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AN ADMISSIBLE SYNTHESIS FOR CONTROL SYSTEMS ON DIFFERENTIABLE MANIFOLDS

by Stefan MIRICĂ (1)

Résumé. — *It is defined the notion of the admissible synthesis for a control system on a differentiable manifold and there are studied its properties. The final result express a sufficient condition (which is also necessary) in the form of the dynamic programming equation or in the form of the maximum principle, for the optimality of the admissible synthesis.*

1. INTRODUCTION

For some control systems the natural phase space are differentiable manifolds ([1], [2]). As it is observed in [2] pp. 454 : « it is this (phase space's) topological complexity that causes the confusion and profusion of switching loci ».

In the present paper we extend the results from [3] to control systems on differentiable manifolds.

The study of Such systems is natural from the point of view of the synthesis namely because in this problem, it is the global dependence on states of the controls that is pointed out. And to understand what a state is, we have to consider it as a point on a differentiable manifold as in classical mechanics ([8], [10]). This fact is clear in situations as the one in the example studied in section 1 of this paper.

But even in the local case — control systems on Euclidian spaces — in the definition of the admissible synthesis we deal with differentiable manifolds.

Finally, differentiable manifolds allow us to write coordinate free relations that express properties of the admissible synthesis. Therefore we deduce that these properties have a « geometric » nature and do not depend on the choice of coordinate systems.

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In the first part of the paper we present shortly an example of control system on two — dimensional cilinder.

Then, using the notions of « piecewise smooth set » and « regular synthesis » introduced by Boltyanskii ([4], [5]), we define an admissible synthesis for control systems on differentiable manifolds. Some properties of the trajectories generated by this admissible synthesis and dual variables are studied using the methods from [6], [7].

In the second part we prove some properties of the value of the performance of the admissible synthesis and we give sufficient conditions for optimality of the admissible synthesis in the form of the functional equation of dynamic programming and in the form of the maximum principle.

We note that the proposition 7.1 was significantly improved with respect to the corresponding result from [3] and this allows us to write in a unitary manner the relations (6.8) and (6.9) that must be satisfied by the dual variables.

The new form of the proposition 7.1 and the fact that dual variables belong to the cotangent manifold of the phase space made more clear the ideas and allowed to describe an algorithm for the admissible synthesis that represent a generalisation and a justification of the R. Isaacs' technique ([17]) for cotrol systems and differential games. The description of this algorithm will be published separately.

2. AN EXAMPLE OF CONTROL SYSTEM ON THE CILINDER

A radar antenna designed to rotate about a vertical axis through its center of mass leads to the following control problem ([1]) :

Find the control $u : [t_0, t_1] \to [-1, 1]$ such that the solution of the system :

$$(2.1)\qquad \begin{cases} \dot{x}^1 = x^2 \\ \dot{x}^2 = u(t) \quad , \quad x^1(t_0) = x_0^1 \quad , \quad x^2(t_0) = x_0^2. \end{cases}$$

reach the « target set » $\mathfrak{F} = \left\{ \left(\frac{2\pi I}{f} n, 0\right) \,\middle|\, n = 0, \pm 1, \pm 2, \dots \right\}$ at a moment $t_1 > t_0$ and such that the integral :

$$(2.2.)\qquad \int_{t_0}^{t_1} [\lambda_1 + \lambda_2 (x^2(t))^2 + \lambda_3 |u(t)|]\, dt$$

is minimal. (I, f, $\lambda_1 > 0$, $\lambda_2, \lambda_3 \geqslant 0$).

In [1] the optimal synthesis for this problem is given using the maximum principle.

We shall present shortly this synthesis then we show that the control system may be described as a control system on the cilinder.

Solve first the problem in the case when the target set is the point (0, 0).

The switching loci for this problem are the following curves (fig. 1) :

$$\eta_+^{(0)} = \left\{ (x^1, x^2) | x^1 = -\frac{1}{2}(x^2)^2, x^2 > 0 \right\},$$

$$\eta_-^{(0)} = \left\{ (x^1, x^2) | x^1 = \frac{1}{2}(x^2)^2, x^2 < 0 \right\}$$

$$\psi_+^{(0)} = \left\{ (x^1, x^2) | x^1 = -\left(\frac{1}{2} + \frac{2\lambda_3}{\lambda_1 - \lambda_2 (x^2)^2}\right)(x^2)^2, x^2 > 0 \right\}$$

$$\psi_-^{(0)} = \left\{ (x^1, x^2) | x^1 = \left(\frac{1}{2} - \frac{2\lambda_3}{\lambda_1 - \lambda_2 (x^2)^2}\right)(x^2)^2, x^2 < 0 \right\}$$

$$\varphi_+^{(0)} = \left\{ (x^1, x^2) | x^2 = \sqrt{\frac{\lambda_1}{\lambda_2}}, x^1 < -\frac{1}{2}\frac{\lambda_1}{\lambda_2} \right\},$$

$$\varphi_-^{(0)} = \left\{ (x^1, x^2) | x^2 = -\sqrt{\frac{\lambda_1}{\lambda_2}}, x^1 > \frac{1}{2}\frac{\lambda_1}{\lambda_2} \right\},$$

We consider in the plane the following sets :

$$c_{+1}^{(0)} = \left\{ \left(-\frac{1}{2}\frac{\lambda_1}{\lambda_2}, \sqrt{\frac{\lambda_1}{\lambda_2}}\right) \right\}$$

$$c_{+2}^{(0)} = \left\{ (x^1, x^2) | (x^1, x^2) \in \eta_+^{(0)}, 0 < x^2 < \sqrt{\frac{\lambda_1}{\lambda_2}} \right\}$$

$$c_{+3}^{(0)} = \left\{ (x^1, x^2) | (x^1, x^2) \in \eta_+^{(0)}, x^2 > \sqrt{\frac{\lambda_1}{\lambda_2}} \right\}$$

$$c_{+4}^{(0)} = \varphi_+^{(0)}$$

$$c_{+5}^{(0)} = \psi_+^{(0)}$$

$c_{+6}^{(0)}$ is the region in the plane to the left of $\eta_+^{(0)}$ and above the line $\varphi_+^{(0)}$;

$c_{+7}^{(0)}$ to the region int he plane to the left of $\eta_+^{(0)}$, below the line $\varphi_+^{(0)}$ and above the curve $\psi_+^{(0)}$;

$c_{+8}^{(0)}$ is the region in the plane to the left of $\eta_-^{(0)}$ and below the curve $\psi_+^{(0)}$.

To the right of the curve $\eta_+^{(0)} \cup \eta_-^{(0)}$ we may define symmetrically the sets $c_{-i}^{(0)}$, $i = 1, 2, ..., 8$.

The optimal synthesis in this case is the function defined as follows :

$$(2.3.)\quad v^{(0)}(x^1, x^2) = \begin{cases} -1 & \text{for } (x^1, x^2) \in c^{(0)}_{+1} \cup c^{(0)}_{+2} \cup c^{(0)}_{+3} \cup c^{(0)}_{+6} \cup c^{(0)}_{+8} \\ 0 & \text{for } (x^1, x^2) \in c^{(0)}_{+4} \cup c^{(0)}_{+7} \cup c^{(0)}_{-4} \cup c^{(0)}_{-7} \\ 1 & \text{for } (x^1, x^2) \in c^{(0)}_{-1} \cup c^{(0)}_{-2} \cup c^{(0)}_{-3} \cup c^{(0)}_{-6} \cup c^{(0)}_{-8} \end{cases}$$

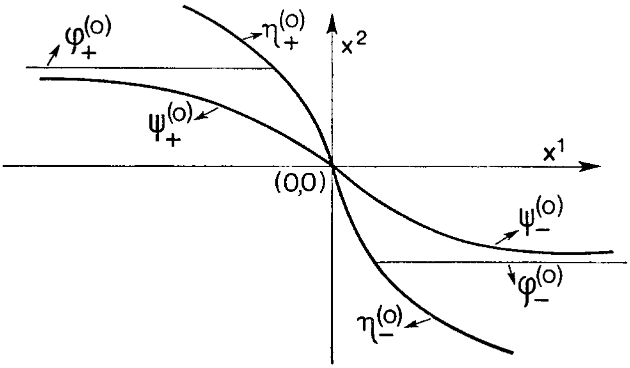

Figure 1

The value of (2.2) for the initial point (x^1, x^2) and for the trajectory generated by this synthesis is the function :

$$W^{(0)}(x^1, x^2) = \begin{cases} (\lambda_1 + \lambda_3)\,|x^2| + \frac{\lambda_2}{3}\,|x^2|^3 \quad \text{for} \quad (x^1, x^2) \in c^{(0)}_{+1} \cup c^{(0)}_{+2} \cup c^{(0)}_{+3} \\ (\lambda_1 + \lambda_3)x^2 + \frac{\lambda_2}{3}(x^2)^3 + (\lambda_1 + \lambda_2(x^2)^2)\left(\frac{|x^1|}{x^2} - \frac{1}{2}x^2\right) \\ \qquad \text{for} \quad (x^1, x^2) \in c^{(0)}_{+4} \cup c^{(0)}_{+5} \cup c^{(0}_{+7} \\ (\lambda_1 + \lambda_3)x^2 + \frac{\lambda_2}{3}(x^2)^2 - \lambda_1\left(\frac{\lambda_1}{\lambda_2}\right)^{1/2} \\ \qquad - 2(\lambda_1\lambda_2)^{1/2}\left(x^1 + \frac{1}{2}\left((x^2)^2 - \frac{\lambda_1}{\lambda_2}\right)\right) \quad \text{for} \quad (x^1, x^2) \in c^{(0}_{+6} \end{cases}$$

and

$$W^{(0)}(x^1, x^2) = (\lambda_1 + \lambda_3)(2\alpha(x^1, x^2) - x^2) + \frac{1}{3}\lambda_2(2\alpha^3(x^1, x^2) - (x^2)^3) +$$

$$+ 2\lambda_3(\lambda_1 + \lambda_2)\,\frac{\alpha^3(x^1, x^2)}{\lambda_1 - \lambda_2\alpha^2(x^1, x^2)} \quad \text{for} \quad (x^1, x^2) \in c^{(0)}_{+8}$$

where $\alpha(\mathfrak{x}^1, \mathfrak{x}^2)$ is given by

$$\alpha^2(\mathfrak{x}^1, \mathfrak{x}^2) = \frac{1}{2\lambda_2}\left\{ \lambda_1 + 2\lambda_3 - \lambda_2\left(\mathfrak{x}^1 - \frac{1}{2}(\mathfrak{x}^2)^2\right) - \left[\left(\lambda_1 + 2\lambda_3 - \lambda_2\left(\mathfrak{x}^1 - \frac{1}{2}(\mathfrak{x}^2)^2\right)\right)^2 + 4\lambda_1\lambda_2\left(\mathfrak{x}^1 - \frac{1}{2}(\mathfrak{x}^2)^2\right)\right]^{1/2} \right\}$$

If $(\mathfrak{x}^1, \mathfrak{x}^2)$ is at the right of the curve $\eta_+^{(0)} \cup \eta_-^{(0)}$ then $(-\mathfrak{x}^1, \mathfrak{x}^2)$ is at the left of this curve and we take $W^{(0)}(\mathfrak{x}^1, \mathfrak{x}^2) = W^{(0)}(-\mathfrak{x}^1, -\mathfrak{x}^2)$.

It is easy to see that the sets $\mathfrak{c}_{\pm\mathfrak{i}}^{(0)}$ $\mathfrak{i} = 1, 2, ..., 8$ and the functions $\mathfrak{v}^{(0)}$ and $W^{(0)}$ represent an optimal synthesis according to the definition from [3] and a « regular synthesis » as in [4], [5].

We consider now the general case of the control problem (2.1), (2.2). The switching loci are the curves $\eta_\pm^{(\mathfrak{n})}$, $\varphi_\pm^{(\mathfrak{n})}$, $\psi_\pm^{(\mathfrak{n})}$ $\mathfrak{n} = 0, \pm 1, \pm 2, ...$ obtained from $\eta_\pm^{(0)}$, $\varphi_\pm^{(0)}$, $\psi_\pm^{(0)}$ by translation along the $\mathfrak{x}^1$-axis with $\frac{2\pi I}{\mathfrak{f}}\mathfrak{n}$.

We define the sets $\mathfrak{c}_{\pm\mathfrak{i}}^{(\mathfrak{n})}$ and the fonctions $\mathfrak{v}^{(\mathfrak{n})}$ and $W^{(\mathfrak{n})}$ as follows :

$$\mathfrak{c}_{\pm\mathfrak{i}}^{(\mathfrak{n})} = \left\{ (\mathfrak{x}^1, \mathfrak{x}^2) \;\middle|\; \left(\mathfrak{x}^1 - \frac{2\pi I}{\mathfrak{f}}\mathfrak{n}, \mathfrak{x}^2\right) \in \mathfrak{c}_{\pm\mathfrak{i}}^{(0)} \right\}$$

$$\mathfrak{v}^{(\mathfrak{n})}(\mathfrak{x}^1, \mathfrak{x}^2) = \mathfrak{v}^{(0)}\left(\mathfrak{x}^1 - \frac{2\pi I}{\mathfrak{f}}\mathfrak{n}, \mathfrak{x}^2\right)$$

$$W^{(\mathfrak{n})}(\mathfrak{x}^1, \mathfrak{x}^2) = W^{(0)}\left(\mathfrak{x}^1 - \frac{2\pi I}{\mathfrak{f}}\mathfrak{n}, \mathfrak{x}^2\right)$$

For this problem there exist the « indiference curves » :

$$\xi_{(\mathfrak{n})} = \{ (\mathfrak{x}^1, \mathfrak{x}^2) | W^{(\mathfrak{n})}(\mathfrak{x}^1, \mathfrak{x}^2) = W^{(\mathfrak{n}+1)}(\mathfrak{x}^1, \mathfrak{x}^2) \}$$

To define the optimal synthesis we consider the sets :

$$D_{(\mathfrak{n})} = \{ (\mathfrak{x}^1, \mathfrak{x}^2) | W^{(\mathfrak{n}-1)}(\mathfrak{x}^1, \mathfrak{x}^2) < W^{(\mathfrak{n})}(\mathfrak{x}^1, \mathfrak{x}^2) < W^{(\mathfrak{n}+1)}(\mathfrak{x}^1, \mathfrak{x}^2) \}$$

and the function v defined as follows :

$$\mathfrak{v}(\mathfrak{x}^1, \mathfrak{x}^2) = \mathfrak{v}^{(\mathfrak{n})}(\mathfrak{x}^1, \mathfrak{x}^2) \quad \text{if} \quad (\mathfrak{x}^1, \mathfrak{x}^2) \in D_{(\mathfrak{n})} \quad \text{and} \quad \mathfrak{v}(\mathfrak{x}^1, \mathfrak{x}^2)$$

is either $\mathfrak{v}^{(\mathfrak{n})}(\mathfrak{x}^1, \mathfrak{x}^2)$ or $\mathfrak{v}^{(\mathfrak{n}+1)}(\mathfrak{x}^1, \mathfrak{x}^2)$ if $(\mathfrak{x}^1, \mathfrak{x}^2)$ is on the indiference curve $\xi_{(\mathfrak{n})}$.

