates 0 and dom (h). However, we have dom $(h) \not\subset H + N$. This contradicts the assumption that 0 and dom (h) are united. Hence, h is quasi-continuous at 0.

(1.7.7) **Theorem**

If the following three conditions are satisfied :

(*a*) there exists $x_0 \in X$ such that the convex functional :

$$\varphi_{x_0} : u \in U \to \varphi(x_0, u) \text{ is quasi-continuous,}$$

(*b*) $\{0\}$ and dom (h) are united,

(*c*) $h(0)$ is finite,

then the problem (P) is inf-dif-stable.

Proof :

By proposition (1.7.4), h is quasi-continuous. Then, by proposition (1.7.6), h is quasi-continuous at 0. Let L be the flat which is spanned by dom (h). As the restriction of h to L is finite and continuous at 0, it is subdifferentiable at this point, i.e. there exists a continuous linear functional $\tilde{v}$ on L such that

$$h(u) \geqslant h(0) + \tilde{v}(u), \quad \text{for all} \quad u \in L \text{ (See [19], 10. } c).$$

Every continuous linear functional $v \in V$ which is an extension of $\tilde{v}$ to U will be clearly an element of $\partial h(0)$. Hence (P) is inf-dif-stable.

(1.7.8) Remark

If $\{0\}$ and dom (h) are united for a particular topology in U, then they are still united for all topologies which are compatible with the duality (since the closed hyperplans are the same). As the inf-dif-stability only depends on the duality, we can give a weaker condition in the theorem (1.7.7) : The condition (a) can be replaced by : « there exists $x_0 \in X$ such that the convex functional $u \in U \to \varphi(x_0, u)$ is quasi-continuous for the $\tau(U, V)$-topology ». The functional h will be quasi-continuous at 0 for this $\tau(U, V)$-topology, but the problem (P) will still be inf-dif-stable.

In the case where the space U of the perturbations is finite dimensional, the theorem (1.7.7) can be simplified :

(1.7.9) **Theorem**

Assume that U is a finite dimensional space. If $h(0)$ is finite and if $0 \in \text{ri}$ (dom (h)), then the problem (P) is inf-dif-stable.

Proof :

The functional h is convex on a finite dimensional space U and its effective domain is non-empty. This domain spans a flat L which is obviously finite co-dimensional and the restriction of h to L is continuous (at every point of ri (dom (h)) ; see [24], p. 82). Hence, h is quasi-continuous. Further, $\{0\}$ and dom (h) are united iff $0 \in$ ri (dom (h)) ; See [24], p. 97).

1.8 Dual conditions for the stability of the minimization problem

Using duality theorems we will give some equivalent conditions for the inf-dif-stability of the problem (P). We recall that a functional $q \in \bar{R}^V$ is said to be inf-compact (See [19], 4.d) if for all $r \in R$ the set :

$$\{v \in V \mid q(v) \leqslant r\}$$

is compact (eventually empty).

We denote again by φ_{x_0} and l_{x_0} the following functionals :

$$\varphi_{x_0} : u \in U \rightarrow \varphi(x_0, u),$$
$$l_{x_0} : v \in V \rightarrow l(x_0, v).$$

We have :

$$l_{x_0} = \varphi^*_{x_0}.$$

(1.8.1) **Theorem**

If $k(0)$ is finite and if there exists $x_0 \in X$ such that the functional l_{x_0} is inf-compact, then the problem (P) is inf-dif-stable (and stable for the $\tau(U, V)$-topology).

Proof :

As l_{x_0} is inf-compact, it is not identically $-\infty$, and φ_{x_0} belongs to $\Gamma_0(U)$. The functional l_{x_0} is a fortiori inf-compact for the $\sigma(V, U)$-topology. Now, it is a fundamental result (See [19]) that $\varphi^*_{x_0}$ is inf-compact for the $\sigma(V, U)$ topology iff φ_{x_0} is finite and continuous at 0 for the $\tau(U, V)$-topology.

Now we have $h(0) \leqslant \varphi_{x_0}(0) < \infty$ and the condition « $k(0)$ finite » implies that $h(0) > -\infty$; hence $h(0)$ is finite. Thus, by theorem (1.7.1), the problem (P) is stable for the $\tau(U, V)$-topology and consequently is inf-dif-stable (for all topologies compatible with the duality; cf. (1.5.5)).

In the same way, the property of quasi-continuity has an equivalent property in the dual space. We shall need the following definition :

(1.8.2) *Definition*

A functional $q \in \bar{R}^V$ is said to be inf-locally compact if for all $r \in R$ the set :

$$\{v \in V | q(v) \leqslant r\}$$

is locally compact.

(One can prove that q is locally compact iff its epigraph is locally compact in $V \times R$.)

Then we have the following property :

(1.8.3) **Proposition**

The following two statements are equivalent :

(i) The functional φ_{x_0} is quasi-continuous for the $\tau(U, V)$ topology.

(ii) The functional l_{x_0} is inf-locally compact for the $\sigma(V, U)$-topology (and is not identically $-\infty$).

Proof :

This result is a direct consequence of the following theorem (Joly [13]) : If $p \in \Gamma_0(U)$ and $q = p^*$, then the functional q is inf-locally compact for the $\sigma(V, U)$-topology iff p is quasi-continuous fot the $\tau(U, V)$-topology.

We have $\varphi_{x0} \in \Gamma(U)$ and $l_{x_0} = \varphi_{x0}^*$. So, we have only to show that (i) or (ii) implies that $\varphi_{x0} \in \Gamma_0(U)$, i.e. actually is not identically $+\infty$. If φ_{x0} is quasi-continuous, dom (φ_{x0}) is non-empty. On the other hand, if (ii) is satisfied, l_{x0} is not identically $-\infty$, hence φ_{x0} is not identically $+\infty$.

Now we will give conditions in the dual space V which are equivalent to : « $\{0\}$ and dom (h) are united ». We need first the following proposition :

(1.8.4) **Proposition**

If $\psi(0, v)$ is not identically $+\infty$, the following two conditions are equivalent :

(i) $\{0\}$ and dom (h) are united,

(ii) $\{0\}$ and dom (h^{**}) are united.

Proof :

Let C be a convex set. Then 0 and C are united iff 0 and $\bar{C}$ are united (as a matter of fact, a closed hyperplane P separates 0 and C iff it separates 0 and $\bar{C}$, and C is contained in P iff $\bar{C}$ is contained in P). Thus we have only to prove that $\overline{\text{dom}\,(h^{**})} = \overline{\text{dom}\,(h)}$.

