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DECOMPOSITION PRINCIPLE FOR LINEAR FRACTIONAL FUNCTIONAL PROGRAMS

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Résumé. — *Le « Principe de décomposition pour des programmes linéaires » est généralisé au cas des programmes fractionnaires. On démontre que pour résoudre un probleme compliqué de programmation ayant une fonction fractionnaire, on doit résoudre une série de problèmes de programmes linéaires qui sont beaucoup plus simples que le problème original. Ce principe peut être appliqué à quelques problèmes spéciaux de programmation fractionnelle.*

INTRODUCTION

From practical point of view, it becomes necessary to consider the large structured programming problems, but even if it is theoretically possible to solve this problem, in practice it is not always so. There are certain limitations which restrict the endeavours of the analyst. Chief among these limitations is the problem of dimensionality. This suggests the idea of developing methods of solution that should not use simultaneously all the data of the problem. One such approach is the decomposition principle due to Dantzig and Wolf [1] for Linear programs. This principle requires the solution of a series of linear programming problems of smaller size than the original problem. The problem considered in [1] has the following structure :

Maximize

$$Z = \sum_{j=1}^{r} \mathbf{C}_j' \mathbf{x}_j \tag{1}$$

Subject to

$$\sum_{j=1}^{r} A_j \mathbf{x}_j = \mathbf{b}_0 \tag{2}$$

$$B j \mathbf{x}_j = \mathbf{b}_j \qquad (j = 1, 2, \dots r) \tag{3}$$

$$\mathbf{x}_j \geqslant 0 \qquad (j = 1, 2, \dots r) \tag{4}$$

where (i) A_j and B_j are matrices of order $(m_0 \times n_j)$ and $(m_j \times n_j)$ respectively.

(ii) $\mathbf{X}_j$ and $\mathbf{C}_j$ are vectors having n_j components each.

(iii) $\mathbf{b}_0$ is an m_0 vector and $\mathbf{b}_j$ $(j = 1, 2, \dots, r)$ is an m_j vector.

Williams [2] gave a treatment of the transportation problem by Decomposition. The decomposition of Non linear programming problems was also considered by a number of authors, mainly, Rosen [3], Whinston [4], Variya [5] and others.

Here a large structured linear fractional functionals programming (L.F.F.P.) problem is considered with the constraints (2), (3) and (4), and it is shown that for solving this problem we have to solve a series of smaller sized linear programming problems. The mathematical formulation of the problem is as follows :

Maximize

$$Z = \frac{\left[\sum_{j=1}^{r} \mathbf{C}_j' \mathbf{x}_j + u\right]}{\left[\sum_{j=1}^{r} \mathbf{d}_j' \mathbf{x}_j + v\right]} \tag{5}$$

Subject to constraints (2) and (3) and (4), where $\mathbf{d}_j$ is a vector having n_j components and u & v are given scalars. Apart from the usual assumptions of L.F.F.P., it is assumed that the set S_j of all $\mathbf{X}_j \geqslant 0$ satisfying $B_j \mathbf{X}_j = \mathbf{b}_j$ is a convex polyhedron. We shall call this as Problem (A).

The paper is dealt into three sections. Section I considers the preliminaries and the main principle is developed in Section II. In Section III, its application is discussed in some special problems.

SECTION I

Preliminaries :

(i) Any point in a convex polyhedron can be written as a convex linear combination of the extreme points and conversely every such point belongs to the polyhedron [6].

(ii) For the L.F.F.P. problem

Maximize

$$Z = \frac{\mathbf{C}'\mathbf{x} + C_0}{\mathbf{d}'\mathbf{x} + d_0}$$

Subject to

$$A\mathbf{x} = \mathbf{b}$$
$$\mathbf{x} \geqslant 0$$

the optimality criterion given by Kantiswarup [7] is that :

$$\Delta_j = [V_2(C_j - Z_j^{(1)}) - V_1(d_j - z_j^{(2)})] \leqslant 0$$

for $j = 1, 2, ..., n$. Here A is a $(m \times n)$ matrix, $\mathbf{C}$ & $\mathbf{d}$ are vectors having n components each, C_0 and d_0 are given scalars. If B is the basis matrix, and $\mathbf{C}_B$ and $\mathbf{d}_B$ are cost vectors corresponding to this basis matrix, then :

$$V_1 = \mathbf{C}_B' B^{-1}\mathbf{b} + C_0 ; \qquad V_2 = \mathbf{d}_B' B^{-1}\mathbf{b} + d_0$$
$$\mathbf{Z}_j^{(1)} = \mathbf{C}_B'\mathbf{y}_j ; \qquad \mathbf{Z}_j^{(2)} = \mathbf{d}_B'\mathbf{y}_j ; \qquad \mathbf{y}_j = B^{-1}\mathbf{a}_j$$
$$A = [\mathbf{a}_1, \mathbf{a}_2, ..., \mathbf{a}_n]$$