In [1] is proved that the synthesis defined above generates optimal trajectories in every point in the plane and the value of the functional (2.2) is given by the formula :

$$W(\mathfrak{x}^1, \mathfrak{x}^2) = W^{(\mathfrak{n})}(\mathfrak{x}^1, \mathfrak{x}^2) \quad \text{if} \quad (\mathfrak{x}^1, \mathfrak{x}^2) \in D_{(\mathfrak{n})} \cup \xi_{(\mathfrak{n})}$$

It is easy to prove that the sets $\mathfrak{c}_{\pm i}^{(\mathfrak{n})} \cap D_{(\mathfrak{n})}$, $\xi_{(\mathfrak{n})}$ $\mathfrak{n} = 0, \pm 1, \pm 2, \ldots$ and the fonctions $\mathfrak{v}$ and W represent an optimal synthesis as in [3].

Let us show that this control system may be considered on the cilinder.

Since the cilinder may be obtained from the plane by identification of the points $(\mathfrak{x}^1, \mathfrak{x}^2)$ and $\left(\mathfrak{x}^1 + \frac{2\pi I}{\mathfrak{f}}\mathfrak{n}, \mathfrak{x}^2\right)$, $\mathfrak{n} = 0, \pm 1, \pm 2, \ldots$ it follows that the right hand side of the system (2.1) (that is the vector $(\mathfrak{x}^2, \mathfrak{u})$) may be considered as a parametric vector field on the cilinder.

Similarly, the function $\mathfrak{f}^0(\mathfrak{x}^1, \mathfrak{x}^2, \mathfrak{u}) = \lambda_1 + \lambda_2(\mathfrak{x}^2)^2 + \lambda_3 |\mathfrak{u}|$ from the integral (2.2) does not depend on $\mathfrak{x}^1$, hence it is a function on the cilinder.

The target set $\mathfrak{F} = \left\{ \left(\frac{2\pi I}{\mathfrak{f}}\mathfrak{n}, 0\right) \middle| \mathfrak{n} = 0, \pm 1, \pm 2, \ldots \right\}$ represents a target point on the cilinder.

Moreover, the synthesis defined above, $\mathfrak{v}$, is also a function defined on the cilinder. Indeed, the switching curves $\eta_\pm^{(\mathfrak{n})}$, $\varphi_\pm^{(\mathfrak{n})}$, $\psi_\pm^{(\mathfrak{n})}$ and the indiference curves $\xi_{(\mathfrak{n})}$, $\mathfrak{n} = 0, \pm 1, \pm 2, \ldots$ define respectively the curves $\eta_\pm$, $\varphi_\pm$, $\psi_\pm$, ξ on the cilinder, as follows :

$$\eta_\pm = \bigcup_{n=0}^{\pm\infty} \left\{ \eta_\pm^{(\mathfrak{n})} \cap \left(\left[-\frac{\pi I}{\mathfrak{f}}, \frac{\pi I}{\mathfrak{f}}\right) \times R\right) \right\}$$

$$\varphi_\pm = \bigcup_{n=0}^{\pm\infty} \left\{ \varphi_\pm^{(\mathfrak{n})} \cap \left(\left[-\frac{\pi I}{\mathfrak{f}}, \frac{\pi I}{\mathfrak{f}}\right) \times R\right) \right\}$$

$$\psi_\pm = \bigcup_{\mathfrak{n}=0}^{\pm\infty} \left\{ \varphi_\pm^{(\mathfrak{n})} \cap \left(\left[-\frac{\pi I}{\mathfrak{f}}, \frac{\pi I}{\mathfrak{f}}\right) \times R\right) \right\}$$

$$\xi = \bigcup_{\mathfrak{n}-\infty}^{+\infty} \left\{ \xi^{(\mathfrak{n})} \cap \left(\left[-\frac{\pi I}{\mathfrak{f}}, \frac{\pi I}{\mathfrak{f}}\right) \times R\right) \right\}$$

For example, the set η_+ is a curve on the cilinder, because for

$$\left(-\frac{\pi I}{\mathfrak{f}}, \mathfrak{x}^2\right) \in \eta_+^{(\mathfrak{n})}$$

we have $\left(\frac{\pi I}{\mathfrak{f}}, \mathfrak{x}^2\right) \in \eta_+^{(\mathfrak{n}+1)}$ that is η_+ is obtained by « patching » the pieces

$$\eta_+^{(\mathfrak{n})} \cap \left(\left[-\frac{\pi I}{\mathfrak{f}}, \frac{\pi I}{\mathfrak{f}}\right) \times R\right)$$

of the curves $\eta_+^{(\mathfrak{n})}$.

The curves $\eta_\pm$, $\varphi_\pm$, $\psi_\pm$, ξ determine on the cilinder the sets :

$$\mathfrak{c}_{\pm i} = \bigcup_{\mathfrak{n}=0}^{\pm\infty} \left\{ D_{(\mathfrak{n})} \cap \mathfrak{c}_\pm^{(\mathfrak{n})} \cap \left(\left[-\frac{\pi I}{\mathfrak{f}}, \frac{\pi I}{\mathfrak{f}}\right) \times R\right) \right\}$$

By definition, the functions $\mathfrak{v}$ and W are periodics on $\mathfrak{x}^1$ with period $\frac{2\pi l}{\mathfrak{k}}$ and therefore are functions on the cilinder.

Moreover, the trajectories of the system :

$$\begin{cases} \dot{\mathfrak{x}}^1 = \mathfrak{x}^2 \\ \dot{\mathfrak{x}}^2 = \mathfrak{v}(\mathfrak{x}^1, \mathfrak{x}^2) \end{cases}$$

become on the cilinder integral curves of the vector field determined by the local representative $(\mathfrak{x}^2, \mathfrak{v}(\mathfrak{x}^1, \mathfrak{x}^2))$. Therefore, the function $\mathfrak{v}(\mathfrak{x}^1, \mathfrak{x}^2)$ is a synthesis for the control problem on the cilinder and $W(\mathfrak{x}^1, \mathfrak{x}^2)$ is the value of the functional (2.2) along the trajectories generated by this synthesis.

REMARK 2.1. The motion of the forced pendulum leads in [2] to the following time optimal control problem :

$$\begin{cases} \dot{\theta} = \mathfrak{z} \\ \dot{\mathfrak{z}} = -\sin\theta - \alpha\mathfrak{z} + \mathfrak{u} \end{cases} \tag{2.4}$$

where $|\mathfrak{u}| \leqslant B$, $B > 0$, $\alpha \geqslant 0$

The target set is $\mathfrak{F} = \{ (2\pi\mathfrak{n}, 0) \mid \mathfrak{n} = 0, \pm 1, \ldots \}$ and the criterion :

$$\int_{t_0}^{t_1} dt = \min \tag{2.5}$$

In [2] the optimal synthesis is given using the maximum principle and it is pointed out that the phase pace of this problem is the cilinder.

3. CONTROL SYSTEMS ON MANIFOLDS. THE BOLTYANSKII'S LEMMAS

In what follows, by differentiable manifold we mean a $C^r (r \geqslant 2)$, Hausdorff, finite-dimensional manifold, which admits the partition of unity and without boundary if we do not specify otherwise.

We shal call « C^p-morphism » a C^p-map from a manifold to another ([8], [9], [10], [11]) and C^p-function » a C^p-morphism from a manifold to R.

A nonautonomous vector field on the differentiable manifold X is a map $\xi : X \times I \to T(X)$, such that for any $t \in I \subset R$, the partial map $\xi_t : X \to T(X)$ is a vector field on X (that is, $\xi_t(\mathfrak{x}) = \xi(\mathfrak{x}, t) \in T_{\mathfrak{x}}X$ for any $\mathfrak{x} \in X$ or $T_X \circ \xi_t = id$ where $T_X : T(X) \to X$ is the tangent bundle (« id » means the « identity »)).

A map $c : I_1 \subset I \to X$ is an integral curve of ξ at the point $\mathfrak{x}_0 \in X$ and the moment to $t_0 \in I_1$ if $c(t_0) = \mathfrak{x}_0$ and $T_t c.1 = \xi(c(t), t)$ for any $t \in I_1 (Tc : T(I_1) \to T(X)$ is the tangent of the map c).

Let us consider X a n-dimensional differentiable manifold called the phase space, Ω a p-dimensional compact manifold, possibly with boundary, called the control space and $\xi : X \times \Omega \to T(X)$ a parametrized C^1-vector field on X ([9]) *called as in* [12], *controllable family of vector fields on X.*

Therefore, ξ is a C^1-morphism from $X \times \Omega$ to $T(X)$ and for any $\omega \in \Omega$ the partial map $\xi_\omega : X \to T(X)$ is a C^1-vector field an X.

We consider $\mathfrak{F} \subset X$ *a closed, k-dimensional submanifold* $(0 \leqslant k \leqslant n-1)$ *possibly with boundary, called the terminal manifold and we consider also the C^1-functions,* $f^0 : X \times \Omega \to R$ and $\mathfrak{g} : \mathfrak{F} \to R$.

Definition 3.1

Let $\mathfrak{x} \in X$ and $I_\mathfrak{u} \subset R$ an interval. The map $\mathfrak{u} : I_\mathfrak{u} \to \Omega$ is called an admissible control corresponding to the point $\mathfrak{x}$ if :

(i) $\mathfrak{u}$ *is a piecewise continuous map ;*

(ii) *the map $\xi_\mathfrak{u} : X \times I_\mathfrak{u} \to T(X)$ defined by the formula :*

$$\xi_\mathfrak{u}(\mathfrak{x}, \mathfrak{t}) = \xi(\mathfrak{x}, \mathfrak{u}(\mathfrak{t})) \qquad \text{for} \qquad (\mathfrak{x}, \mathfrak{t}) \in X \times I_\mathfrak{u} \tag{3.1}$$

is a nonautonomous vector field on X with the following property : there exists a point $\mathfrak{t}_0 \in I_\mathfrak{u}$ so that the integral curve $\varphi_{(\mathfrak{t}_0,\mathfrak{x})}$ of $\xi_\mathfrak{u}$ at the point $\mathfrak{x}$ and the moment $\mathfrak{t}_0$, intersects $\mathfrak{F}$ in a finite time, that is, there exists a moment $\mathfrak{t}_1 > \mathfrak{t}_0$, $\mathfrak{t}_1 \in I_\mathfrak{u}$ such that $\varphi_{(\mathfrak{t}_0,\mathfrak{x})}(\mathfrak{t}) \in X \setminus \mathfrak{F}$ for $\mathfrak{t} \in [\mathfrak{t}_0, \mathfrak{t}_1)$ and $\mathfrak{x}_1 = \varphi_{(\mathfrak{t}_0,\mathfrak{x})}(\mathfrak{t}_1) \in \mathfrak{F}$

The curve $\varphi_{(\mathfrak{t}_0,\mathfrak{x})}$ is called admissible trajectory through the point $\mathfrak{x} \in X$ (corresponding to the admissible control $\mathfrak{u}$).

Let $\mathcal{U}_\mathfrak{x}$ denote the set of all admissible controls corresponding to the point $\mathfrak{x} \in X$ and $\mathcal{U} = \bigcup_{\mathfrak{x} \in X} \mathcal{U}_\mathfrak{x}$.

For every admissible control $\mathfrak{u} \in \mathcal{U}_\mathfrak{x}$ (hence for every admissible trajectory through $\mathfrak{x} \in X$) we define the real number :

$$P(\mathfrak{x}, \mathfrak{u}) = P(\mathfrak{x}, \varphi_{(\mathfrak{t}_0,\mathfrak{x})}) = \mathfrak{g}(\mathfrak{x}_1) + \int_{\mathfrak{t}_0}^{\mathfrak{t}_1} \mathfrak{f}^0(\varphi_{(\mathfrak{t}_0,\mathfrak{x})}(\mathfrak{t}), \mathfrak{u}(\mathfrak{t}))\, d\mathfrak{t} \tag{3.2}$$

called the performance of the control $\mathfrak{u} \in \mathcal{U}_\mathfrak{x}$ (or the performance of the trajectory $\varphi_{(\mathfrak{t}_0,\mathfrak{x})}$).

The map $P : X \times \mathcal{U} \to R$ is called the performance.

Definition 3.2

$S = (X, \Omega, \xi, \mathfrak{F}, \mathcal{U}, P)$ is a preferential control system on X.

Definition 3.3

The admissible control $\tilde{\mathfrak{u}} \in \mathcal{U}_\mathfrak{x}$ is an optimal control corresponding to $\mathfrak{x} \in X$ if we have.

$$P(\mathfrak{x}, \tilde{\mathfrak{u}}) \leqslant P(\mathfrak{x}, \mathfrak{u}) \qquad \text{for all} \qquad \mathfrak{u} \in \mathcal{U}_\mathfrak{x}$$

The control problem for the system S is to find an optimal control for any point $\mathfrak{x} \in X$ (or to find an optimal control for a given point $\mathfrak{x}_0 \in X$).

Modifing in a suitable manner the notions of « curvilinear polyhedron » and « piecewise smooth set » the main lemma from [4] ([5]) is also true for the control system on differentiable manifold.

Definition 3.4

Let R^s be the real euclidian s-dimensional space, $K \subset R^s$ a convex, bounded, closed, s-dimensional polyhedron, $V \subset R^s$ an open neighborhood of K and $\varphi : V \to X$ a C^1-injective immersion on K. Then, the set $\varphi(K) \subset X$ is a s-dimensional curvilinear polyhedron in X.

Definition 3.5

$M \subset X$ is a piecewise smooth set of dimension s if the following conditions hold :

(i) *M is a union of curvilinear polyhedra in X;*

(ii) *Every compact subset of X intersects only a finite number of such polyhedra;*

(iii) *There exists in M a s-dimensional curvilinear polyhedron and the others are of dimension $\leqslant s$.*

Using the theorems on vector fields from [11], the theorem on smooth maps of differentiable manifolds from [14] and the classical theorems on differential equations from [15] and [16] we can prove the following lemmas (3.1-3.4) — the Bolthyankii's lemmas—as in [4] or [5].

Lemma 3.1.

Let $V : X \to R$ be a C^1-function such that for every $\mathfrak{x} \in X$ the following inequality holds :

$$T_{\mathfrak{x}}V \cdot \xi(\mathfrak{x}, \omega) + \mathfrak{f}^0(\mathfrak{x}, \omega) \geqslant 0 \qquad \text{for all} \qquad \omega \in \Omega \tag{3.3}$$

Then, for every admissible control $\mathfrak{u} \in \mathfrak{U}_{\mathfrak{x}}$ *the following* inequality holds :

$$V(\mathfrak{x}) \leqslant P(\mathfrak{x}, \mathfrak{u}) \tag{3.4}$$

Lemma 3.2.

(i) *Let $V : X \to R$ be a C^0-function, $M \subset X$ be a closed set such that the map $V_1 = V \mid X \backslash M$ is a C^1-function and for every $\mathfrak{x} \in X \setminus M$ we have :*

$$T_{\mathfrak{x}}V_1\xi(\mathfrak{x}, \omega) + \mathfrak{f}^0(\mathfrak{x}, \omega) \geqslant 0 \qquad \text{for all } \omega \in \Omega \tag{3.5}$$

(ii) *If $\mathfrak{u} \in \mathfrak{U}_{\mathfrak{x}}$ is an admissible control such that the corresponding admissible trajectory $\varphi_{(t_0, \mathfrak{x})}$ intersects M only in a finite number of points, then* (3.4) *holds.*

Lemma 3.3.