As $\psi(0, v) = g(v) = h^*(v)$ is not identically $+\infty$, h has at least one continuous affine minorant and we have $h^{**} = \bar{h}$ (See [19], 5.e). Then by (1.1.4) we have :

$$\text{epi}\,(h^{**}) = \text{epi}\,(\bar{h}) = \overline{\text{epi}\,(h)}.$$

Using the continuity of the projection p of $U \times R$ onto U, we have :

$$\text{dom}\,(h^{**}) = p\,(\text{epi}\,(h^{**})) = p\,\overline{(\text{epi}\,(h))} \subset \overline{p\,(\text{epi}\,(h))} = \overline{\text{dom}\,(h)}.$$

Obviously, we have also $\overline{\text{dom}\,(h)} \subset \overline{\text{dom}\,(h^{**})}$.

Hence we have $\overline{\text{dom}\,(h)} = \overline{\text{dom}\,(h^{**})}$.

We recall the definitions of the recession cone and of the recession functional : Given a non empty closed convex subset C of V, we denote by C_∞ the recession cone of C, i.e. the set of all $y \in V$ such that $x + \lambda y \in C$, for all $x \in C$ and all $\lambda > 0$. It is a closed convex cone with vertex 0. We have also :

$$C_\infty = \bigcap_{\lambda > 0} \lambda(C - c) \quad , \quad \text{with } c \in C.$$

(This intersection does not depend on the particular element $c \in C$.)

Given $g \in \Gamma_0(V)$, we will denote by g_∞ the recession functional of g, i.e. the functional defined by :

$$g_\infty(v) = \operatorname*{Sup}_{\lambda > 0} \frac{g(v_0 + \lambda v) - g(v_0)}{\lambda} \quad \text{with } v_0 \in \operatorname{dom}(g).$$

(The definition does not depend on the particular choice of $v_0 \in \operatorname{dom}(g)$.)

The functional g_∞ belongs to $\Gamma_0(V)$. The preceding two notions are strongly related : The epigraph of the recession functional g_∞ of $g \in \Gamma_0(V)$ is equal to the recession cone of the epigraph of g.

We will use the following property :

(1.8.5) **Proposition**

If $g \in \Gamma_0(V)$, the following two statements are equivalent :

(i) $\{0\}$ and $\operatorname{dom}(g^*)$ are united,

(ii) $g_\infty(v) \geqslant 0$, for all $v \in V$ and

$g_\infty(v) = 0$ implies $g_\infty(-v) = 0$.

Proof :

We will use the following fundamental formulae ([19], 8.k)

$$g_\infty(v) = \operatorname*{Sup}_{u \in \operatorname{dom}(g^*)} (u, v).$$

The condition (ii) is equivalent to

$$g_\infty(v) \leqslant 0 \qquad \text{implies} \qquad g_\infty(-v) = g_\infty(v) = 0, \qquad \text{i.e.}$$

$$\operatorname*{Sup}_{u \in \operatorname{dom}(g^*)} (u, v) \leqslant 0 \text{ implies } \operatorname*{Sup}_{u \in \operatorname{dom}(g^*)} (u, -v) = \operatorname*{Sup}_{u \in \operatorname{dom}(g^*)} (u, v) = 0.$$

In other words, if the closed half space $\{u \in U \mid (u, v) \leqslant 0\}$ contains $\operatorname{dom}(g^*)$, then $\operatorname{dom}(g^*)$ is contained in the hyperplane $\{u \in U \mid (u, v) = 0\}$, i.e. $\{0\}$ and $\operatorname{dom}(g^*)$ are united.

(1.8.6) REMARKS

(i) The condition « $g_\infty(v) \geqslant 0$, for all $v \in V$ » is equivalent to « $0 \in \overline{\operatorname{dom}(g^*)}$ ».

(ii) We have here $g(v) = \psi(0, v)$. If $\psi(0, v)$ is not identically $+\infty$, then $g_\infty(v) = \psi_\infty(0, v)$, where ψ_∞ is the recession functional of $\psi \in \Gamma_0(Y \times V)$, i.e. with respect to the two variables.

The preceding equivalences lead to the following theorem :

(1.8.7) **Theorem**

If the following three conditions are satisfied :

(*a*) There exists $x_0 \in X$ such that the functional

$$l_{x_0} : v \in V \to l(x_0, v)$$

is inf-locally compact,

(*b*) The set $\{v \in V \mid g_\infty(v) = 0\}$ is a linear subspace,

(*c*) $k(0)$ is finite,

then the problem (P) is inf-dif-stable.

Proof :

As $k(0) = -h^{**}(0) > -\infty$ (See (1.3.4) and (1.3.5)), we have $0 \in \mathrm{dom}\,(h^{**})$, and by the remark (1.8.6.i), the condition « $g_\infty(v) \geqslant 0$, for all $v \in V$ » is satisfied. As $k(0) < +\infty$, $\psi(0, v) = g(v)$ is not identically $+\infty$ (i.e. $g \in \Gamma_0(V)$) and using proposition (1.8.5), $\{0\}$ and $\mathrm{dom}\,(g^*)$ are united. Finally, using proposition (1.8.4) and the fact that $g^* = h^{**}$, we see that $\{0\}$ and $\mathrm{dom}\,(h)$ are united.

The functional l_{x_0} is a fortiori inf-locally compact for the $\sigma(V, U)$-topology. Then, by proposition (1.8.3), the functional

$$\varphi_{x_0} : u \in U \to \varphi(x_0, u)$$

is quasi-continuous for the $\tau(U, V)$-topology.

By proposition (1.7.6), h is quasi-continuous at $0 \in U$, hence $h(0) < \infty$. The condition $k(0) < \infty$ (i.e. $g = h^*$ non identically $+\infty$) is equivalent to the fact that h has at least one continuous affine minorant, hence $h(0) > -\infty$. Finally we have « $h(0)$ finite ». By theorem (1.7.7), the problem (P) is inf-dif-stable for the $\tau(U, V)$-topology, hence (See remark (1.5.5)) inf-dif-stable for all topologies on U which are compatible with the duality.

Obviously, we can mix the preceding conditions to obtain other theorems for the inf-dif-stability of (P). For example the following theorem is certainly very useful for the applications :

(1.8.8) **Corollary**

If the following three conditions are satisfied

(*a*) there exists $x_0 \in X$ such that the convex functional

$$\varphi_{x_0} : u \in U \to \varphi(x_0, u)$$

is quasi-continuous,

(*b*) the set $\{v \in V \mid g_\infty(v) = 0\}$ is a linear subspace,

(*c*) $k(0)$ is finite,

then the problem (P) is inf-dif-stable.

In the case where U is a finite dimensional space, we obtain the following theorem :

(1.8.9) **Theorem**

Assume that U is finite dimensional. If $k(0)$ is finite and if the set $\{v \in V \mid g_\infty(v) = 0\}$ is a linear subspace, then the problem (P) is inf-dif-stable.

APPLICATIONS

In this second part, we will study several types of perturbations for different problems in optimization and approximation theory.