Let $V : X \to R$ be a C^0-function and $M \subset X$ be a subset satisfing the condition (i) *from lemma* (3.2).

(i) *Let* $\mathfrak{u} \in \mathfrak{U}_{\mathfrak{x}}$ *be an admissible control with the following property : for every neighborhood $G \subset X$* of $\mathfrak{x}$, *there exists a point* $\mathfrak{x}_1 \in G$ *such that, there exists an integral curve* $\varphi_{(t_0, \mathfrak{x}_1)}$ *of the nonautonomous vector field* $\xi_{\mathfrak{u}}$ (3.1) *such that* $\varphi_{(t_0, \mathfrak{x}_1)}$ *is defined on* $[t_0, t_1]$ *and intersects M only in a finite number of points.*

Then (3.4) *holds.*

Lemma 3.4.

Let $V : X \to R$ be a C^0-function, $M \subset X$ be a closed piecewise smooth set.

If the condition (i) *from lemma* (3.2) *holds then the condition* (i) *from lemma* (3.3) *and hence the inequality* (3.4) *holds.*

4. THE DEFINITION OF THE ADMISSIBLE SYNTHESIS

The following definition is obtained from the definition of the « regular synthesis » ([4], [5]) by omitting the condition that the « marked trajectories » satisfy the maximum principle.

Let $N, P^k, P^{k+1}, \ldots P^{n-1} \subset X$ be piecewise smooth sets such that

$$P^i (i = k, k+1, \ldots n-1)$$

is of dimension *i*, *N* is of dimension smaller than *n* and such that

$$\mathfrak{F} \subset P^k \subset P^{k+1} \subset \ldots \subset P^{n-1} \subset X$$

We denote $P^{k-1} = \mathfrak{F}$, $P^n = X$

The sets $N, P^k, \ldots P^{n-1}$ and a map $\mathfrak{v} : X \to \Omega$ represent an admissible synthesis for the control system *S* if the following requirements are fulfilled :

A. (i) The connected components of the sets $P^i \setminus (P^{i-1} U N)$ $i = k, k+1, \ldots n$ are differentiable submanifolds of *X*, of dimension *i*; we call them *i*-dimensional cells. The connected components of the target set $\mathfrak{F} = P^{k-1}$ are also *k*-dimensionnal cells.

(ii) The restriction $\mathfrak{v}_{\mathfrak{c}} = \mathfrak{v}|\mathfrak{c}$ is a C^1-morphism from the cell $\mathfrak{c}$ to the manifold Ω. Moreover, there exists a neighborhood $\tilde{\bar{\mathfrak{c}}} \subset X$ of the closure $\bar{\mathfrak{c}}$ of the cell $\mathfrak{c}$ and a smooth extension $\tilde{\mathfrak{v}}_{\mathfrak{c}} : \tilde{\mathfrak{c}} \to \Omega$ of the map $\mathfrak{v}_{\mathfrak{c}}$ (that is $\tilde{\mathfrak{v}}_{\mathfrak{c}}$ is a C^1-morphism such that $\tilde{\mathfrak{v}}_{\mathfrak{c}}(\mathfrak{x}) = \mathfrak{v}_{\mathfrak{c}}(\mathfrak{x})$ for $\mathfrak{x} \in \mathfrak{c}$);

B. Every cell is either of type I or of type II.

(i) The *n*-dimensional cells are of type I, the *k*-dimensional ones of type II.

(ii) If $\mathfrak{c}$ is an i-dimensional cell of type I then, through any point $\mathfrak{x} \in \mathfrak{c}$ there passes a unique integral curve of the piecewise smooth vector field $\bar{\xi} : X \to T(X)$ defined by

$$\bar{\xi}(\mathfrak{x}) = \xi(\mathfrak{x}, \mathfrak{v}(\mathfrak{x})) \qquad \text{for } \mathfrak{x} \in X. \tag{4.1}$$

There exists a unique $(i - 1)$-dimensional cell $\Pi(\mathfrak{c})$ (of type I or II) such that the integral curve $\varphi_{\mathfrak{x}}$ of $\bar{\xi}$ starting at $\mathfrak{x} \in \mathfrak{c}$, leaves $\mathfrak{c}$ after a finite time and reaches $\Pi(\mathfrak{c})$ transversally (that is in the incidence point $\varphi_{\mathfrak{x}}(\mathfrak{t}') = \mathfrak{x}' \in \Pi(\mathfrak{c})$ we have $\lim\limits_{\substack{\mathfrak{t} \to \mathfrak{t}' \\ \mathfrak{t} < \mathfrak{t}'}} \bar{\xi}(\varphi_{\mathfrak{x}}(\mathfrak{t})) \notin T\mathfrak{J}(T_{\mathfrak{x}'}\ \Pi(\mathfrak{c}))$ where $\mathfrak{J} : \Pi(\mathfrak{c}) \to X$ is the inclusion map, and $T_{\mathfrak{x}'}\,\Pi(\mathfrak{c})$ is the tangent space at $\mathfrak{x}'$ to the submanifold $\Pi(\mathfrak{c})$;

(iii) If $\mathfrak{c}$ is an i-dimensional cell of type II and $\mathfrak{c} \not\subset \mathfrak{F}$, then there exists a unique $(i + 1)$-dimensional cell $\Sigma(\mathfrak{c})$ of type I such that from any point $\mathfrak{x} \in \mathfrak{c}$ there starts a unique integral curve of the vector field $\bar{\xi}$ entering $\Sigma(\mathfrak{c})$ and having in $\mathfrak{c}$ only the start point.

Moreover the set $\mathfrak{c}' = \mathfrak{c}\ U\ \Sigma(\mathfrak{c})$ is a differentiable submanifold, possibly with boundary and $\mathfrak{v}\,|_{\mathfrak{c}'}$ is a C^1-morphism.

C. (i). Every integral curve of the vector field $\bar{\xi}$ reaches $\mathfrak{F}$, transversally, in a finite time and intersects only a finite number of cells.

(ii). From the points in N may start several integral curves of $\bar{\xi}$. The integral curves of $\bar{\xi}$ starting at points in N do not remain in N but enter in a cell of type I.

We denote by $\varphi_{\mathfrak{x}}$ an integral curve of $\bar{\xi}$ starting at $\mathfrak{x} \in X$ and we call it marked trajectory ([5]). If $\mathfrak{x} \in X \setminus N$ then $\varphi_{\mathfrak{x}}$ is unique.

If $\mathfrak{t}_F$ is the first moment when the trajectory $\varphi_{\mathfrak{x}}$ reaches $\mathfrak{F}$ and $\mathfrak{x}_F = \varphi_{\mathfrak{x}}(\mathfrak{t}_F)$, then for the point $\mathfrak{x} \in X$ and for the marked trajectory $\varphi_{\mathfrak{x}}$ we may define the real number :

$$P(\mathfrak{x}, \varphi_{\mathfrak{x}}) = \mathfrak{g}(\mathfrak{x}_F) + \int_0^{\mathfrak{t}_F} \mathfrak{f}^0(\varphi_{\mathfrak{x}}(\mathfrak{t}), \mathfrak{v}(\varphi_{\mathfrak{x}}(\mathfrak{t})))\, d\mathfrak{t} \tag{4.2}$$

D. The number $P(\mathfrak{x}, \varphi_{\mathfrak{x}})$ is the same for any marked trajectory starting at $\mathfrak{x} \in N$.

The function $W : X \to R$ defined by $W(\mathfrak{x}) = P(\mathfrak{x}, \varphi_{\mathfrak{x}})$ is continuous and we call it the value of the synthesis.

REMARK 4.1. The map $\bar{\mathfrak{u}} = \mathfrak{v} \circ \varphi_{\mathfrak{x}}$ is an admissible control corresponding to the point $\mathfrak{x} \in X$ according to the definition (3.1) and the marked trajectory $\varphi_{\mathfrak{x}}$ is the corresponding admissible trajectory for $\bar{\mathfrak{u}}$. It follows that the admissible synthesis generates admissible controls at every point of X.

REMARK 4.2. It is easy to show that according to this definition, the synthesis defined in the example from the section 2 on the cilinder is an admissible synthesis.

Indeed, the sets

$$N = \xi,\ P^0 = \{ (0, 0) \} \cup \mathfrak{c}_{+1} \cup \mathfrak{c}_{-1},\ P^1 = \left(\bigcup_{i=1}^{5} \mathfrak{c}_{+i}\right) \cup \left(\bigcup_{i=1}^{5} \mathfrak{c}_{-i}\right)$$

are piecewise smooth sets and the cells are $\mathfrak{c}_{\pm i}$, $i = 1, 2, \ldots 8$. The restrictions $\mathfrak{v}\,|\mathfrak{c}_{\pm i}$ are C^1-morphisms and may be extended to the C^1-morphisms of some neighborhoods of the cells $\mathfrak{c}_{\pm i}$. The points $\mathfrak{c}_{+1}$, $\mathfrak{c}_{-1}$ and the curves $\mathfrak{c}_{+5}$, $\mathfrak{c}_{-5}$ are the cells of type II and the curves $\mathfrak{c}_{\pm 2}$, $\mathfrak{c}_{\pm 3}$, $\mathfrak{c}_{\pm 4}$ and the « pieces » of the cilinder $\mathfrak{c}_{\pm 6}$, $\mathfrak{c}_{\pm 7}$, $\mathfrak{c}_{\pm 8}$ are the cells of type I. The value W of the synthesis is continuous on the cilinder and hence we have an admissible synthesis.

In the same way it can be proved that the synthesis given in [2] for the example from the Remark 2.1 is an admissible synthesis.

5. MARKED TRAJECTORIES

We consider a cell $\mathfrak{c} \subset X$ of type I of the admissible synthesis defined in the section 4.

Since from every point $\mathfrak{x} \in \mathfrak{c}$ there starts a unique marked trajectory $\varphi_{\mathfrak{x}}$ staying in $\mathfrak{c}$ during a time interval it follows that $\bar{\xi}$ defined by (4.1) is a tangent vector field to the submanifold $\mathfrak{c} \subset X$ and hence from the proposition 6.7, chap. III from [10] it follows that there exists a unique vector field on $\mathfrak{c}$

$$\bar{\xi}_{\mathfrak{c}} : \mathfrak{c} \to T(\mathfrak{c})$$

such that :

(5.1) $T\mathfrak{J}_{\mathfrak{c}} \circ \bar{\xi}_{\mathfrak{c}} = \bar{\xi} \circ \mathfrak{J}_{\mathfrak{c}}$ where $\mathfrak{J} : \mathfrak{c} \to X$ is the inclusion map and $T\mathfrak{J}_{\mathfrak{c}} : T(\mathfrak{c}) \to T(X)$ is its tangent map.

Therefore, every integral curve of the vector field $\bar{\xi}_{\mathfrak{c}}$ is a piece of a marked trajectory and conversely, every marked trajectory which passes through any point of $\mathfrak{c}$ is an integral curve of the vector field $\bar{\xi}_{\mathfrak{c}}$.

We say that $\bar{\xi}_{\mathfrak{c}}$ is the vector field defined by $\bar{\xi}$ on the submanifold $\mathfrak{c}$.

From the property A. (ii) of the admissible synthesis we know that there exists a neighborhood $\tilde{\mathfrak{c}}$ of $\bar{\mathfrak{c}} \subset X$ and a C^1-extension $\tilde{\mathfrak{v}}_{\mathfrak{c}} :\ \tilde{\mathfrak{c}} \to \Omega$ of the C^1-morphism $\mathfrak{v}_{\mathfrak{c}} = \mathfrak{v}|\mathfrak{c}$.

Since the set $\tilde{\mathfrak{c}} \subset X$ is an open submanifold and $\xi : X \times \Omega \to T(X)$ is a parametrized C^1-vector field, the map $\tilde{\xi}_{\mathfrak{c}} : \tilde{\mathfrak{c}} \to T(X)$ given by :

$$(5.2) \qquad \tilde{\xi}_{\mathfrak{c}}(\mathfrak{x}) = \xi(\mathfrak{x}, \tilde{\mathfrak{v}}_{\mathfrak{c}}(\mathfrak{x})) \qquad \text{for} \qquad \mathfrak{x} \in \tilde{\mathfrak{c}}$$

is a C^1-vector field on $\tilde{\mathfrak{c}}$.

Moreover, $\tilde{\xi}$ is equal to ξ on $\mathfrak{c}$ and hence $\tilde{\xi}$ is tangent to the submanifold $\mathfrak{c} \subset \tilde{\mathfrak{c}}$ and defines on $\mathfrak{c}$ the same vector field $\bar{\xi}_\mathfrak{c}$ as $\bar{\xi}$.

Therefore, every integral curve of the vector field $\tilde{\xi}_\mathfrak{c}$ which starts from a point of $\mathfrak{c}$ coincides « locally » with a piece of a marked trajectory.

From the definition of the admissible synthesis we deduce :

— either there exists a cell $\mathfrak{c}_0$ of type II such that from every point of $\mathfrak{c}_0$ starts a marked trajectory which enters $\mathfrak{c}$ and hence $\mathfrak{c} = \Sigma(\mathfrak{c}_0)$,

— or the marked trajectories reach $\mathfrak{c}$ from another cell of type I.

In the first case we have $\mathfrak{c}_0 \subset \bar{\mathfrak{c}}$ and hence the submanifold (possibly with boundary) $\mathfrak{c}' = \mathfrak{c}_0 \cup \mathfrak{c}$ is also a submanifold of $\tilde{\mathfrak{c}}$.

Since $\mathfrak{v}|_{\mathfrak{c}'}$ is of class C^1 we deduce that $\tilde{\xi}_\mathfrak{c}(\mathfrak{x}) = \tilde{\xi}(\mathfrak{x})$ for $\mathfrak{x} \in \mathfrak{c}_0$ and hence the integral curve of $\tilde{\xi}_\mathfrak{c}$ passing through any point of $\mathfrak{c}' = \mathfrak{c}_0 \cup \mathfrak{c}$ coincides « locally » with the marked trajectory.

I what follows we shall consider the submanifold $\mathfrak{c}' \subset X$ which is either the cell $\mathfrak{c}$ of type I if does not exist a cell $\mathfrak{c}_0$ of type II such that $\mathfrak{c} = \Sigma(\mathfrak{c}_0)$ or the union $\mathfrak{c}_0 \cup \mathfrak{c}$ if there exists a cell $\mathfrak{c}_0$ of type II such that $\mathfrak{c} = \Sigma(\mathfrak{c}_0)$.

From the condition B. (iii) of the definition of the admissible synthesis it follows that for any point $\mathfrak{x} \in \mathfrak{c}'$ there exists a number $\tau(\mathfrak{x}) > 0$ such that the marked trajectory $\varphi_\mathfrak{x}$ reaches the cell $\Pi(\mathfrak{c})$ at the moment $\tau(\mathfrak{x})$, that is we have $\chi(\mathfrak{x}) = \varphi_\mathfrak{x}(\tau(\mathfrak{x})) \in \Pi(\mathfrak{c})$ and $\varphi_\mathfrak{x}(\mathfrak{t}) \in \mathfrak{c}'$ for $0 \leqslant \mathfrak{t} < \tau(\mathfrak{x})$.