2.1. Horizontal perturbations

The first example was our starting point for writing the general approach. In fact we obtained the results, in a first version, by using the properties of the inf-convolution of two convex functionals. This example was studied directly (in a slightly different form, using concave functionals and a linear transformation) by R. T. Rockafellar [23] ; see also [24] for the case of finite dimensional spaces.

Consider the following minimization problem :

$$(P) \qquad \alpha = \operatorname*{Inf}_{x \in X} (a(x) + b(x))$$

with $a, b \in \Gamma_0(X)$. We have $f = a + b$ in the notation of § 1.2. This formulation includes the classical problem of the minimization of a convex functional on a convex set : If $b = \chi_C$, where C is a closed convex set (see (1.1.3)), then we have :

$$\alpha = \operatorname*{Inf}_{x \in C} a(x).$$

For the family of perturbed problems, we take :

$$(P_u) \qquad h(u) = \operatorname*{Inf}_{x \in X} (a(x) + b(x - u))$$

where $u \in U = X$. We have :

$$(2.1.1) \qquad \varphi(x, u) = a(x) + b(x - u).$$

For example, if $b = \chi_C$, then $h(u)$ is the minimum of a over C_u which is obtained by translating C of u. A very simple calculation shows that :

$$(2.1.2) \qquad \psi(y, v) = a^*(v + y) + b^*(-v).$$

Thus the dual problem of (P), (relatively to the type of perturbations we introduced) is :

$$(Q) \qquad \beta = \operatorname*{Inf}_{v \in V} (a^*(v) + b^*(-v))$$

and the perturbed dual problem (for the perturbation $y \in Y$) is :

$$(Q_y) \qquad k(y) = \operatorname*{Inf}_{v \in V} (a^*(y + v) + b^*(-v)).$$

(We have here $Y = V$.)

(2.1.3) REMARK

Given two convex functionals α, $\beta \in \Gamma_0(X)$, it is classical (see J. J. Moreau [19]) to consider the functional :

$$\gamma(x) = \operatorname*{Inf}_{\substack{x_1, x_2 \in X \\ x_1 + x_2 = x}} (\alpha(x_1) + \beta(x_1)) = \operatorname*{Inf}_{x_1 \in X} (\alpha(x_1) + \beta(x - x_1)).$$

We denote $\gamma = \alpha \nabla \beta$, the inf-convolution of α and β. The functional γ is convex, but in general it does belong to $\Gamma_0(X)$. We have (see [19], 3.b) :

$$\operatorname{dom}(\gamma) = \operatorname{dom}(\alpha) + \operatorname{dom}(\beta).$$

If we denote by $\hat{b}$ the functional defined by $\hat{b}(x) = b(-x)$, we have :

$$h = a \nabla \hat{b}.$$

In the same way we have :

$$k = a^* \nabla b^*.$$

Thus, some of the results we will give (proposition (2.1.4) for example) could be deduced from the properties of the inf-convolution.

We will rather use the general theory. The theorem (2.6.1) gives the following condition for the stability of the problem (P) :

(2.1.4) **Proposition** (Rockafellar [20])

If $\alpha = h(0)$ is finite and if there exists $x_0 \in X$ such that a is finite at x_0 and b is finite and continuous at x_0, then (P) is stable (hence inf-dif-stable).

(2.1.5) REMARK

As $h(u)$ can be written

$$h(u) = \operatorname*{Inf}_{x \in X} (b(x) + a(x + u))$$

if $\alpha = h(0)$ is finite and if there exists $x_0 \in X$ such that a is finite and continuous at x_0 and b is finite at x_0, then the problem (P) is stable.

The convex functional b will be said *continuous* if it is continuous at every point of the (non-empty) interior of its effective domain (It is well known that b is continuous iff it is continuous at one point x_0 of its effective domain). Thus the conditions of the proposition (2.1.4) can be written in the following form :

If (α) b is continuous,

(β) $\text{dom}(a) \cap \text{int}(\text{dom}(b)) \neq \emptyset$,

(γ) $\alpha = h(0)$ is finite,

then (P) is stable.

We will apply theorem (1.7.7) in order to obtain weaker conditions of the same form :

(2.1.6) **Proposition**

If the following three conditions are satisfied :

(α) b is quasi-continuous,

(β) dom (a) and dom (b) are united,

(γ) $\alpha = h(0)$ is finite,

then the problem (P) is inf-dif-stable.

Proof :

We have only to prove that the conditions (α), (β) and (γ) imply the conditions of theorem (1.7.7) :

If b is quasi-continuous, obviously $u \to b(x_0 - u)$ is quasi-continuous and if $x_0 \in \text{dom}(a)$, $u \to a(x_0) + b(x_0 - u)$ is also quasi-continuous.

By the remark (2.1.3), we have :

$$\text{dom}(h) = \text{dom}(a) + \text{dom}(\hat{b}) = \text{dom}(a) - \text{dom}(b).$$

Let us consider a closed hyperplane H_0 which separates 0 and dom (h) (the condition (γ) implies that H_0 contains 0). Thus we have :

$$H_0 = \{u \in U \mid \langle u, v \rangle = 0\}$$

and

$$\langle u, v \rangle \leqslant 0, \qquad \text{for all} \qquad u \in \text{dom}(a) - \text{dom}(b).$$

Therefore, there exists c such that :

$$\sup_{u_1 \in \text{dom}(a)} \langle u_1, v \rangle \leqslant c \leqslant \inf_{u_2 \in \text{dom}(b)} \langle u_2, v \rangle.$$

Thus the closed hyperplane :

$$H = \{u \in U \mid \langle u, v \rangle = c\}$$

separates dom (a) and dom (b). The condition (β) implies that H contains both dom (a) and dom (b), hence that H_0 contains dom (h). We have proved that $\{ 0 \}$ and dom (h) are united.

The following characterization theorem can be easily deduced from theorem (1.6.2) :

(2.1.7) **Proposition** (Rockafellar [20])

If the problem (P) is inf-dif-stable, then an element $\bar{x} \in X$ is a solution of (P) iff there exists $\bar{v} \in V$ such that $\bar{v} \in \partial a(\bar{x})$ and $-\bar{v} \in \partial b(\bar{x})$. (Such an element $\bar{v}$ is necessarily a solution of (Q)).