Since $\Pi(\mathfrak{c}) \subset X$ is a submanifold and $\tilde{\mathfrak{c}} \subset X$ is an open submanifold, the set $\tilde{\Pi}(\mathfrak{c}) = \Pi(\mathfrak{c}) \cap \tilde{\mathfrak{c}}$ is a submanifold both for X and for $\tilde{\mathfrak{c}}$ and an open submanifold for $\Pi(\mathfrak{c})$.

Therefore, there exist the maps :

$$\tau : \mathfrak{c}' \to R \quad \text{and} \quad \chi : \mathfrak{c}' \to \tilde{\Pi}(\mathfrak{c}) \tag{5.3}$$

such that $\varphi_\mathfrak{x}(\tau(\mathfrak{x})) = \chi(\mathfrak{x}))$ for any $\mathfrak{x} \in \mathfrak{c}'$ and $\varphi_\mathfrak{x}(\mathfrak{t}) \in \mathfrak{c}'$ for $0 \leqslant \mathfrak{t} < \tau(\mathfrak{x})$.

Proposition 5.1

The maps (5.3) *are C^1-morphisms.*

Proof. We shall use the C^1-vector field $\tilde{\xi}_\mathfrak{c} : \tilde{\mathfrak{c}} \to T(\tilde{\mathfrak{c}})$ which defines a C^1-maximal flow ([10], [11]), $\tilde{\psi}_\mathfrak{c} : \tilde{D}_\mathfrak{c} \to \tilde{\mathfrak{c}}$ where $\tilde{D}_\mathfrak{c} \subset R \times \tilde{\mathfrak{c}}$ is an open subset.

For any $\mathfrak{x} \in \tilde{\mathfrak{c}}$, the partial map $\psi_{\mathfrak{c},\mathfrak{x}} : (\mathfrak{t}^-(\mathfrak{x}), \mathfrak{t}^+(\mathfrak{x})) \to \tilde{\mathfrak{c}}$ is the maximal integral curve in $\mathfrak{x}$ of the vector field $\tilde{\xi}_\mathfrak{c}$. Moreover, if $\tilde{D}_{\mathfrak{c},\mathfrak{t}} = \{ \mathfrak{x} \in \tilde{\mathfrak{c}} \mid (\mathfrak{t}, \mathfrak{x}) \in \tilde{D}_\mathfrak{c} \}$ then the partial map $\tilde{\psi}_{\mathfrak{c}.\mathfrak{t}} : \tilde{D}_{\mathfrak{c},\mathfrak{t}} \to \tilde{\mathfrak{c}}$ is a local diffeomorphism in every point $\mathfrak{x} \in \tilde{D}_{\mathfrak{c},\mathfrak{t}}$.

According to the above mentioned properties the integral curve $\tilde{\psi}_{c,x}$ coincides « locally » with the marked trajectory φ_x, that is we have :

(5.4) $\tilde{\psi}_{c,x}(t) = \varphi_x(t)$ for any t for which $\varphi_x(t) \in c'$ and in particular for $t \in [0, \tau(x)]$.

We shall show first that for $t = \tau(x)$ (5.4) holds; it is sufficient to prove that $(t^-(x), t^+(x)) \supset [0, \tau(x)]$.

Since φ_x is continuous $\varphi_x([0, \tau(x)]) \subset \tilde{c}$ is a compact subset and from $\varphi_x(t) = \tilde{\psi}_{c,x}(t)$ for $t \in [0, \tau(x)]$ it follows that $\tilde{\psi}_{c,x}$ does not leave the compact $\varphi([_x0, \tau(x)])$ for $t \in [0, \tau(x)]$.

Hence, from the theorem 4 chap. IV in [11] it follows that

$$(t^-(x), t^+(x)) \supset [0, \tau(x)].$$

Therefore $(\tau(x), x), (\tau(x), \chi(x)) \in \tilde{D}_c$ and

$$\varphi_x(\tau(x)) = \chi(x) = \tilde{\psi}_{c,x}(\tau(x)) = \tilde{\psi}_c(\tau(x), x)$$

To prove that the maps $\chi : c' \to \tilde{\Pi}(c)$, $\tau : c' \to R$ are C^1-morphisms we must prove that at every point $x_0 \in c'$ there exist the charts (U, α), (U', α') on c, the chart (V, β) at $\chi(x)$ on $\Pi(c)$ and there exists the interval $I_{x_0} \ni \tau(x_0)$ such that $\beta(U) \subset V$, $\tau(U') \subset I_{x_0}$ and the local representatives of the maps χ and τ, $\beta \circ \chi \circ \alpha^{-1} : \alpha(U) \to \beta(V)$, $\tau \circ (\alpha')^{-1} : \alpha'(U') \to I_{x_0}$ respectively are C^1-morphisms.

Since the flow $\tilde{\psi}_c : D_c \to \tilde{c}$ is a C^1-morphism it follows that there exists a chart $(I_2 \times U_2, \mathrm{id} \times \alpha_2)$ at $(\tau(x_0), x_0) \in \tilde{D}_c$ (where (U_2, α_2) is a chart at x_0 on $\tilde{c}$) and there exists a chart (V_2, β_2) at the point $\chi(x_0) = \tilde{\psi}_c(\tau(x_0), x_0)$ on $\tilde{c}$ such that $\tilde{\psi}_c(I_2 \times U_2) \subset V_2$ and the map

$$\beta_2 \circ \tilde{\psi} \circ (id \times \alpha_2^{-1}) : I_2 \times \alpha_2(U) \to \beta_2(V_2)$$

is of class C^1.

On the other hand, since $c' \subset \tilde{c}$ is a submanifold (possibly with boundary) say of dimension r $(k + 1 \leqslant r \leqslant n)$ and $\tilde{\Pi}(c) = \Pi(c) \cap \tilde{c}$ is a $(r - 1)$-dimensional submanifold, it follows that there exists a chart (U_1, α_1) at $x_0 \in c'$ with the submanifold property for c' ([11]) and there exists a chart (V_1, β_1) at the point $\chi(x_0)$ on $\tilde{c}$ with the submanifold property for $\Pi(c)$.

That means that if the local coordinates in $U_1 \subset \tilde{c}$ are $\alpha = (\alpha^1, \alpha^2, \ldots, \alpha^n)$ and in $V_1 \subset \tilde{c}$ the local coordinates are $\beta = (\beta^1, \ldots, \beta^n)$ then the local coordinates in $U_1 \cap c'$ are $(\alpha^1, \ldots \alpha^r, 0, \ldots 0)$ where $\alpha^r \geqslant 0$ and the local coordinates in $V_1 \cap \tilde{\Pi}(c)$ are $(\beta^1, \ldots, \beta^{r-1}, 0, \ldots, 0)$.

We consider now the charts $(I_3 \times U_3, \text{id} \times \alpha_3)$ and (V_3, β_3) instead of the charts $(I_2 \times U_2, \text{id} \times \alpha_2)$ and (V_2, β_2) respectively, where

$$U_3 = U_1 \cap U_2, \alpha_3 = \alpha_2 | U_3, V_3 = V_1 \cap V_2, \beta_3 = \beta_2 | V_3,$$
$$I_3 = I_2 \cap (P_R^r \tilde{\psi}_c^{-1}(V_3))$$

We denote by $\psi_c = (\psi_c^1, \psi_c^2, ..., \psi_c^n) = \beta_3 \circ \tilde{\psi}_c \circ (id \times \alpha_3^{-1})$ the local representative of $\tilde{\psi}_c$ with respect to the charts $(I_3 \times U_3, \text{id} \times \alpha_3)$ and (V_3, β_3) (ψ^i, $i = 1, 2, ... n$ are real functions of class C^1 on $I_3 \times \alpha_3(U_3)$).

Then, for any $(t; \alpha^1, ..., \alpha^n) \in I_3 \times \alpha_3(U_3)$ we have

$$\psi_c(t; \alpha^1, ..., \alpha^n) = (\psi_c^1(t; \alpha^1, ..., \alpha^n), ..., \psi_c^n(t; \alpha^1, ..., \alpha^n)) \in \beta_3(V_3)$$

but $\psi_c(t; \alpha^1, ..., \alpha^n) \in \beta_3(V_3 \cap \tilde{\Pi}(c))$ means that

$$\psi_c^i(t; \alpha^1, ..., \alpha^n) = 0 \text{ for } i = r, r+1, ... n$$

(We have chosen the charts $(I_3 \times U_3, \text{id} \times \alpha_3)$ and (V_3, β_3) having the submanifolds properties for the local representative of $\tilde{\psi}_c$).

It is obvious that to prove the proposition (5.1.) it is sufficient to prove that there exists a neighborhood $U'' \subset R^r$ of the point $(0, 0, ..., 0) \in R^r$ and there exist the real functions $t : U'' \to I_3$, $\beta_i : U'' \to R$, $i = 1, 2, ..., r-1$, such that the following conditions hold :

$$(5.5) \quad \begin{cases} t(0, 0, ..., 0) = \tau(x_0) \\ \beta_i(0, 0, ..., 0) = 0 \qquad i = 1, 2, ..., r-1 \end{cases}$$

$$(5.6) \quad \begin{cases} \psi_c^i(t(\alpha^1, ..., \alpha^r); \alpha^1, ..., \alpha^r, 0, ..., 0) = \beta_i(\alpha^1, ..., \alpha^r) \quad i = 1, 2, ..., r-1 \\ \psi_c^j(t(\alpha^1, ..., \alpha^r); \alpha^1, ..., \alpha^r; 0, ..., 0) = 0 \qquad j = r, r+1, ..., n \end{cases}$$

for any point $(\alpha^1, ..., \alpha^r) \in U''$

Therefore, we must prove that the functions :

$$(5.7) \quad \psi_c^i(t; \alpha^1, ..., \alpha^r; 0, ..., 0) - \beta^i, \psi_c^j(t; \alpha^1, ..., \alpha^r, 0, ... 0),$$
$$i = 1, 2, ... r-1; j = r, r+1, ..., n$$

satisfy the conditions of the implicit function theorem at the point $t = \tau(x_0)$, $\alpha^1 = \alpha^2 = ... = \alpha^r = 0$, $\beta^1 = \beta^2 = ... = \beta^r = 0$

The functions (5.7) vanish at this point because we can take the charts $(I_3 \times U_3, \text{id} \times \alpha_3)$ and (V_3, β_3) such that

$$\alpha_3(x_0) = (0, ..., 0), \beta_3(\chi(x_0)) = (0, ..., 0)$$

and we have $\tilde{\psi}(\tau(x_0), x_0) = \chi(x_0)$

To prove that the Jacobi matrix of the functions

$\psi_c^i(t;\alpha^1,\dots,\alpha^r,0,\dots,0)-\beta^i,\ i=1,2,\dots,r-1,\ \psi_c^j(t;\alpha^1,\dots\alpha^r,0,\dots,0),$

$$j=r,r+1,\dots n$$

with respect to the variables $t,\beta^1,\dots,\beta^{r-1}$ at the point $t=\tau(x_0),\alpha^i=0$, $\beta^i=0$, $i=1,2,\dots r$, $j=1,2,\dots,r-1$ has the maximum rank r, we must prove that the vectors $\frac{\partial\psi_c}{\partial t}(\tau(x_0);0,\dots 0), e_1,\dots e_{r-1}$ are linearly independant where

$$\frac{\partial\psi}{\partial t}(\tau(x_0);0,\dots,0)=\begin{pmatrix}\frac{\partial\psi_c^1}{\partial t}(\tau(x_0);0,\dots,0)\\ \vdots\\ \vdots\\ \frac{\partial\psi_c^n}{\partial t}(\tau(x_0);0,\dots,0)\end{pmatrix}\quad\text{and}\quad e_i=\begin{pmatrix}0\\ \vdots\\ 0\\ 1\\ 0\\ \vdots\\ 0\end{pmatrix}-(i)$$

The chart (V_3,β_3) at the point $\chi(x_0)$ on the open submanifold $\tilde{c}\subset X$ induces a linear topological isomorphism $\beta_{3,\chi(x_0)}:T_{\chi(x_0)}\tilde{c}\to R^n$ between the tangent space $T_{\chi(x_0)}\tilde{c}$ and R^n ([11]).

The vector $\frac{\partial\psi}{\partial t}(\tau(x_0);0,\dots,0)$ is the immage by this isomorphism of the vector $T_{\tau(x_0)}\tilde{\psi}_{c,x_0}\cdot 1=\tilde{\xi}(\chi(x_0))\in T_{\chi(x_0)}\tilde{c}$ but the vectors $e_1,\dots,e_{r-1}$ are the immages by the same isomorphism of a basis of the space $T\mathfrak{J}_{\tilde{\Pi}(c)}(T_{\chi(x_0)}\tilde{\Pi}(c)$ where $\mathfrak{J}_{\tilde{\Pi}(c)}:\tilde{\Pi}(c)\to\tilde{c}$ the inclusion map.

The condition B. (ii) from the definition of the admissible synthesis states that $\lim_{t\nearrow\tau(x_0)}\tilde{\xi}(\varphi_{x_0}(t))\notin T\mathfrak{J}_{\tilde{\Pi}(c)}(T_{\chi(x_0)}\tilde{\Pi}(c))$ and hence $\tilde{\xi}(\chi(x_0)\notin T\mathfrak{J}_{\tilde{\Pi}(c)}(T_{\chi(x_0)}\tilde{\Pi}(c))$ because from the definition we have

$$\tilde{\xi}(\chi(x_0))=\lim_{t\nearrow\tau(x_0)}\tilde{\xi}(\tilde{\psi}_{c,x_0}(t))=\lim_{t\nearrow\tau(x_0)}\tilde{\xi}(\varphi_{x_0}(t)$$

This implies that the vectors $\frac{\partial\psi_c}{\partial t}(\tau(x_0);0\dots 0), e_1\dots e_{r-1}$ are linearly independent and the proposition (5.1) is completely proved.

Let us consider now the cells of type I, $c_1=c, c_2,\dots,c_q$ through which passes every marked trajectory starting in c' and such that $\Pi(c_q)\subset\mathfrak{F}$.

From the definition of the admissible synthesis it follows that every marked trajectory passes from the cell c_{i-1} either directly to the cell c_i if $\Pi(c_{i-1})$ is of type I (and hence $c_i = \Pi(c_{i-1})$ or by « crossing » the type II cell $\Pi(c_{i-1})$ when $c = \Sigma(\Pi(c_{i-1}))$.

For every cell of type I, c_i, the submanifold c'_i is either the cell c_i (if $\Pi(c_i)$ is of type I) or the submanifold $c_i \cup \Pi(c_i)$ if $\Pi(c_i)$ is of type II.

For every cell c_i we obtain the neighborhood $\tilde{c}_i$ of the closure $\bar{c}_i$, the vector field $\tilde{\xi}_i$ of class C^1 on $\tilde{c}_i$, the maximal flow $\tilde{\psi}_i : \tilde{D}_i \subset R \times \tilde{c}_i \to \tilde{c}_i$ and the C^1-morphisms $\tau^i : c'_i \to R$, $\chi^i : c'_i \to \tilde{\Pi}(c_i)$ such that

$$\chi^i(x) = \tilde{\psi}_i(\tau^i(x), x) \text{ for any } x \in c'_i. \tag{5.8}$$

Moreover, the set $\tilde{\Pi}(c_i) = \Pi(c_i) \cap \tilde{c}_i$ is a submanifold for $\tilde{c}_i$, an open submanifold for $\Pi(c_i)$ and also a submanifold for $\tilde{c}_{i+1}$ because if $\Pi(c_i)$ is of type II we have $\Pi(c_i) \subset \bar{c}_{i+1} \subset \tilde{c}_{i+1}$ and if $\Pi(c_i)$ is of type I we have $\Pi(c_i) = c_{i+1} \subset \tilde{c}_{i+1}$

From the definition of the admissible synthesis it follows that for any point $x \in c'$ the marked trajectory φ_x reaches $\Pi(c_i)$ at a moment $\tau_i(x)$ and let denote $\chi_i(x) = \varphi_x(\tau_i(x))$ (we note that τ_i, χ_i are not the same as τ^i, χ^i defined above). In particular we have $t_F = \tau_q(x)$, $x_F = \chi_q(x)$.

Since the marked trajectory φ_x is uniquely determined we have :

$$\varphi_{\chi_{i-1}(x)}(t - \tau_{i-1}(x)) = \varphi_x(t) \qquad \text{for} \qquad t \in [\tau_{i-1}(x), \tau_i(x)] \tag{5.9}$$

and $i = 2, \ldots, q$. If we define the maps $\tau_0 : c' \to R$, $\chi_0 : c' \to c'$ by

$$\tau_0(x) = 0, \ \chi_0(x) = x \tag{5.10}$$

for any $x \in c'$ then (5.9) holds also for $i = 1$.

From the definition of the maps τ^i, χ^i, τ_i, χ_i we obtain :

$$\tau^i(\chi_{i-1}(x)) + \tau_{i-1}(x) = \tau_i(x), \qquad \chi^i(\chi_{i-1}(x)) = \chi_i(x) \tag{5.11}$$

for $x \in c'$ and $i = 1, 2, \ldots, q$.

The formulae (5.11) represent recurrence relations for τ_i, χ_i. Since τ_0, χ_0 and τ^i, χ^i are C^1-morphisms (see proposition 5.1) we deduce that the maps τ_i, χ_i, $i = 1, 2, \ldots, q$ are also C^1-morphisms.

We denote by $J_i(x) = (t_i^-(\chi_{i-1}(x)), t_i^+(\chi_{i-1}(x)))$ the interval of definition of the maximal integral curve $\tilde{\psi}_{i, \chi_{i-1}(x)}$ of the vector field $\tilde{\xi}_i$ at the point $\chi_{i-1}(x) \in \tilde{c}_i$

$$\text{Let } \bar{D}_i = \{ (t, x) \mid x \in c', t - \tau_{i-1}(x) \in J_i(x) \}$$

Since $\tau_0(x) - \tau_{i-1}(x) = \tau^i(\chi_{i-1}(x))$ and $[0, \tau^i(\chi_{i-1}(x))] \subset J_i(x)$ it follows that $(t, x) \in \bar{D}_i$ for any $x \in c'$ and $t \in [0, \tau_i(x) - \tau_{i-1}(x)]$

We define the maps $\bar{\psi}_i : \bar{D}_i \to \tilde{c}_i$ by :

$$(5.12) \qquad \bar{\psi}_i(t, x) = \tilde{\psi}_i(t - \tau_{i-1}(x), \chi_{i-1}(x)), \qquad i = 1, 2, \dots q$$

In particular, from (5.11) we deduce that $\bar{D}_1 = \tilde{D}_1$ and $\bar{\psi}_1 = \tilde{\psi}_1$.

Since every marked trajectory coincides « locally » with and integral curve of the vector field $\tilde{\xi}$ we have

$$\varphi_x(t) = \tilde{\psi}_i(t - \tau_{i-1}(x), \chi_{i-1}(x)) = \varphi_{\chi_{i-1}(x)}(t - \tau_{i-1}(x))$$

for $t \in (\tau_{i-1}(x), \tau_i(x))$ and hence

$$(5.13) \qquad \varphi_x(t) = \bar{\psi}_x(t, x) \text{ for } x \in c' \text{ and } t \in [\tau_{i-1}(x), \tau_i(x)]$$

In particular, for $t = \tau_{i-1}(x)$ and for $t = \tau_i(x)$ we obtain respectively :

$$(5.14) \qquad \chi_{i-1}(x) = \bar{\psi}_i(\tau_{i-1}(x), x), \qquad \chi_i(x) = \bar{\psi}_i(\tau(x), x), \qquad i = 1, 2, \dots, q$$

Proposition 5.2.

The subsets $\bar{D}_i \subset R \times c'$, $i = 1, 2, \dots, q$ are open and the maps $\bar{\psi}_i : \bar{D}_i \to \tilde{c}_i$ defined by (5.12) *are C^1-morphisms.*

Moreover, the following relations hold :

$$(5.15) \qquad T_t\bar{\psi}_{i,x} \cdot 1 = \tilde{\xi}_i(\bar{\psi}_i(t, x)) \qquad \text{for} \qquad (t, x) \in \bar{D}_i$$

$$(5.16) \qquad T_x\bar{\psi}_{i,\tau_{i-1}(x)} \cdot v = -(T_x\tau \cdot v) \cdot \tilde{\xi}_i(\chi_{i-1}(x)) + T_{\chi_{i-1}(x)}\mathfrak{I}_{(i-1)}T_x\chi_{i-1} \cdot v$$

$$(5.17) \qquad T_x\bar{\psi}_{i,\tau_i(x)} \cdot v = -(T_x\tau_i \cdot v)\tilde{\xi}_i(\chi_i(x)) + T_{\chi_i(x)}\mathfrak{I}_{(i)}T_x\chi_i \cdot v$$

for every $v \in T_x c'$ and every $x \in c'$, where

$$\mathfrak{I}_{(i-1)} : \tilde{\Pi}(c_{i-1}) \to \tilde{c}, \qquad \mathfrak{I}_{(i)} : \tilde{\Pi}(c_i) \to \tilde{c}$$

are the inclusion maps.

Proof

We shall prove first that for any point $(t_0, x_0) \in \bar{D}_i$ there exists a real number $\varepsilon > 0$ and a neighborhood $U \subset c'$ of the point x_0 such that

$$\left(t_0 - \frac{\varepsilon}{2}, t_0 + \frac{\varepsilon}{2}\right) \times U \subset \bar{D}_i$$

If $J_i(x_0) = (t_i^-(\chi_{i-1}(x_0)), t_i^+(\chi(x_0)))$ is the interval of definition of the maximal integral curve $\tilde{\psi}_{i,\chi_{i-1}(x_0)}$, we have seen that if $(t_0, x_0) \in \bar{D}_i$ then $t - \tau_{i-1}(x_0) \in J_i(x_0)$. We denote

$$\tilde{t}_i^-(x_0) = t_i^-(\chi_{i-1}(x_0)) \text{ and } \tilde{t}_i^+(x_0) = t_i^+(\chi_{i-1}(x_0))$$

Since $J_i(x_0)$ is an open interval, there exists an $\varepsilon > 0$ such that :

(5.18) $\tilde{t}_{i-1}^-(x_0) < t_0 - \tau_{i-1}(x_0) - 2\varepsilon < t_0 + \tau_{i-1}(x_0) + 2\varepsilon < \tilde{t}_i^+(x_0)$

Since the map $\tau_{i-1} : c' \to R$ is continuous, the set

$$U_1 = \tau_{i-1}^{-1}(\tau_{i-1}(x_0) - \varepsilon, \tau_{i-1}(x_0) + \varepsilon)$$

is an open neighborhood of the point x_0 in c'.

Let $\delta_1 = t_0 - \tau_{i-1}(x_0) - \varepsilon$ and $\delta_2 = t_0 - \tau_{i-1}(x_0) + \varepsilon$.

For any point $(t, x) \in \left(t_0 - \frac{\varepsilon}{2}, t_0 + \frac{\varepsilon}{2}\right) \times U_1$ we have $t - \tau_{i-1}(x) \in (\delta_1, \delta_2)$ because for $x \in U_1$, $\tau(x) \in (\tau_{i-1}(x_0) - \varepsilon, \tau_{i-1}(x_0) + \varepsilon)$.

Moreover, from (5.18) it follows that $[\delta_1, \delta_2] \subset (\tilde{t}_i^-(x_0), \tilde{t}_i^+(x_0))$

From the corollary of the theorem 6 chap. IV in [11] it follows that if $[\delta_1, \delta_2] \subset (t_i^-(y_0), t_i^+(y_0))$ there exists a neighborhood $V \subset^{\circ} \tilde{c}_i$ of y_0 such that $[\delta_1, \delta_2] \subset (t_i^-(y), t_i^+(y))$ for any $y \in V$.

If we take $y_0 = \chi_{i-1}(x_0)$ then, by using the continuity of the map $\chi_{i-1} : c' \to \tilde{\Pi}(c_{i-1})$ we obtain a neighborhood $U_2 \subset c'$ of the point x_0 such that $\chi_{i-1}(U_2) \subset V \cap \tilde{\Pi}(c_{i-1})$

It is obvious that if $U = U_1 \cap U_2$ we have $(t_0 - \varepsilon/2, t_0 + \varepsilon/2) \times U \subset \bar{D}$; and hence $\bar{D}_i$ is open.

From the relation (5.11) and the fact that $\tau_{i-1}, \chi_{i-1}, \tilde{\psi}_i$ are C^1-morphisms it follows immediately that $\bar{\psi}_i$ is a C^1-morphism.

Using the functorial properties of the tangent of a C^1-morphism ([11]), from (5.12) we obtain (5.15) if we observe that

$$T_{t-\tau_{i-1}(x)}\tilde{\psi}_{i,\chi_{i-1}(x)} \cdot 1 = \tilde{\xi}_i(\tilde{\psi}_{i,\chi_{i-1}(x)}(t - \tau_{i-1}(x)))$$

To prove the relations (5.16) and (5.17) we observe that the relations (5.14) may be written

$$\bar{\psi}_i \circ (\tau_{i-1}, \mathrm{id}_{c'}) = \mathfrak{I}_{(i-1)} \circ \chi_{i-1} \tag{5.19}$$

$$\bar{\psi}_i \circ (\tau_i, \mathrm{id}_{c'}) = \mathfrak{I}_{(i)} \circ \chi_i$$

where $\mathfrak{I}_{(i-1)} : \tilde{\pi}(c_{i-1}) \to \tilde{c}_i$, $\mathfrak{I}_{(i)} : \tilde{\pi}(c_i) \to \tilde{c}_i$ are the inclusion maps.

Considering the tangent to the maps in (5.19) and using the relations (5.15) proved above, we obtain immediatly the relations (5.16) and (5.17).

6. DUAL VARIABLES

In this section we shall use as above, some notations from [9] and [11] and also some results on symplectic manifolds from [8] and [10].

We shall denote (U, α) a chart an X at the point $\mathfrak{x} \in X$ where $\alpha(U) \subset R^n$ and $\alpha_{\mathfrak{x}} : T_{\mathfrak{x}}X \to R^n$ the toplinear isomorphism induced by the chart (U, α) between the tangent space $T_{\mathfrak{x}}X$ and R^n

The natural projection $\mathrm{T}_X : T(X) \to X$ is the tangent bundle and the natural charts an $T(X)$ are the pairs $(T(U), T\alpha)$ where $T(U) = \bigcup_{\mathfrak{x} \in U} T_{\mathfrak{x}}X$ and $T\alpha :$ $T(U) \to \alpha(U) \times R^n$ is the diffeomorphism given by the relation

$$T\alpha(\dot{\mathfrak{x}}) = (\alpha(\mathfrak{x}), \alpha\,(\dot{\mathfrak{x}})) \qquad \text{for any} \qquad \dot{\mathfrak{x}} \in T_{\mathfrak{x}}X$$

That means that of on $U \subset X$ we have the local coordinates $\alpha = (\alpha^1, \ldots, \alpha^n)$ (that is $\alpha(\mathfrak{x}) = (\alpha^1, \ldots, \alpha^n) \in \alpha(U)$ for $\mathfrak{x} \in U$) then $\dot{\mathfrak{x}} \in T_{\mathfrak{x}}X$ has the local coordinates $(\alpha^1, \ldots, \alpha^n, v_1, \ldots, v_n)$ where $v = (v_1, \ldots, v_n) = \alpha_n(\dot{\mathfrak{x}})$.

We write $v = \sum_{i=1}^{n} v_n \frac{\partial}{\partial \alpha_i}$ and say that v is the coordinate expression of the vector $\dot{\mathfrak{x}} \in T_{\mathfrak{x}}X$.

If $T_{\mathfrak{x}}^*X$ is the dual space of the vector space $T_{\mathfrak{x}}X$ (that is the space of the linear continuous functionals on $T_{\mathfrak{x}}X$) then, the set $T^*(X) = \bigcup_{\mathfrak{x} \in X} T_{\mathfrak{x}}^*X$ is the cotangent manifold and the natural projection $T_X^* : T^*(X) \to X$ is the cotangent bundle.

The natural charts an $T^*(X)$ are the pairs $(T^*(U), T^*\alpha)$ where $T^*(U) = \bigcup_{\mathfrak{x} \in U} T_{\mathfrak{x}}^*X$ and $T^*\alpha : T\ (U) \to \alpha(U) \times (R^n)^*$ is defined by :

$$T^*\alpha(\mathfrak{y}) = (\alpha(\mathfrak{x}), \lambda) \qquad \text{for} \qquad \mathfrak{y} \in T_{\mathfrak{x}}^*X$$

where $\lambda \in (R^n)^*$ is given by $\lambda = \mathfrak{y} \circ \alpha_{\mathfrak{x}}^{-1}$

The map $\alpha_{\mathfrak{x}}^* : T_{\mathfrak{x}}^*X \to (R^n)^*$ given by

$$\alpha_{\mathfrak{x}}^*(\mathfrak{y}) = \lambda = \mathfrak{y} \circ \alpha_{\mathfrak{x}}^{-1}$$

is a toplinear isomorphism.

If the local coordonates in U on X are $(\alpha^1, \ldots, \alpha^n) = \alpha$ then $(\alpha^1, \ldots, \alpha^n,$ $\lambda_1, \ldots, \lambda_n)$ are the local coordinates in $T^*(U)$ on $T^*(X)$.

We write $\lambda = \sum_{i=1}^{n} \lambda_i \, d\alpha^i$ and say that this is the coordinate expression of the linear form $\mathfrak{y} \in T_{\mathfrak{x}}^*X$ if $\alpha_{\mathfrak{x}}^*(\mathfrak{y}) = \lambda = (\lambda_1, \ldots, \lambda_n)$

We shall use the well known fact that the manifold $T^*(X)$ with the second canonic differential form on X represent a symplectic manifold.