Proof :

An element $\bar{x} \in X$ is a solution of (P) iff there exists $\bar{v} \in V$ such that :

$$\varphi(\bar{x}, 0) + \psi(0, \bar{v}) = 0$$

i.e. $\quad a(\bar{x}) + b(\bar{x}) + a^*(\bar{v}) + b^*(-\bar{v}) = 0$. As we have always :

$$a(\bar{x}) + a^*(\bar{v}) \geqslant \langle \bar{x}, \bar{v} \rangle$$

$$b(\bar{x}) + b^*(-\bar{v}) \geqslant \langle \bar{x}, -\bar{v} \rangle$$

this is equivalent to :

$$a(\bar{x}) + a^*(\bar{v}) = \langle \bar{x}, \bar{v} \rangle, \quad \textit{i.e. } \bar{v} \in \partial a(\bar{x})$$

$$b(\bar{x}) + b^*(-\bar{v}) = \langle \bar{x}, -\bar{v} \rangle, \quad \textit{i.e. } -\bar{v} \in \partial b(\bar{x})$$

(2.1.8) REMARK

The same theorem could be obtained with the Lagrangian of (P) :

$$l(x, v) = \begin{cases} b^*(-v) + \langle x, v \rangle - a(x) \text{ if } x \in \operatorname{dom}(a) \\ -\infty \text{ if } x \notin \operatorname{dom}(a). \end{cases}$$

If (P) is inf-dif-stable, an element $\bar{x} \in$ dom (a) is a solution of (P) iff there exists $\bar{v} \in V$ such that :

$$b^*(-\bar{v}) + \langle x, \bar{v} \rangle - a(x) \leqslant b^*(-v) + \langle \bar{x}, \bar{v} \rangle - a(\bar{x}) \leqslant b^*(-v) + \langle \bar{x}, v \rangle - a(\bar{x})$$

for all $x \in$ dom (a) and all $v \in V$.

The first inequality implies that $\bar{v} \in \partial a(\bar{x})$ and the second inequality implies that $\bar{x} \in \partial b^*(-\bar{v})$, *i.e.* $-\bar{v} \in \partial b(\bar{x})$.

Existence of spline functions

Given a minimization problem (P), if one of its dual problems (Q) is inf-dif-stable, then the problem (P) has solutions. Thus the sufficient conditions for the inf-dif-stability of (Q) give conditions for the existence of solutions for (P). We will use this method for the problem of spline functions. Let X and Z be two real Hilbert spaces and T be a continuous linear operator of X onto Z with a finite dimensional null space $N = (\mathcal{N}T)$.

If we take $Y = X$, the spaces X and Y are in duality with respect to the inner product $\langle x \mid y \rangle$. Let C be a non empty closed convex set of X. We consider the following minimization problem which includes most of the problems related to spline functions (see [2], [16]) :

$$(P') \qquad \alpha = \operatorname*{Inf}_{x \in C} \|T(x)\|.$$

A solution of (P') is called a spline function (relatively to T and C). The perturbed problem we consider is the minimization of $\| T(x) \|$ over a translate of C. The spaces U and V are taken both equal to X with the duality defined by the inner product.

The function φ is :

$$\varphi(x, u) = \begin{cases} \| T(x) \| \text{ if } x - u \in C, \\ + \infty \text{ elsewhere.} \end{cases}$$

This problem is obviously a particular case of the problem we studied above :

$$\begin{aligned} a(x) &= \| T(x) \|, \\ b(x) &= \chi_C(x), \\ \varphi(x, u) &= \| T(x) \| + \chi_C(x - u). \end{aligned}$$

In this case we obtain for the conjugate functionals of a and b the following expressions :

$$a^*(y) = \begin{cases} 0 \text{ if } y \in N^{\perp} \text{and } \|T'^{-1}(y)\| \leqslant 1, \\ + \infty \text{ elsewhere,} \end{cases}$$

where T' denotes the adjoint operator of T, and :

$$b^*(y) = \operatorname*{Sup}_{x \in C} \langle x \mid y \rangle.$$

Thus the functional ψ is equal to :

$$\psi(y, v) = \begin{cases} \operatorname*{Sup}_{x \in C} \langle x \mid -v \rangle \text{ if } y + v \in N^{\perp} \text{and } \|T'^{-1}(y + v)\| \leqslant 1, \\ + \infty \text{ elsewhere.} \end{cases}$$

The dual problem is :

$$(Q') \qquad \beta = \operatorname*{Inf}_{\substack{v \in N^{\perp} \\ \|T'^{-1}(v)\| \leq 1}} \left(\operatorname*{Sup}_{x \in C} \langle x \mid -v \rangle \right)$$

and the perturbed dual problem :

$$(Q'_y) \qquad k(y) = \operatorname*{Inf}_{\substack{y+v \in N^{\perp} \\ \|T'^{-1}(y+v)\| \leq 1}} \left(\operatorname*{Sup}_{x \in C} \langle x \mid -v \rangle \right)$$

As the functional $a(x) = \| T(x) \|$ is continuous, using the theorem (2.6.1), we see that the problem (P') is stable. Hence we have $\alpha = - \beta$ and the problem (Q') has solutions.

In order to obtain existence of solutions for (P') we will give sufficient conditions for the inf-dif-stability of (Q') (using theorem (2.7.8)) in which we permute the roles of (P) and (Q)).

(2.1.9) **Proposition**

If the subset $C_\infty \cap N$ is a linear subspace, then the problem (Q') is inf-dif-stable (and consequently (P') has solutions).

Proof :

We will use the theorem (1.8.8), but this time for the problem (Q'). The number $\alpha = - \beta$ is finite, hence the condition (iii) is already satisfied. We have to show that there exists v_0 such that the functional

$$\psi_{v_0} : y \in Y \rightarrow \psi(y, v_0)$$

is quasi-continuous.

This functional is equal to the constant $\operatorname*{Sup}_{x \in C} \langle x \mid - v_0 \rangle$ if

$$y + v_0 \in N^{\perp} \text{ and } \|T'^{-1}(y + v_0)\| \leq 1.$$

The restriction of ψ_{v_0} to $- v_0 + N^{\perp}$ (which is a finite codimensional flat) is continuous : actually it is constant on a convex set which has a non-empty (relative) interior.

Now, we have to show that the set

$$E = \{ x \in X \mid \varphi_\infty(x, 0) = 0 \}$$

is a linear subspace.

As the functional $x \in X \rightarrow \| T(x) \|$ is positively homogeneous, we have

$$\varphi_\infty(x, 0) = \| T(x) \| + \chi_{C_\infty}(x),$$

and $E = N \cap C_\infty$; this gives the condition of our theorem.

2.2. Vertical perturbations

In this section, X and Y still denote two local convex Hausdorff linear topological spaces in duality. We denote by w a function defined on $X \times \Omega$ with values in $\bar{R}$, where Ω is a compact set. We suppose that for all $t \in \Omega$, the functional w_t belongs to $\Gamma_0(X)$ (where $w_t(x) = w(x, t)$) and for all $x \in X$ the functional w_x is continuous (where $w_x(t) = w(x, t)$). Let us consider the following minimization problem :

$$(P) \qquad \alpha = \operatorname*{Inf}_{x \in C} f_0(x),$$

with $f_0 \in \Gamma_0(X)$ and $C = \{ x \in X \mid w_t(x) \leqslant 0, \text{ for all } t \in \Omega \}$.