It is known (see [8], [10]) that for every C^1-function $H : T^*(X) \to R$ there exists a unique C^1-vector field on $T^*(X)$: $(dH)^{\#} = \xi_H : T^*(X) \to T(T^*(X))$ such that the corresponding principal part with respect to the natural charts on $T^*(X)$ and $T(T^*(X))$ has the following coordinate expression :

$$\sum_{i=1}^{n} \left(\frac{\partial H_\alpha}{\partial \lambda_i} \frac{\partial}{\partial \alpha^i} - \frac{\partial H_\alpha}{\partial \alpha^i} \frac{\partial}{\partial \lambda_i} \right) \tag{6.1}$$

where $H_\alpha = H \circ (T^*\alpha)^{-1}$ is the local representative of the function H with respect to the chart $(T^*(U), T^*\alpha)$.

Therefore, the local representative of an integral curve of the vector field ξ_H is a solution of the differential system :

$$\begin{aligned} \frac{d\alpha^i}{dt} &= \frac{\partial H_\alpha}{\partial \lambda_i}(\alpha^1, \ldots, \alpha^n, \lambda_1, \ldots, \lambda_n) \\ \frac{d\lambda^i}{dt} &= -\frac{\partial H_\alpha}{\partial \alpha^i}(\alpha^1, \ldots, \alpha^n, \lambda_1, \ldots, \lambda_n) \end{aligned} \tag{6.2}$$

Moreover, the function H is a first integral for the vector field ξ_H that is H is constant along to any integral curve of ξ_H.

For the admissible synthesis defined in section 4 we shall define a function H of class C^1 on every cell and we shall study the integral curves of the vector field ξ_H. As we shall see in this section these integral curves which are curves in the cotangent manifold are in a certain connection with the marked trajectories.

Let us consider as in the preceeding section the cell $\mathfrak{c}$ of type I, the neighborhood $\tilde{\mathfrak{c}} \subset X$ of the closure $\bar{\mathfrak{c}}$, the C^1-extension $\tilde{\mathfrak{v}}_\mathfrak{c} : \tilde{\mathfrak{c}} \to \Omega$ of the restriction $\mathfrak{v}_\mathfrak{c} = \mathfrak{v}|_\mathfrak{c}$ and the vector field $\tilde{\xi}_\mathfrak{c}$ on $\tilde{\mathfrak{c}}$ (5.2).

For the given C^1-function $\mathfrak{f}^0 : X \times \Omega \to R$ we define the C^1-function $\tilde{\mathfrak{f}}^0_\mathfrak{c} : \tilde{\mathfrak{c}} \to R$ by

$$\tilde{\mathfrak{f}}^0_\mathfrak{c}(\mathfrak{x}) = \mathfrak{f}^0(\mathfrak{x}, \tilde{\mathfrak{v}}_\mathfrak{c}(\mathfrak{x})) \quad \text{for any} \quad \mathfrak{x} \in \tilde{\mathfrak{c}} \tag{6.3}$$

We define now the the function $\tilde{H}_\mathfrak{c} : T^*\tilde{\mathfrak{c}} \to R$ by :

$$\tilde{H}_\mathfrak{c}(\mathfrak{y}) = \tilde{\mathfrak{f}}_\mathfrak{c}^0(\mathfrak{x}) + \mathfrak{y} \cdot \tilde{\xi}_\mathfrak{c}(\mathfrak{x}) \quad \text{for} \quad \mathfrak{y} \in T^*_\mathfrak{x}\tilde{\mathfrak{c}} \quad \text{and} \quad \mathfrak{x} \in \tilde{\mathfrak{c}} \tag{6.4}$$

Using the local representative it follows immediatly that the function $\tilde{H}_\mathfrak{c}$ is of class C^1.

Indeed, if (U, α) is a chart on $\tilde{\mathfrak{c}}$ at $\mathfrak{x} \in \tilde{\mathfrak{c}}$ where the local coordinates are $(\alpha^1, \ldots, \alpha^n) = \alpha$ and if $(T^*(U), T^*\alpha)$ is the corresponding natural chart on $T^*\tilde{\mathfrak{c}}$

where the local coordinates are $(\alpha^1, ..., \alpha^n, \lambda_1, ..., \lambda_n)$ then, the local representative of the function $\tilde{H}_c$, $\tilde{H}_{c,\alpha} = \tilde{H}_c \circ (T^*\alpha)^{-1}$ is given by :

$$(6.5) \quad \tilde{H}_{c,\alpha}(\alpha^1, ..., \alpha^n, \lambda_1, ..., \lambda_n) = \tilde{f}^0_{c,\alpha}(\alpha^1, ..., \alpha^n) + \sum_{i=1}^{n} \lambda_i \tilde{f}_{c,\alpha}(\alpha^1, ..., \alpha^n)$$

where $\tilde{f}^0_{c,\alpha} = \tilde{f}^0_c \circ \alpha^{-1}$ is the local representative of the function $\tilde{f}^0_c$ and $\tilde{f}_{c,\alpha} = (\tilde{f}^1_{c,\alpha}, ..., \tilde{f}^n_{c,\alpha})$ is the principal part of the local representative of the vector field $\tilde{\xi}_c$ with respect to the charts (U, α) and $(T(U), T\alpha)$.

Hence we may associate a unique vector field $\xi_{\tilde{H}_c} = (d\tilde{H}_c)^{\#}$ on $T^*\tilde{c}$ which has the principal part

$$(6.6) \quad \sum_{i=1}^{n} \left(\frac{\partial \tilde{H}_{c,\alpha}}{\partial \lambda_i} \frac{\partial}{\partial \alpha^i} - \frac{\partial \tilde{H}_{c,\alpha}}{\partial \alpha^i} \frac{\partial}{\partial \lambda_i} \right)$$

with respect to the charts $(T^*(U), T^*\alpha)$, $(T(T^*(U)), T(T^*\alpha))$.

Therefore, the local representative of an integral curve of the vector field $\xi_{\tilde{H}_c}$ is a solution of the differential system :

$$(6.7) \quad \begin{cases} \dfrac{d\alpha^i}{dt} = \tilde{f}^i_{c,\alpha}(\alpha^1, ..., \alpha^n) \\ \dfrac{d\lambda_i}{dt} = -\dfrac{\partial \tilde{f}^0_{c,\alpha}}{\partial \alpha^i}(\alpha^1, ..., \alpha^n) - \displaystyle\sum_{i=1}^{n} \lambda_i \frac{\partial \tilde{f}^i_{c,\alpha}}{\partial \alpha^i}(\alpha^1, ..., \alpha^n) \quad i = 1, 2, ..., n. \end{cases}$$

Since the first equations are independent and since the right hand side of this subsystem is the principal part of the vector field $\tilde{\xi}_c$ we deduce :

— at every point $\eta \in T^*(\tilde{c})$ there exists a unique integral curve $\tilde{\Phi}_{c,\eta}$ of the vector field $\xi_{\tilde{H}_c}$;

— the projection of the integral curve $\tilde{\Phi}_{c,\eta}$ an the manifold $\tilde{c}$ by the cotangent bundle $T^*_{\tilde{c}} : T^*(\tilde{c}) \to \tilde{c}$ is the integral curve $\tilde{\psi}_{c,x}$ of the vector field $\tilde{\xi}_c$ if $\eta \in T^*_x\tilde{c}$ (or $T^*_{\tilde{c}}(\eta) = x$).

Moreover, the integral curve $\tilde{\Phi}_{c,\eta}$ is defined on the whole interval of definition $(t^-_c(x), t^+_c(x))$ of the maximal integral curve $\tilde{\psi}_{c,x}$ of $\tilde{\xi}_c$;

— $\tilde{H}_c(\tilde{\Phi}_{c,\eta}(t))$ is constant for any t in the interval of definition of the curve $\tilde{\Phi}_{c,\eta}$.

We consider again the submanifold c' which is either the cell c of type I or the union $c_0 \cup c$ if there exists a cell of type II c_0 such that $c = \Sigma(c_0)$ and we consider the cells of type I $c_1 = c, c_2, ... c_2$ through which pass the marked trajectories starting in c' and $\Pi(c_q) \subset \mathfrak{F}$.

For every cell c_i we consider as above the neighborhood $\tilde{c}_i$, the function $\tilde{H}_i = \tilde{H}_{c_i}$ and the vector field $\xi_{\tilde{H}_i} : T^*(\tilde{c}_i) \to T(T^*\tilde{c}_i)$

Proposition 6.1.

For every marked trajectory $\varphi_{\mathfrak{x}} : [0, t_F] \to X$ *which starts at the points* $\mathfrak{x} \in X \backslash N$ *there exists a functional* $\mathfrak{y}(\mathfrak{x}) \in T^*_{\mathfrak{x}}X$ *and a curve* $\Phi_{\mathfrak{y}(\mathfrak{x})} : [0, t_F] \to T^*X$ *with the following properties :*

(i) $\Phi_{\mathfrak{y}(\mathfrak{x})}(0) = \mathfrak{y}(\mathfrak{x}), \Phi_{\mathfrak{y}(\mathfrak{x})}(t) \in T^*_{\varphi_{\mathfrak{x}}(t)}X \qquad \text{for } t \in [0, t_F]$

(ii) *on every interval* $[\tau_{i-1}(\mathfrak{x}), \tau_i(\mathfrak{x})]$ *the function* $\Phi_{\mathfrak{y}(\mathfrak{x})}$ *is an integral curve of the vector field* $\xi_{\tilde{H}_i}$.

(iii) $\Phi_{\mathfrak{y}(\mathfrak{x})}$ *is continuous to the right and its one sided limits at the points* $t = \tau_i(\mathfrak{x}), \mathfrak{y}_i^- = \Phi_{\mathfrak{y}(\mathfrak{x})}(\tau_i(\mathfrak{x}) - 0), i = 1, 2, \ldots, q$

satisfy the following relations :

$$(6.8) \qquad \begin{cases} \mathfrak{y}_q^- \circ T_{\chi_q(\mathfrak{x})}\mathfrak{I}_{\pi(\mathfrak{c}_q)} = T_{\chi_q(\mathfrak{x})}\mathfrak{g} \\ \tilde{H}_q(\mathfrak{y}_q^-) = 0 \end{cases}$$

$$(6.9) \qquad \begin{cases} \mathfrak{y}_i^- \circ T_{\chi_i(\mathfrak{x})}\mathfrak{I}_{\pi(\mathfrak{c}_i)} = \mathfrak{y}_i^+ \circ T_{\chi_i(\mathfrak{x})}\mathfrak{I}_{\tilde{\pi}(\mathfrak{c}_i)} \\ \tilde{H}_i(\mathfrak{y}_i^-) = 0 \end{cases}$$

for $i = 1, 2, \ldots, q - 1$

where $\mathfrak{I}_{\tilde{\pi}(\mathfrak{c}_i)} : \tilde{\pi}(\mathfrak{c}_i) \to \tilde{\mathfrak{c}}_i$ *are the inclusion maps and* $\mathfrak{y}_i^+ = \Phi_{\mathfrak{y}(\mathfrak{x})}(\tau_i(\mathfrak{x}))$

Proof

As we have observed above, for every point $\mathfrak{y} \in T^*(\tilde{\mathfrak{c}}_i)$ there exists an integral curve $\tilde{\Phi}_{i,\mathfrak{y}}$ of the vector field $\xi_{\tilde{H}_i}$ and if $\mathfrak{y} \in T^*_{\mathfrak{x}}(\tilde{\mathfrak{c}}_i)$ where $\mathfrak{x} \in \mathfrak{c}_i$ then the projection of the curve $\Phi_{i,\mathfrak{y}}$ by the cotangent bundle $T^*_{\tilde{\mathfrak{c}}_i}$ is a piece of the marked trajectory $\varphi_{\mathfrak{x}}$.

The « dual » trajectory $\Phi_{\mathfrak{y}(\mathfrak{x})}$ will be obtained by « sticking » such integral curves which corresponds to the cells $\mathfrak{c}_1, \mathfrak{c}_2, \ldots, \mathfrak{c}_q$.

Let us suppose that there exist the cotangent vectors $\mathfrak{y}_q^-, \mathfrak{y}_{q-1}^-, \ldots, \mathfrak{y}_1^-$ which satisfy the relations (6.8) and (6.9) when $\mathfrak{y}_{q-1}^+, \ldots, \mathfrak{y}_1^+$ are given.

For the points $\mathfrak{y}_i^- \in T_{\chi_i(\mathfrak{x})}\tilde{\mathfrak{c}}_i$ $i = 1, 2, \ldots, q$ there exist the integral curves $\tilde{\Phi}_{i,\mathfrak{y}_i^-}$ of the vector fields $\xi_{\tilde{H}_i}$, $i = 1, 2, \ldots, q$ respectively, which are defined on the intervals $(t_i^-(\chi_i(\mathfrak{x})), t_i^+(\chi_i(\mathfrak{x})))$ of the maximal integral curve $\tilde{\psi}_{i,\chi_i(\mathfrak{x})}$.

According to (5.7) we have $\varphi_{\mathfrak{x}}(t) = \varphi_{\chi_{i-1}(\mathfrak{x})}(t - \tau_{i-1}(\mathfrak{x})), t \in [\tau_{i-1}(\mathfrak{x}), \tau_i(\mathfrak{x})]$ and according to (5.4) and (5.9), we have $\varphi_{\chi_{i-1}(\mathfrak{x})}(t) = \tilde{\psi}_{i,\chi_{i-1}(\mathfrak{x})}(t)$ for $t \in [0, \tau_i(\mathfrak{x}) - \tau_{i-1}(\mathfrak{x})]$ hence, it follows that

$$(6.10) \qquad \varphi_{\mathfrak{x}}(t) = \tilde{\psi}_{i,\chi_{i-1}(\mathfrak{x})}(t - \tau_{i-1}(\mathfrak{x})) \qquad \text{for} \qquad t \in [\tau_{i-1}(\mathfrak{x}), \tau_i(\mathfrak{x})]$$

Since $\tilde{\psi}_{i,\chi_{i-1}(x)}(\tau(x) - \tau_{i-1}(x)) = \chi_i(x)$ we have

$$\tilde{\psi}_{i,\chi_{i-1}(x)}(t) = \tilde{\psi}_{i,\chi_i(x)}(t - \tau_i(x) + \tau_{i-1}(x)) \text{ for } t \in [0, \tau_i(x) - \tau_{i-1}(x)]$$

and from (6.10) we obtain :

$$(6.11) \qquad \varphi_x(t) = \tilde{\psi}_{i,\chi_i(x)}(t - \tau_i(x)) \qquad \text{for} \qquad t \in [\tau_{i-1}(x), \tau_i(x)]$$

Therefore the integral curve $\tilde{\Phi}_{i,\eta_i^-}$ is defined on the interval

$$[\tau_{i-1}(x) - \tau_i(x), 0]$$

(as the integral curve $\tilde{\psi}_{i,\chi_i(x)}$) and we denote $\eta_i^+ = \tilde{\Phi}_{i,\eta_i^-}(\tau_{i-1}(x) - \tau_i(x))$ for $i = 1, 2, ..., q-1$ and $\eta(x) = \tilde{\Phi}_{1,\eta_1^-}(-\tau_1(x))$

It is obvious that the map $\Phi_{\eta(x)}$ defined by

$$\Phi_{\eta(x)}(t) = \tilde{\Phi}_{i,\eta_i^-}(t - \tau(x)) \qquad \text{for} \qquad t \in [\tau_{i-1}(x), \tau_i(x)), \Phi_{\eta(x)}(\tau_q(x)) = \eta_q^-$$

satisfies the conditions from the proposition.

We must prove now that there exist the points

$$\eta_q^- \in T^*_{\chi_q(x)}\tilde{c}_q = T^*_{\chi_q(x)}X \qquad \text{and} \qquad \eta_i^- \in T^*_{\chi_i(x)}\tilde{c}_i = T^*_{\chi_i(x)}X$$

which satisfy the relations (6.8) and (6.9) respectively.

Let us denote by $\hat{\xi}_{i(x)}$ the one-dimensional subspace of $T_{\chi_i(x)}\tilde{c}_i$ generated by the vector $\tilde{\xi}_i(\chi_i(x)$ From the hypothesis, we have $\tilde{\xi}_i(\chi_i(x)) \neq 0$ and $\tilde{\xi}_i(\chi_{i(x)}) \notin T\mathfrak{I}_{\tilde{\pi}(c_i)}(T_{\chi_i}\Pi((x)c_i))$ and hence $\hat{\xi}_i(x) + T\mathfrak{I}_{\pi(c_i)}(T_{i\chi(x)}\Pi(c_i))$ is $(k_i + 1)$-dimensional subspace of $T_{\chi_i(x)}\tilde{c}_i$ if $\Pi(c_i)$ is k_i-dimensional.

On the other hand the relations (6.8) and (6.9) mean that we must find the functional $\eta_i^- : T_{\chi_i(x)}\tilde{c}_i \to R$ when we know its action on the $(k_i + 1)$-dimensional vector subspace $\hat{\xi}_i(x) + T\mathfrak{I}_{\tilde{\pi}(c_i)}(T_{\chi_i(x)}\Pi(c_i))$ of the n-dimensional vector space $T_{\chi_i(x)}\tilde{c}_i$

Since $0 \leqslant k_i \leqslant n - 1$, hence $1 \leqslant k_i + 1 \leqslant n$, it is well known that such a functional exists always, moreover, the set of these functionals represent a $(n - k_i - 1)$-dimensional vector space. (In fact to find such a functional is to find a solution of a linear, $n \times (k_i + 1)$ algebraic system with matrix of the maximum rank $k_i + 1$.)

REMARK 6.1.

Since $\tilde{H}_i$ is a first integral from $\xi_{\tilde{H}_i}$ and from (6.8), (6.9) we obtain that $\tilde{H}_i(\Phi_{\eta(x)}(t)) = 0$ for $t \in [\tau_{i-1}(x), \tau_i(x)]$

REMARK 6.2.

If the dimension of the terminal manifold $\mathfrak{F}$ is $k = \mathfrak{n} - 1$, then $\mathfrak{y}_q^-, \mathfrak{y}_{q-1}^-, \dots, \mathfrak{y}_1^-$ are uniquely determined and hence the point $\mathfrak{y}(\mathfrak{x})$ is also uniqualy determined. When $0 \leqslant k < \mathfrak{n} - 1$ the one-sided limits $\mathfrak{y}_i^-$ $(i = 1, 2, \dots, q)$ are not unique and we may deduce that the point $\mathfrak{y}(\mathfrak{x})$ is not unique. However, in the next section we shall prove that when $\mathfrak{x}$ belongs to a $\mathfrak{n}$-dimensional cell, $\mathfrak{y}(\mathfrak{x})$ is uniquely determined even if the dimension of $\mathfrak{F}$ is $0 \leqslant k < \mathfrak{n} - 1$.

7. THE VALUE OF THE ADMISSIBLE SYNTHESIS AND SUFFICIENT CONDITIONS OF OPTIMALITY

For any $\mathfrak{x} \in X \backslash N$ there exists a submanifold $\mathfrak{c}' \subset X$ which is either a cell of type I $\mathfrak{c}$ or the union $\mathfrak{c}_0 \cup \mathfrak{c}$ where $\mathfrak{c}_0$ is a cell of type II such that $\mathfrak{c} = \Sigma(\mathfrak{c}_0)$ and such that $\mathfrak{x} \in \mathfrak{c}'$. Therefore, there exist the cells of type I $\mathfrak{c}_1 = \mathfrak{c}, \mathfrak{c}_2, \dots, \mathfrak{c}_q$ such that $\Pi(\mathfrak{c}_q) \subset \mathfrak{F}$ and the marked trajectory $\varphi_{\mathfrak{x}}$ passes through $\mathfrak{c}_1, \mathfrak{c}_2, \dots, \mathfrak{c}_q$. Then, the value of the admissible synthesis at the point $\mathfrak{x}$ is :

$$
\begin{aligned}
W(\mathfrak{x}) &= \mathfrak{g}(\chi_q(\mathfrak{x})) + \int_0^{\tau_q(x)} \mathfrak{f}^0(\varphi_{\mathfrak{x}}(t), \mathfrak{v}(\varphi_{\mathfrak{x}}(t)))\, dt \\
&= \mathfrak{g}(\chi_q(\mathfrak{x})) + \sum_{i=1}^{q} \int_{\tau^i_{-1}(\mathfrak{x})}^{\tau_i(\mathfrak{x})} \mathfrak{f}^0(\varphi_{\mathfrak{x}}(t), \mathfrak{v}(\varphi_{\mathfrak{x}}(t)))\, dt \qquad (7.1)\\
&= \mathfrak{g}(\chi_q(\mathfrak{x})) + \sum_{i=1}^{q} \int_{\tau^i_{-1}(\mathfrak{x})}^{\tau_i(\mathfrak{x})} \tilde{\mathfrak{f}}_i^0(\bar{\psi}_i(t, \mathfrak{x}))\, dt
\end{aligned}
$$

where the maps $\tilde{\mathfrak{f}}_i^0$, $\bar{\psi}_i$ are given by (6.3) and (6.10) respectively.

If we denote $M = N \cup \left(\bigcup_{i=k-1}^{\mathfrak{n}-1} P^i \right)$ then, the set $X \backslash M$ is the union of the all n-dimensional cells and hence an open submanifold of X (generally $X \backslash M$ is not connected).

Proposition 7.1.

(i) *For every above defined submanifold $\mathfrak{c}' \subset X$ the restriction $W|_{\mathfrak{c}'}$ is a C^1-function.*

(ii) *For every $\mathfrak{x} \in \mathfrak{c}'$ we have :*

$$
T_{\mathfrak{x}} W_{\mathfrak{c}'} = \mathfrak{y}(\mathfrak{x}) \circ T_{\mathfrak{x}} \mathfrak{i}_{\mathfrak{c}'} \qquad (7.2)
$$

where $\mathfrak{y}(\mathfrak{x})$ is the functional defined in the proposition 6.1. It follows that for $\mathfrak{x} \in X \backslash M$, $\mathfrak{y}(\mathfrak{x})$ is unique ($T_{\mathfrak{x}} W = \mathfrak{y}(\mathfrak{x})$).

Proof

If we denote

$$I_j(x) = \int_{\tau_{j-1}(x)}^{\tau_j(x)} \bar{f}_j^0(t, x)\, dt \qquad j = 1, 2, \ldots, q \tag{7.3}$$

where

$$\bar{f}_j^0(t, x) = \tilde{f}_j^0(\bar{\psi}_j(t, x)) \quad \text{for} \quad (t, x) \in \bar{D}_j \tag{7.4}$$

then, (7.1) becomes :

$$W(x) = g(\chi_q(x)) + \sum_{j=1}^{q} I_j(x) \tag{7.5}$$

for every $x \in c'$.

Since the maps $\bar{f}_j^0$, τ_{j-1}, τ_j are of class C^1 it is immediatly seen (using the local representatives) that the integral I_j given by (7.3) is a C^1 map and

$$T_x I_j \cdot v = (T_x \tau_j \cdot v)\bar{f}_j^0(\tau_j(x), x) - (T_x \tau_{j-1}(x) \cdot v)\bar{f}_j^0(\tau_{j-1}(x), x) + \int_{\tau_{j-1}(x)}^{\tau_j(x)} (T_x \bar{f}_{j,t}^0 \cdot v)\, dt \tag{7.6}$$

Hence, the first part of the proposition is proved.

To prove the second part of the proposition we need the following lemma :

Lemma 7.1

The map defined for every $x \in c'$ and $v \in T_x c'$ by : $t \to \Phi_{\eta(x)}(t) \cdot (T_x \bar{\psi}_{j,t} \cdot v)$ *for every* $t \in (\tau_{j-1}(x), \tau_j(x))$ *is a* C^1*-function and the following relation holds :*

$$\frac{d}{dt}(\Phi_{\eta(x)}(t) \cdot (T_x \bar{\psi}_{j,t} \cdot v)) + T_x \bar{f}_{j,t}^0 \cdot v = 0 \tag{7.7}$$

for every $t \in (\tau_{i-1}(x), \tau_i(x))$ *where* $\bar{\psi}_{i,t}$, $\bar{f}_{i,t}^0$ *are the partial maps of* $\bar{\psi}_i$, $\bar{f}_i^0$ *respectively and* $\Phi_{\eta(x)}$ *is the curve from the proposition* (6.1).

Proof of the lemma 7.1.

We shall use the local representative $\bar{\psi}_{i,\alpha,\beta} = \beta \circ \bar{\psi}_i \circ (\mathrm{id} \times \alpha^{-1})$ of the map $\bar{\psi}_i : \bar{D}_i \to \tilde{c}_i$ with respect to the charts $(I \times U, \mathrm{id} \times \alpha)$ at $(t_0, x_0) \in \bar{D}_i$ and (V, β) at $\bar{\psi}_i(t_0, x_0) \in \tilde{c}_i$ ((U, α) is a chart at $x_0 \in c'$ on c').

Then the local representative of the tangent $x \mapsto T_x \bar{\psi}_{i,t}$ for $x \in c'$ and $t \in (\tau_{i-1}(x), \tau_i(x))$ fixed, is the derivative

$$(\alpha^1, \alpha^2, \ldots, \alpha^n) \mapsto D_2 \bar{\psi}_{i,\alpha,\beta}(t; \alpha^1, \ldots, \alpha^n)$$

On the other hand we have $\Phi_{\eta(x)}(t) \in T^*_{\varphi_x(t)}X$ for every $t \in [0, \tau_q(x)]$ and since we have $\varphi_x(t) = \bar{\psi}_i(t, x)$ for every $t \in (\tau_{i-1}(x), \tau_i(x))$ it follows that $\Phi_{\eta(x)}(t) \in T^*_{\bar{\psi}_i(t,x)}\tilde{c}_i = T^*_{\bar{\psi}_i(t,x)}X$

From the proposition (6.1) since the local representative of the curve $\Phi_{\mathfrak{y}(\mathfrak{x})}$ with respect to the chart $(T^*(V), T^*\beta)$ an $T^*(\tilde{\mathfrak{c}}_{\mathfrak{i}})$ is :

$$\mathfrak{t} \mapsto T^*\beta(\Phi_{\mathfrak{y}(\mathfrak{x})}(\mathfrak{t})) = (\bar{\psi}_{\mathfrak{i},\alpha,\beta}(\mathfrak{t}\ ;\ \alpha', ..., \alpha\ ), \lambda(\mathfrak{t})) \in \beta(V) \times (R^{\mathfrak{n}})^*$$

the map $\mathfrak{t} \mapsto (\bar{\psi}_{\mathfrak{i},\alpha,\beta}(\mathfrak{t}\ ;\ \alpha', ..., \alpha^{\mathfrak{n}}), \lambda(\mathfrak{t}))$ is a solution of the differential system :

$$(7.8)\quad \begin{cases} D_1\bar{\psi}_{\mathfrak{i},\alpha,\beta}(\mathfrak{t}\ ;\ \alpha^1, \alpha^2, ..., \alpha^{\mathfrak{n}}) \cdot 1 = \tilde{\mathfrak{f}}_{\mathfrak{i},\beta}(\bar{\psi}_{\mathfrak{i},\alpha,\beta}(\mathfrak{t}\ ;\ \alpha^1, ..., \alpha^{\mathfrak{n}})) \\ D\lambda(\mathfrak{t}) \cdot 1 = -D_2\bar{\mathfrak{f}}^0_{\mathfrak{i},\beta}(\mathfrak{t}\ ;\ \alpha^1, ..., \alpha^{\mathfrak{n}}) - \lambda(\mathfrak{t}) \circ D_2\bar{\mathfrak{f}}_{\mathfrak{i},\beta}(\mathfrak{t}\ ;\ \alpha^1, ..., \alpha^{\mathfrak{n}}) \end{cases}$$

where

$$\bar{\mathfrak{f}}^0_{\mathfrak{i},\beta}(\mathfrak{t}\ ;\ \alpha^1, ..., \alpha^{\mathfrak{n}}) = \tilde{\mathfrak{f}}^0_{\mathfrak{i},\beta}(\bar{\psi}_{\mathfrak{i},\alpha,\beta}(\mathfrak{t}\ ;\ \alpha^1, ..., \alpha^{\mathfrak{n}}))$$

and

$$\bar{\mathfrak{f}}_{\mathfrak{i},\beta}(\mathfrak{t}\ ;\ \alpha^1, ..., \alpha^{\mathfrak{n}}) = \tilde{\mathfrak{f}}_{\mathfrak{i},\beta}(\bar{\psi}_{\mathfrak{i},\alpha,\beta}(\mathfrak{t}\ ;\ \alpha^1, ..., \alpha^{\mathfrak{n}}))$$

$\tilde{\mathfrak{f}}_{\mathfrak{i},\beta}$ being the principal part of the vector field $\tilde{\xi}_{\mathfrak{i}}$ with respect to the charts (V, β), $(T(V), T\beta)$ and $\tilde{\mathfrak{f}}^0_{\mathfrak{i},\beta}$ the local representative of the map $\tilde{\mathfrak{f}}^0_{\mathfrak{i}}$ with respect to the chat (V, β).

In particular we deduce that the map $\mathfrak{t} \to \lambda(\mathfrak{t})$ is of class C^1.

We may observe that the system (7.8) is the same with (6.7).