According to the notations of § 1.2, we have here :

$$f(x) = \begin{cases} f_0(x) \text{ if } w_t(x) \leqslant 0, \text{ for all } t \in \Omega, \\ + \infty \text{ elsewhere.} \end{cases}$$

We take $U = C(\Omega)$, the space of continuous functions on Ω with the norm :

$$\|u\| = \max_{t \in \Omega} |u(t)|.$$

For each $u \in C(\Omega)$ we consider the perturbed problem :

$$(P_u) \qquad h(u) = \operatorname*{Inf}_{x \in C_u} f_0(x),$$

with $C_u = \{ x \in X \mid w_t(x) \leqslant u(t), \text{ for all } t \in \Omega \}$. This corresponds to the following functional φ :

$$\varphi(x, u) = \begin{cases} f_0(x) \text{ if } w_t(x) \leqslant u(t), \text{ for all } t \in \Omega, \\ \infty \text{ elsewhere.} \end{cases}$$

As space V take for example the dual $C(\Omega)'$ of $C(\Omega)$ with the weak topology $\sigma(C(\Omega)', C(\Omega))$.

We will say that $v \in V$ is positive if :

$$(u, v) \geqslant 0, \text{ for all } u \in C(\Omega) \text{ satisfying } u(t) \geqslant 0, \text{ for all } t \in \Omega.$$

We denote by K^+ the set of all positive $v \in V$ (and in the same way by $K^- = - K^+$ the set of all negative $v \in V$). Then we obtain :

$$\psi(y, v) = \begin{cases} \operatorname*{Sup}_{x \in X} (\langle x, y \rangle - f_0(x) + (w_x, v)) \text{ if } v \in K^-, \\ + \infty \text{ elsewhere.} \end{cases}$$

Thus the dual problem of (P) is :

$$(Q) \qquad \beta = \inf_{v \in K^-} (\sup_{x \in X} (-f_0(x) + (w_x, v))).$$

The perturbed dual problem is :

$$(Q_y) \qquad k(y) = \inf_{v \in K^-} (\sup_{x \in X} (\langle x, y \rangle - f_0(x) + (w_x, v))).$$

As an example we will apply theorem (2.6.1) to obtain a condition for the inf-dif-stability of (P) :

(2.2.1) **Proposition**

If $\alpha = h(0)$ is finite, and if there exists $x_0 \in X$ such that $f_0(x_0)$ is finite and $w_{x_0}(t) < 0$ for all $t \in \Omega$ then the problem (P) is stable (hence inf-dif-stable).

Proof :

$$\text{Let } -\varepsilon = \max_{t \in \Omega} w_{x_0}(t). \text{ We have } \varepsilon > 0.$$

For every u in the following neighborhood of 0 :

$$\mathcal{U} = \{ u \in U \mid \|u\| < \varepsilon \}$$

we have $w_{x_0}(t) = w(x_0, t) \leqslant u(t)$, for all $t \in \Omega$, i.e., $x_0 \in C_u$, hence

$$\varphi(x_0, u) = f_0(x_0).$$

Thus there exists $x_0 \in X$ such that the functional $u \in U \to \varphi(x_0, u)$ is finite and continuous at $0 \in U$. We can apply theorem (1.7.1).

We have the following characterization of a solution of the problem (P) :

(2.2.2) **Proposition**

If the problem (P) is inf-dif-stable, then an element $\bar{x} \in C$ is a solution of (P) iff there exists $\bar{v} \in K^-$ such that :

$$f_0(\bar{x}) - (w_{\bar{x}}, \bar{v}) = \min_{x \in X} (f_0(x) - (w_x, \bar{v})) \text{ and } (w_{\bar{x}}, \bar{v}) = 0.$$

(such an element $\bar{v}$ is necessarily a solution of (Q)).

Proof :

Using the theorem (1.6.2), $\bar{x} \in C$ is a solution of (P) iff there exists $\bar{v} \in K^-$ such that :

$$\text{(i)} \qquad f_0(\bar{x}) + \sup_{x \in X} (-f_0(x) + (w_x, \bar{v})) = 0 \cdot$$

As $w_{\bar{x}} \in C(\Omega)$ is negative, we have $(w_{\bar{x}}, \bar{v}) \geqslant 0$ and thus :

$$-f_0(\bar{x}) \leqslant -f_0(\bar{x}) + (w_{\bar{x}}, \bar{v}) \leqslant \operatorname*{Sup}_{x \in X} (-f_0(x) + (w_x, \bar{v})).$$

The equation (i) then implies :

$$-f_0(\bar{x}) + (w_{\bar{x}}, \bar{v}) = \max_{x \in X} (-f_0(x) + (w_x, \bar{v})) \text{ and } (w_{\bar{x}}, \bar{v}) = 0.$$

(2.2.3) REMARK

We could obtain the same theorem by using the Lagrangian of (P) which is :

$$l(x, v) = \begin{cases} -f_0(x) + (w_x, v) \text{ if } v \in K^- \text{ and } x \in \operatorname{dom}(f_0), \\ +\infty \text{ if } v \notin K^- \text{ and } x \in \operatorname{dom}(f_0), \\ -\infty \text{ if } x \notin \operatorname{dom}(f_0). \end{cases}$$

If (P) is inf-dif-stable, then $\bar{x} \in \operatorname{dom}(f_0)$ is a solution of (P) iff :

$$-f_0(x) + (w_x, \bar{v}) \leqslant -f_0(\bar{x}) + (w_{\bar{x}}, \bar{v}) \leqslant -f_0(\bar{x}) + (w_{\bar{x}}, v),$$

for all $x \in \operatorname{dom}(f_0)$ and all $v \in K^-$. The first inequality is equivalent to :

$$f_0(\bar{x}) - (w_{\bar{x}}, \bar{v}) = \min_{x \in X} (f_0(x) - (w_x, \bar{v})).$$

The second inequality is equivalent to :

$$(w_{\bar{x}}, \bar{v} - v) \leqslant 0, \text{ for all } v \in K^-$$

which is equivalent to

$$(w_{\bar{x}}, \bar{v}) = 0 \text{ and } \bar{x} \in C.$$

EXEMPLE 1 :

Now, let us consider the case where Ω consists of just m elements $t_1, \ldots, t_m$. We denote :

$$f_i(x) = w_{t_i}(x) \quad , \quad i = 1, \ldots, m.$$

In that case we have :

$$C = \{ x \in X \mid f_i(x) \leqslant 0 \quad , \quad i = 1, \ldots, m \}.$$

If we put $u_i = u(t_i)$, $i = 1, \ldots, m$, a perturbation is now an element $u = [u_1, \ldots, u_m] \in R^m$.

We have :

$$\varphi(x, u) = \begin{cases} f_0(x) \text{ if } f_i(x) \leqslant u_i, \ i = 1, \ldots, m, \\ + \infty \text{ elsewhere} \end{cases}$$

$$\psi(y, v) = \begin{cases} \operatorname*{Sup}_{x \in X} \left(\langle x, y \rangle - f_0(x) + \sum_{i=1}^{m} v_i f_i(x)\right) \text{ if } v_i \leqslant 0, \ i = 1, \ldots, m, \\ + \infty \text{ elsewhere.} \end{cases}$$

and the problems (P) and (Q) become :

$$(P') \qquad \alpha = \operatorname*{Inf}_{x \in C} f_0(x)$$

$$(Q') \qquad \beta = \operatorname*{Inf}_{\substack{v_i \leqslant 0 \\ i=1,\ldots,m}} \left(\operatorname*{Sup}_{x \in X} \left(-f_0(x) + \sum_{i=1}^{m} v_i f_i(x)\right)\right)$$

The propostion (2.2.1) becomes :

If $\alpha = h(0)$ is finite, and if there exists $x_0 \in X$ such that $f_0(x_0)$ is finite and $f_i(x_0) < 0$, $i = 1, \ldots, m$ then the problem (P) is stable.

But in the present case, as the space U is finite dimensional, we can apply the theorems (1.7.9) or (1.8.9). This last theorem gives the following condition for the inf-dif-stability of (P).

(2.2.4) **Proposition**

If $\beta = k(0)$ is finite and if there exists λ such that the subset :

$$K_\lambda = \{ v \in R^m \mid v_i \leqslant 0, i = 1, \ldots, m, \operatorname*{Sup}_{x \in X} [-f_0(x) + \sum_{i=1}^{m} v_i f_i(x)] \leqslant \lambda \}$$

is non-empty and bounded, then the problem (P') is inf-dif-stable.

Proof :

As $k(0)$ is finite, we have $0 \in \operatorname{dom}(h^{**})$ and by the remark (1.8.6. i), $g_\infty(v) \geqslant 0$, for all $v \in V$.

Thus, we have :

$$\{ v \in V \mid g_\infty(v) = 0 \} = \{ v \in V \mid g_\infty(v) \leqslant 0 \}.$$

This set, which is called the recession cone of g, is equal to the recession cone of all sets :

$$K_\lambda = \{ v \in V \mid g(v) \leqslant \lambda \}$$

which are non-empty (see [24], Th. 8.7). If there exists λ such that K_λ is non-empty and bounded, then $(K_\lambda)_\infty = \{ v \in V \mid g_\infty(v) = 0 \}$ is equal to $\{ 0 \}$ and by theorem (1.8.9), the problem (P') is inf-dif-stable.

(2.2.5) REMARKS

(i) As $K_\lambda \subset \{ v \in R^m \mid v_i \leqslant 0, i = 1, ..., m \}$, if the recession cone of K_λ is a linear subspace, it is actually reduced to $\{ 0 \}$ and K_λ is bounded (cf. [24], p. 64).

(ii) There are several simple conditions which imply that $k(0)$ is finite. For example if $\varphi(x, 0)$ and $\psi(0, v)$ are both non identically $+ \infty$, then $\alpha = h(0)$ and $\beta = k(0)$ are both finite.

The proposition (2.2.2) becomes for the problem (P') :

(2.2.6) Proposition

If the problem (P') is inf-dif-stable, then an element $\bar{x} \in C$ is a solution of (P') iff there exists $\bar{v} \in R^m$, $\bar{v}_i \leqslant 0$, $i = 1, ..., m$ such that

$$f_0(\bar{x}) - \sum_{i=1}^{m} \bar{v}_i f_i(\bar{x}) = \min_{x \in X} \left(f_0(x) - \sum_{i=1}^{m} \bar{v}_i f_i(x)\right) \quad \text{and,}$$

$$\bar{v}_i f_i(\bar{x}) = 0, i = 1, ..., m.$$

Spline functions

As a practical example of problem (P') we take again the problem of spline functions (cf. M. Attéia [1], K. Ritter [20]) but with a different kind of perturbation.

We take the notation of § 2.1 but we shall define the convex C more explicitely : Given m elements $k_1, ..., k_m \in X$, and $2m$ real numbers

$$\alpha_i, \beta_i (\alpha_i < \beta_i), i = 1, ..., m),$$

we consider the convex set :

$$C = \{ x \in X \mid \alpha_i \leqslant \langle k_i \mid x \rangle \leqslant \beta_i, i = 1, ..., m \}$$

and the minimization problem :

$$(P') \qquad \alpha = \operatorname*{Inf}_{x \in C} \| T(x) \|.$$

We take $Y = X$ and the duality defined by the inner product. The space of the perturbations will be R^{2m}. Given a perturbation

$$u = [a_1, ..., a_m, b_1, ..., b_m] \in R^{2m},$$

we consider the perturbed convex subset

$$C_u = \{ x \in X \mid \alpha_i - a_i \leqslant \langle k_i \mid x \rangle \leqslant \beta_i + b_i, i = 1, \ldots, m \}$$

and the perturbed minimization problem :

$$(P'_u) \qquad h(u) = \operatorname*{Inf}_{x \in C_u} \| T(x) \| .$$

In the notation of this paragraph, we have :

$$\varphi(x, u) = \begin{cases} \| T(x) \| \text{ if } \alpha_i - a_i \leqslant \langle k_i \mid x \rangle \leqslant \beta_i + b_i, i = 1, \ldots, m, \\ + \infty \text{ elsewhere.} \end{cases}$$

We obtain for the functional

$$\psi(y, v) = \begin{cases} \operatorname*{Sup}_{x \in X} \left(\langle x \mid y + \sum_{i=1}^{m} (\mu_i - \rho_i) k_i \rangle - \| T(x) \| \right) + \sum_{i=1}^{m} \rho_i \alpha_i - \sum_{i=1}^{m} \mu_i \beta_i \\ \qquad \text{if } \rho_i \leqslant 0, \mu_i \leqslant 0, i = 1, \ldots, m, \\ + \infty \text{ elsewhere,} \end{cases}$$

with $v = [\rho_1, \ldots, \rho_m, \mu_1, \ldots, \mu_m] \in R^{2m}$.