If the submanifold $\mathfrak{c}'$ is r-dimensional, $1 \leqslant r \leqslant \mathfrak{n}$ and $\alpha_{\mathfrak{x}} : T_{\mathfrak{x}}\mathfrak{c}' \to R^r$ is the toplinear isomorphism induced by the chart (U, α) between the tangent space $T_{\mathfrak{x}}\mathfrak{c}'$ and R^r and *if* $v \in T_{\mathfrak{x}}\mathfrak{c}'$, $\mathfrak{u} = \alpha_{\mathfrak{x}}v$ then the following relation may be immediately proved :

$$(7.9)\qquad \Phi_{\mathfrak{y}(\mathfrak{x})}(\mathfrak{t}) \cdot (T_{\mathfrak{x}}\bar{\psi}_{\mathfrak{i},\mathfrak{t}} \cdot v) = \lambda(\mathfrak{t}) \cdot D_2\bar{\psi}_{\mathfrak{i},\alpha,\beta}(\mathfrak{t}\ ;\ \alpha^1, ..., \alpha\ ) \cdot \mathfrak{u}$$

Therefore, we must prove that the map

$$\mathfrak{t} \mapsto \lambda(\mathfrak{t})D_2\bar{\psi}_{\mathfrak{i},\alpha,\beta}(\mathfrak{t}\ ;\ \alpha) \cdot \mathfrak{u} \qquad \text{is of class } C^1$$

Since the C^1-morphism $\bar{\psi}_{\mathfrak{i}}$ is defined by (5.10) with the flow $\tilde{\psi}$ of the C^1-vector field $\tilde{\xi}_{\mathfrak{i}}$ it follows ([14]) that the local representative $\bar{\psi}_{\mathfrak{i},\alpha,\beta}$ is a C^1-function with respect to the variables $\mathfrak{t}$ and α and its mixed second ordre partial derivatives exist and are equal, that is :

$$(7.10)\qquad D_1D_2\bar{\psi}_{\mathfrak{i},\alpha,\beta}(\mathfrak{t}, \alpha) \cdot \mathfrak{u} \cdot s = D_2D_1\bar{\psi}_{\mathfrak{i},\alpha,\beta}(\mathfrak{t}, \alpha) \cdot s \cdot \mathfrak{u}$$

for every $s \in R$ and $\mathfrak{u} \in R^r$

Since the maps $\mathfrak{t} \mapsto \lambda(\mathfrak{t})$, $\mathfrak{t} \mapsto D_2\bar{\psi}_{\mathfrak{i},\alpha,\beta}(\mathfrak{t}, \alpha)$ are of class C^1 it immediatly follows that the map $\mathfrak{t} \mapsto \lambda(\mathfrak{t}) \cdot D_2\bar{\psi}_{\mathfrak{i},\alpha,\beta}(\mathfrak{t}, \alpha) \cdot \mathfrak{u}$ is also of class C^1. Hence the first part of the lemma 7.1. is proved.

To prove the relation (7.7) we observe that from (7.9) and (7.10) it follows :

$$\frac{d}{dt}(\Phi_{\eta(x)}(t)\cdot(T_x\bar{\psi}_{i,t}\cdot v))=\frac{d}{dt}[\lambda(t)\cdot D_2\bar{\psi}_{i,\alpha,\beta}(t,\alpha)\cdot u]$$

$$=(D\lambda(t)\cdot 1)(D_2\bar{\psi}_{i,\alpha,\beta}(t,\alpha)\cdot u)+\lambda(t)\cdot D_1D_2\bar{\psi}_{i,\alpha,\beta}(t,\alpha)\cdot u\cdot 1$$

$$=(D\lambda(t)\cdot 1)(D_2\bar{\psi}_{i,\alpha,\beta}(t,\alpha)\cdot u)+\lambda(t)D_2D_1\bar{\psi}_{i,\alpha,\beta}(t,\alpha)\cdot 1\cdot u$$

From (7.8) we deduce :

$$D_2D_1\bar{\psi}_{i,\alpha,\beta}(t,\alpha)\cdot 1\cdot u=D_2\bar{f}_{i,\beta}(t,\alpha)\cdot u$$

where

$$\bar{f}_{i,\beta}(t,\alpha)=\tilde{f}_{i,\beta}(\bar{\psi}_{i,\alpha,\beta}(t,\alpha))$$

and :

$$D\lambda(t)\cdot 1=-D_2\bar{f}^0_{i,\beta}(t,\alpha)-\lambda(t)\cdot D\bar{f}_{i,\beta}(t,\alpha)$$

Therefore, we have :

$$\frac{d}{dt}(\Phi_{\eta(x)}(t)\cdot T_x\bar{\psi}_{i,t}\cdot v)=-D_2\bar{f}^0_{i,\alpha}(t,\alpha)\cdot u$$

Using the local representative of the tangent $T\bar{f}^0_{i,t}$ that is

$$D_2\bar{f}^0_{i,t}(t,\alpha)=T_x\bar{f}^0_{i,t}\circ\alpha_x^{-1}$$

and the fact that $\alpha_x^{-1}\cdot u=v$ we obtain the relation (7.7) and lemma (7.1) is proved.

To prove the relation (7.2) let us suppose the marked trajectory φ_x passes through the cells of type I $c_1, c_2, \ldots, c_q$ where $\Pi(c_q)\subset\mathfrak{F}$.

From (7.5) we have :

$$T_xW_{c'}\cdot v=T_{\chi_q(x)}g\cdot T_x\chi_q\cdot v+\sum_{i=1}^{q}T_xI_i\cdot v \tag{7.11}$$

for any $v\in T_xc'$, where $T_xI_i\cdot v$ is given by the formula (7.6).

To compute the integral from (7.6) we use the relation (7.7) and we obtain :

$$\int_{\tau_{i-1}(x)}^{\tau_i(x)}(T_x\bar{f}^0_{i,t}\cdot v)\,dt$$
$$=\Phi_{\eta(x)}(\tau_{i-1}(x)+0)\cdot T_x\bar{\psi}_{i,\tau_{i-1}(x)}\cdot v-\Phi_{\eta(x)}(\tau_i(x)-0)T_x\bar{\psi}_{i,\tau_i(x)}\cdot v$$

From (5.15) and (5.16) we have :

$$\int_{\tau-1(x)}^{\tau_i(x)}(T_x\bar{f}^0_{i,t}\cdot v)\,dt=\eta_i^+\cdot[-(T_x\tau_{i-1}\cdot v)\tilde{\xi}_i(\chi_{i-1}(x))+T_{\chi-1(x)}\mathfrak{I}_{(i-1)}$$
$$\cdot T_x\chi_{i-1}\cdot v]-\eta_i^-\cdot[-(T_x\tau_i\cdot v)\tilde{\xi}_i(\chi_i(x))+T_{\chi_i(x)}\mathfrak{I}_{(i)}\cdot T_x\chi_i\cdot v]$$

From (6.8) and (6.9) it follows that :

$$\mathfrak{y}_i^+ \cdot T_{\chi_i(\mathfrak{x})}\mathfrak{J}_{(i)}T_{\mathfrak{x}}\chi_i \cdot v - \mathfrak{y}_i^- \cdot T_{\chi_i(\mathfrak{x})}\mathfrak{J}_{(i)}T_{\mathfrak{x}}\chi_i \cdot v = 0 \qquad i = 1, 2, \ldots, q_{-1}$$

$$(T_{\mathfrak{x}}\tau_i \cdot v(\bar{\mathfrak{f}}_i^0(\tau_i(\mathfrak{x}), \mathfrak{x}) + \mathfrak{y}_i^- \cdot [(T_{\mathfrak{x}}\tau_i \cdot v)\bar{\xi}_i(\chi_i(\mathfrak{x}))]$$
$$= (T_{\mathfrak{x}}\tau_i \cdot v)[\bar{\mathfrak{f}}_i^0(\tau_i(\mathfrak{x}), \mathfrak{x}) + \mathfrak{y}_i^- \cdot \tilde{\xi}_i(\chi_i(\mathfrak{x}))] = 0, \qquad i = 1, 2, \ldots, q$$
$$-(T_{\mathfrak{x}}\tau_i \cdot v)\bar{\mathfrak{f}}_{i+1}^0(\tau_i(\mathfrak{x}), \mathfrak{x}) - \mathfrak{y}_i^+ \cdot [(T_{\mathfrak{x}}\tau_i \cdot v)\tilde{\xi}_{i+1}(\chi_i(\mathfrak{x}))]$$
$$= -(T_{\mathfrak{x}}\tau_i \cdot v)[\bar{\mathfrak{f}}_{i+1}^0(\tau_i(\mathfrak{x}), \mathfrak{x}) + \mathfrak{y}_i^+\tilde{\xi}_{i+1}(\chi_i(\mathfrak{x}))] = 0 \qquad i = 1, 2, \ldots, q_{-1}$$

$$T_{\chi_q(\mathfrak{x})}\mathfrak{g} \cdot T_{\mathfrak{x}}\chi_q \cdot v - \mathfrak{y}_q^- \cdot T_{\chi_q}\mathfrak{J}_{(q)} \cdot T_{\mathfrak{x}}\chi_q \cdot v =$$
$$= (T_{\chi_q(\mathfrak{x})}\mathfrak{g} - \mathfrak{y}_q^- \circ T_{\chi_q(\mathfrak{x})}\mathfrak{J}_{(q)}) \cdot T_{\mathfrak{x}}\chi_q \cdot v = 0$$

since $\Phi_{\mathfrak{y}(\mathfrak{x})}(\tau_0(\mathfrak{x})) = \Phi_{\mathfrak{y}(\mathfrak{x})}(0) = \mathfrak{y}(\mathfrak{x})$ and $T_{\mathfrak{x}}\bar{\psi}_{1,0} = T_{\mathfrak{x}}\,\mathfrak{i}_{c'}$ from (7.11) we obtain (7.2) and the proposition (7.1) is completely proved.

Theorem 7.1

If at every point $\mathfrak{x} \in X\backslash M$ *the following inequality holds :*

(7.12) $$T_{\mathfrak{x}}W \cdot \xi(\mathfrak{x}, \omega) + \mathfrak{f}^0(\mathfrak{x}, \omega) \geqslant 0 \qquad \textit{for any } \omega \in \Omega$$

then the marked trajectories are optimal.

Proof

Since $M = N \cup \left(\bigcup_{i=t-1}^{n-1} P^i \right)$ is a pricewise smooth set and $X\backslash M$ is an open submanifold (generally non connected) we may apply the lemma 3.4 and we deduce that for every admissible control $\mathfrak{u}_{\mathfrak{x}}$, the following inequality holds :

$$W(\mathfrak{x}) \leqslant P(\mathfrak{x}, \mathfrak{u}_{\mathfrak{x}}) = P(\mathfrak{x}, \varphi_{(t_0,\mathfrak{x})})$$

where $\varphi_{(t_0,\mathfrak{x})}$ is the admissible trajectory corresponding to the control $\mathfrak{u}_{\mathfrak{x}}$.

Since for the marked trajectory $\varphi_{\mathfrak{x}}$ (or for the admissible control $\bar{\mathfrak{u}}(t) = \mathfrak{v}(\varphi_{\mathfrak{x}}(t))$ we have :

$$W(\mathfrak{x}) = P(\mathfrak{x}, \bar{\mathfrak{u}}) = P(\mathfrak{x}, \varphi_{\mathfrak{x}})$$

it follows that the marked trajectories are optimal.

We define the map $H : T^*X \times \Omega \to R$ by :

(7.13) $$H(\mathfrak{y}, \omega) = \mathfrak{y} \cdot \xi(\mathfrak{x}, \omega) + \mathfrak{f}^0(\mathfrak{x}, \omega)$$

for $\mathfrak{y} \in T_{\mathfrak{x}}^*X$, $\omega \in \Omega$ and $\mathfrak{x} \in X$.

From (6.4) it is obvious that $H(\mathfrak{y}, \mathfrak{v}(T_X^*(\mathfrak{y}))) = \tilde{H}_c(\mathfrak{y})$ if $T_X^*(\mathfrak{y}) = \mathfrak{x} \in c$ where c is a cell of the admissible synthesis.

Therefore, from the Remark 6.1 it follows that $H(\Phi_{\mathfrak{y}(\mathfrak{x})}(\mathfrak{t}), \mathfrak{v}(\varphi_{\mathfrak{x}}(\mathfrak{t}) = 0)$ for $\mathfrak{t} \in [0, \tau_q(\mathfrak{x})]$ in particular $H(\mathfrak{y}(\mathfrak{x}), \mathfrak{v}(\mathfrak{x})) = 0$ for any $\mathfrak{x} \in X \backslash M$.

Theorem 7.2

If $\mathfrak{y}(\mathfrak{x}) \in T^*_{\mathfrak{x}}X$ is the functional defined in the proposition 6.1 and for every $\mathfrak{x} \in X \backslash M$ the following inequality holds :

$$(7.14) \qquad H(\mathfrak{y}(\mathfrak{x}), \omega) \geqslant H(\mathfrak{y}(\mathfrak{x}), v(\mathfrak{x})) = 0 \qquad \textit{for any } \omega \in \Omega$$

then the marked trajectories are optimal.

Proof

From (7.13), the inequality (7.14) becomes :

$$\mathfrak{y}(\mathfrak{x}) \cdot \xi(\mathfrak{x}, \omega) + \mathfrak{f}^0(\mathfrak{x}, \omega) \geqslant 0$$

and using (7.2) we obtain :

$$T_{\mathfrak{x}} W \cdot \xi(\mathfrak{x}, \omega) + \mathfrak{f}^0(\mathfrak{x}, \omega) \geqslant 0 \qquad \text{for any } \omega \in \Omega$$

The theorem follows from the theorem 7.1.

REMARK 7.1.

The condition (7.12) which may be written :

$$\min_{\omega \in \Omega} [T_{\mathfrak{x}} W \cdot \xi(\mathfrak{x}, \omega) + \mathfrak{f}^0(\mathfrak{x}, \omega)] = 0 \qquad \text{for every } \mathfrak{x} \in X \backslash M$$

represents the functional equation of the dynamic programming for the control system on a manifold.

The condition (7.14) may be also written as a special form of the maximum principle if we observe that it is contained in the stronger condition :

$$\min_{\omega \in \Omega} H(\Phi_{\mathfrak{y}(\mathfrak{x})}(\mathfrak{t}), \omega) = 0 \qquad \text{for every } \mathfrak{x} \in X \backslash M \text{ and for every } \mathfrak{t} \in [0, \mathfrak{t}_F]$$

REMARK 7.2.

Analogous results may be obtained for the « non autonomous » control system $S = (I, X, \Omega, \xi, \mathfrak{F}, \mathfrak{U}, P)$ (definition 3.2) where :

$I \subset R$ is an open interval, X a $\mathfrak{n}$-dimensional differentiable manifold; $\xi : I \times X \times \Omega \to T(X)$ a « nonautonomous parametrized C^1-vector field on X (that is, for every $\mathfrak{t} \in I$, $\omega \in \Omega$ the partial map $\xi_{\mathfrak{t},\omega} : X \to T(X)$ is a C^1-vector field on X, the partial map $\xi_{\mathfrak{t}} : X \times \Omega \to T(X)$ is C^1 and for every $\mathfrak{x} \in X$, the partial map $\xi_{(\mathfrak{x},\omega)} : I \to T(X)$ is continuous); $\mathfrak{F} \subset I \times X$ is a k-dimensional $(0 \leqslant k \leqslant \mathfrak{n} + 1)$ closed submanifold; $\mathfrak{U}$ and P are defined as in the definition (3.1) and the relation (3.2) where $\mathfrak{f}^0 : I \times R \times \Omega \to R$ is a C^1-function in $(\mathfrak{x}, \omega) \in X \times \Omega$ and a C^0-function in $\mathfrak{t} \in I$.

The changes in the analogous formulae (and in their proofs) appear due to the « nonautonomous » vector field ξ (5.2) which is C^1 in x and only continuous in t.

Since from classical theorems on differential equations ([14], [16]) and using « globalization » techniques from [11] we may deduce the existence and uniquness of the C^1-flow for this vector field, then, we may reduce the « nonautonomous » control system to the « autonomous » one by an evident change of the phase space.

In the « local case » (that is, the case when the phase space is an open subset of the Euclidian space) the nonautonomous » control system is studied in [3].

In this paper, the autonomous control system was preffered because of the simplicity of the notations and the formulae.

It is useful to remark that a control system with fixed time duration has to be considered as a « nonautonomous » control system even if the parametrized vector field is an « autonomous » one.

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