As the conjugate functional of $f_0(x) = \| T(x) \|$ is (cf. § 2.1) :

$$f_0^*(y) = \begin{cases} 0 \text{ if } y \in N^{\perp} \quad \text{and} \quad \| T'^{-1}(y) \| \leqslant 1, \\ + \infty \text{ elsewhere,} \end{cases}$$

we obtain finally :

$$\psi(y, v) = \begin{cases} \sum_{i=1}^{m} \rho_i \alpha_i - \sum_{i=1}^{m} \mu_i \beta_i \text{ if } y + \sum_{i=1}^{m} (\mu_i - \rho_i) k_i \in N^{\perp}, \\ \qquad \| T'^{-1} (y + \sum_{i=1}^{m} (\mu_i - \rho_i) k_i) \| \leqslant 1 \\ \qquad \text{and} \quad \rho_i \leqslant 0, \mu_i \leqslant 0, i = 1, \ldots, m \\ + \infty \text{ elsewhere.} \end{cases}$$

The dual problem of (P') is :

$$(Q') \qquad \beta = \operatorname*{Inf}_{\substack{\rho_i \leqslant 0 \ \mu_i \leqslant 0, \\ \lambda_i = \mu_i - \rho_i}} \left(\sum_{i=1}^{m} \rho_i \alpha_i - \sum_{i=1}^{m} \mu_i \beta_i \right).$$

$$\sum_{i=1}^{m} \lambda_i k_i \in N^{\perp}$$

$$\| T'^{-1} \left(\sum_{i=1}^{m} \lambda_i k_i \right) \| \leqslant 1.$$

Applying proposition (2.2.1), we see that (P') is stable, hence inf-dif-stable. We can apply proposition (2.1.9) to deduce the existence of solutions for the problem (P'). Here C_∞ is a linear subspace. Hence the problem (P') has solutions.

Thus by corollary (1.5.4), we obtain :

(2.2.7) **Proposition**

The preceding spline-function problem (P') and its dual (Q') are both inf-dif-stable.

EXEMPLE 2 :

Let us consider now the case where space X is finite dimensional ($X = R^n$) and the functional w has the following form :

$$w(x, t) = \sum_{i=1}^{n} x_i c_i(t) - c_0(t)$$

where the c_i are continuous functions on Ω.

Suppose that $f_0(x) = \langle \gamma \mid x \rangle = \sum_{i=1}^{n} \gamma_i x_i$. The dual problem becomes :

$$(Q'') \qquad \beta = \operatorname*{Inf}_{v \leqslant 0} -(c_0, v).$$

$$(c_i, v) = \gamma_i,\ i = 1, \ldots, n.$$

The space Y of the perturbations for (Q'') is then n-dimensional. The perturbed dual problem is the following :

$$(Q''_y) \qquad k(y) = \operatorname*{Inf}_{v \leqslant 0} -(c_0, v)$$

$$(c_i, v) = \gamma_i - y_i,\ i = 1, \ldots, n.$$

We can apply theorem (1.8.9) to find conditions for the inf-dif-stability of (Q'') :

(2.2.7) **Proposition**

If $\{ x \in R^n \mid \sum_{i=1}^{n} \gamma_i x_i = 0 \} \cap \{ x \in R^n \mid \sum_{i=1}^{n} x_i c_i(t) \leqslant 0, \text{ for all } t \in \Omega \}$

is a linear subspace, and if α is finite then the problem (Q'') is inf-dif-stable (consequently $\alpha = -\beta$ and the problem (P) has solutions).

Proof :

By theorem (1.8.9) (applied to the problem (Q'')), if $h(0) = \alpha$ is finite and if $\{ x \in X \mid f_\infty(x) = 0 \}$ is a linear subspace, then (Q'') is inf-dif-stable. But we have here :

$$f_\infty(x) = \begin{cases} \sum_{i=1}^{n} \gamma_i x_i \text{ if } \sum_{i=1}^{n} x_i c_i(t) \leqslant 0, \text{ for all } t \in \Omega \\ +\infty \text{ elsewhere.} \end{cases}$$

2.3. Mixed perturbations

We will consider a general problem of approximation with constraints in a normed space.

Let X be a normed space and Y be its dual space (with the $\sigma(Y, X)$ topology). We denote by W a closed linear subspace of X. We consider the following closed convex subset :

$$C = \{ x \in X \mid \langle x - x_1, h \rangle \leqslant 0, \text{for all } h \in H \}$$

where H is a compact convex set of Y and $x_1 \in X$ is fixed. If we denote

$$q(x) = \max_{h \in H} \langle x, h \rangle,$$

the convex set C can be written

$$C = \{ x \in X \mid q(x - x_1) \leqslant 0 \}.$$

We put

$$p(x) = \max_{k \in K} \langle x, k \rangle,$$

where K is a symmetrical compact convex set of Y. The functional p is a continuous semi norm on X.

We will study the following problem :

$$(P) \qquad \alpha = \inf_{x \in C \cap W} p(x - x_0),$$

where $x_0 \in X$ is fixed.

As the space of perturbations, we will take : $U = X \times X \times R$.

Associated with a perturbation $u = [u_0, u_1, \mu]$, we have the perturbed problem :

$$(P_u) \qquad h(u) = \inf_{x \in C_{u_1,\mu} \cap W} p(x - (x_0 - u_0)),$$

where $C_{u_1,\mu} = \{ x \in X \mid q(x - (x_1 - u_1)) \leqslant \mu \}$.

Thus the functional φ is :

$$\varphi(x, u) = \begin{cases} p(x - (x_0 - u_0)) \text{ if } q(x - (x_1 - u_1)) \leqslant \mu \text{ and } x \in W, \\ + \infty \text{ elsewhere,} \end{cases}$$

where $u = [u_0, u_1, \mu] \in X \times X \times R$. The dual of U is $V = Y \times Y \times R$. A direct computation gives for ψ the following expression :

$$\psi(y, v) = \begin{cases} \langle x_0, v_0 \rangle + \langle x_1, v_1 \rangle \text{ if } v_0 \in K, v_1 = -\lambda h_1, h_1 \in H, \\ \qquad\qquad \lambda \leqslant 0 \text{ and } y - v_0 - v_1 \in W^{\perp}, \\ + \infty \text{ elsewhere,} \end{cases}$$

where $v = [v_0, v_1, \lambda] \in Y \times Y \times R$.

The dual problem is the following one :

$$(Q) \qquad \beta = \operatorname*{Inf}_{\substack{v_0 \in K \\ v_1 = -\lambda h_1 \\ h_1 \in H \\ \lambda \leqslant 0 \\ v_0 + v_1 \in W^{\perp}.}} (\langle x_0, v_0 \rangle + \langle x_1, v_1 \rangle)$$

This kind of dual is very useful for the computation of the solution. It is the basis for a generalization of the Remes' Algorithm (see [14]).

The perturbed dual problem is

$$(Q_y) \qquad k(y) = \operatorname*{Inf}_{\substack{v_0 \in K \\ v_1 = -\lambda h_1 \\ h_1 \in H \\ \lambda \leqslant 0 \\ y - v_0 - v_1 \in W^{\perp}}} (\langle x_0, v_0 \rangle + \langle x_1, v_1 \rangle).$$

The theorem (2.6.1) gives the following conditions for the inf-dif-stability of (P) :

(2.3.1) **Proposition**

If there exists $\tilde{x} \in W$ such that $q(\tilde{x} - x_1) < 0$, then the problem (P) is stable (hence inf-dif-stable).

Proof

The function $u_0 \rightarrow p(\tilde{x} - (x_0 - u_0))$ is continuous. Put $-\varepsilon = q(\tilde{x} - x_1)$. If $|\mu| < \frac{\varepsilon}{2}$ and $q(u_1) < \frac{\varepsilon}{2}$, then we have

$$q(\tilde{x} - (x_1 - u_1)) \leqslant \mu.$$

Thus there exists a neighborhood $\mathcal{U}$ of $0 \in X$ such that for all $u_1 \in \mathcal{U}$ and all $\mu \in]-\frac{\varepsilon}{2}, +\frac{\varepsilon}{2}[$, we have :

$$\varphi(\tilde{x}, u) = p(\tilde{x} - (x_0 - u_0)),$$

(where $u = [u_0, u_1, \mu]$), which is continuous with respect to u.

Let us write the corresponding characterization theorem using theorem (2.5.2) :

(2.3.2) **Proposition**

If the problem (P) is inf-dif-stable, then an element $\bar{x} \in C \cap W$ is a solution of (P) iff there exist $\bar{v} = [\bar{v}_0, \bar{v}_1, \bar{\lambda}] \in Y \times Y \times R$ such that $\bar{v}_0 \in K, \bar{v}_1 = -\bar{\lambda} h_1$ with $h_1 \in H$ and $\bar{\lambda} \leqslant 0$,
$\bar{v}_0 + \bar{v}_1 \in W^{\perp}$,
$\langle \bar{x} - x_0, \bar{v}_0 \rangle = p(\bar{x} - x_0)$
and $\bar{\lambda} \langle \bar{x} - x_1, h_1 \rangle = 0$. (This element $\bar{v}$ is a solution of the problem Q).

Proof :

An element $\bar{x} \in X$ is a solution of (P) iff there exists $\bar{v} \in V$ such that $\varphi(\bar{x}, 0) + \psi(0, \bar{v}) = 0$. In our case, $\bar{x} \in C \cap W$ is a solution iff there exist $\bar{v}_0, \bar{v}_1, \bar{\lambda}$ (satisfying $\bar{v}_0 \in K, \bar{v}_1 = -\bar{\lambda} h_1, \bar{\lambda} \leqslant 0, h_1 \in H, \bar{v}_0 + \bar{v}_1 \in W^{\perp}$) such that :

$$p(\bar{x} - x_0) + \langle x_0, \bar{v}_0 \rangle + \langle x_1, \bar{v}_1 \rangle = 0$$

i.e. since $\bar{v}_0 + \bar{v}_1 \in W^{\perp}$

(i) $$p(\bar{x} - x_0) - \langle \bar{x} - x_0, \bar{v}_0 \rangle - \langle \bar{x} - x_1, \bar{v}_1 \rangle = 0.$$

As we have :

$$\langle \bar{x} - x_0, \bar{v}_0 \rangle \leqslant p(\bar{x} - x_0),$$

$$\langle \bar{x} - x_1, h_1 \rangle \leqslant 0,$$

hence :

$$p(\bar{x} - x_0) - \langle \bar{x} - x_0, \bar{v}_0 \rangle \geqslant 0,$$

$$- \langle \bar{x} - x_1, \bar{v}_1 \rangle) \geqslant 0.$$

these two inequalities imply that (i) is equivalent to :

$$p(\bar{x} - x_0) - \langle \bar{x} - x_0, \bar{v}_0 \rangle = 0,$$

$$\bar{\lambda} \langle \bar{x} - x_1, h_1 \rangle = 0.$$

(2.3.3) REMARK

The same theorem could be obtained by using the Lagrangian of (P) which is :

$$l(x, v) = \begin{cases} \text{if } x \in W \begin{cases} -\langle x - x_0, v_0\rangle - \langle x - x_1, v_1\rangle \text{ if } v_0 \in K, v_1 = -\lambda h_1, \\ \qquad h_1 \in H, \lambda \leqslant 0 \\ +\infty \text{ elsewhere,} \end{cases} \\ \text{if } x \notin W : -\infty. \end{cases}$$

If (P) is inf-dif-stable, then an element $\bar{x} \in W$ is a solution of (P) iff there exists $\bar{v} = [\bar{v}_0, \bar{v}_1, \bar{\lambda}] \in V$ satisfying $\bar{v}_0 \in K, \bar{v}_1 = -\bar{\lambda} h_1, h_1 \in H, \bar{\lambda} \leqslant 0$ such that :

$$-\langle x - x_0, \bar{v}_0 \rangle - \langle x - x_1, \bar{v}_1 \rangle \leqslant -\langle \bar{x} - x_0, \bar{v}_0 \rangle - \langle \bar{x} - x_1, \bar{v}_1 \rangle$$
$$\leqslant -\langle \bar{x} - x_0, v_0 \rangle - \langle \bar{x} - x_1, v_1 \rangle$$

for all $x \in W$, $v_0 \in K$, $v_1 = -\lambda h_1$, $\lambda \leqslant 0$, $h_1 \in H$. The first inequality is equivalent to $\bar{v}_0 + \bar{v}_1 \in W^\perp$. The second inequality is equivalent to :

$q(\bar{x} - x_1) \leqslant 0$, $\langle \bar{x} - x_0, \bar{v}_0 \rangle = p(\bar{x} - x_0 \rangle$ and $\bar{\lambda} \langle \bar{x} - x_1, h_1 \rangle = 0$.

(2.3.4) **Proposition**

Assume that W is finite-dimensional. If there exists $\tilde{x} \in W$ such that $q(\tilde{x} - x_1) < 0$ and if the set $\{ x \in X \mid x \in W, p(x) = 0 \text{ and } \langle x, h\rangle \leqslant 0, \text{ for all } h \in H \}$ is a linear subspace then, both problems (P) and (Q) are inf-dif-stable (consequently $-\beta = \alpha$ and (P) and (Q) have solutions).

Proof :

By (2.3.1), (P) is inf-dif-stable. Hence $k(0)$ is also finite. We shall apply corollory (1.8.8) for the problem (Q). The functional $\psi_{\tilde{v}} : y \in Y \rightarrow \psi(y, \tilde{v})$ is clearly quasi-continuous. The subset $\{ x \in X \mid f_\infty(x) = 0 \}$ have to be a linear subspace. But we have :

$$f_\infty(x) = \begin{cases} p(x) \text{ if } x \in W \cap C_\infty \\ +\infty \text{ elsewhere} \end{cases}$$

with $C_\infty = \{ x \in X \mid \langle x, h\rangle \leqslant 0, \text{ for all } h \in H \}$. This gives the condition of the proposition.

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