SECTION II

Let $\mathbf{x}_{kj}^0$ denote the extreme points of the convex set of feasible solutions to $B_j\mathbf{x}_j = \mathbf{b}_j, \mathbf{x}_j \geqslant 0$ $(j = 1, 2, ... r)$. Since the set S_j is a convex polyhedron, we have $\mathbf{x}_j \in S_j$ if and only if there exist γ_{kj} such that

$$\mathbf{x}_j = \sum_{k=1}^{h_j} \gamma_{kj}\mathbf{x}_{kj}^0$$
$$\gamma_{kj} \geqslant 0 \qquad (k = 1, 2, ..., h_j)$$
$$\sum_{k=1}^{h_j} \gamma_{kj} = 1 \tag{6}$$

In view of (5), the problem (A) is equivalent to the following problem :

Maximize

$$Z = \frac{\left[\sum_{j=1}^{r} \sum_{k=1}^{h_j} \gamma_{kj}\mathbf{C}_j'\mathbf{x}_{kj}^0 + u\right]}{\left[\sum_{j=1}^{r} \sum_{k=1}^{h_j} \gamma_{kj}\mathbf{d}_j'\mathbf{x}_{kj}^0 + v\right]}$$

Subject to,

$$\sum_{j=1}^{r} \sum_{k=1}^{h_j} \gamma_{kj} A_j \mathbf{x}_{kj}^0 = \mathbf{b}_0$$

$$\sum_{k=1}^{h_j} \gamma_{kj} = 1, \qquad \gamma_{kj} \geqslant 0 \tag{7}$$

Let us write :

$$\mathbf{p}_{kj} = A_j \mathbf{x}_{kj}^0 ; \qquad f_{kj}^{(1)} = \mathbf{C}_j' \mathbf{x}_{kj}^0$$

$$f_{kj}^{(2)} = \mathbf{d}_j' \mathbf{x}_{kj}^0 \qquad (j = 1, 2, \dots r ; k = 1, 2, \dots h_j)$$

$$\mathbf{q}_{kj} = (\mathbf{p}_{kj}, \mathbf{e}_j) ; \qquad \mathbf{b} = (\mathbf{b}_0, \mathbf{1}')$$

Where e_j is the j^{th} unit vector having r – components and $1'$ is a r – component vector, each component being 1. Then the above problem becomes :

Maximize

$$Z = \frac{\left[\sum_{j=1}^{r} \sum_{k=1}^{h_j} f_{kj}^{(1)} \gamma_{kj} + u\right]}{\left[\sum_{j=1}^{r} \sum_{k=1}^{h_j} f_{kj}^{(2)} \gamma_{kj} + v\right]}$$

Subject to,

$$\sum_{j=1}^{r} \sum_{k=1}^{h_j} \gamma_{kj} \mathbf{q}_{kj} = \mathbf{b}$$

$$\gamma_{kj} \geqslant 0$$

We call this as problem (B). It is to be noted that problem (B) has $(m_0 + r)$ constraints while problem (A) has $\sum_{j=0}^{r} m_j$ constraints, thus in general requiring a smaller basis than problem (A). We shall show that it is not necessary to generate every extreme point, before the problem is solved. $\mathbf{p}_{kj}$, $f_{kj}^{(1)}$ and $f_{kj}^{(2)}$ can be generated, as they are used.

We now apply result (ii) of Section I to the problem (B). Let B is any basis matrix (of order $(m_0 + r)$) for problem (B) and $\boldsymbol{\gamma}_B = B^{-1}\mathbf{b}$ is the basic feasible solution. Let :

$$\boldsymbol{\sigma}^{(1)} = (\boldsymbol{\sigma}_1^{(1)}, \boldsymbol{\sigma}_2^{(1)}) = (\mathbf{f}_B^{(1)})' B^{-1}$$

and

$$\boldsymbol{\sigma}^{(2)} = (\boldsymbol{\sigma}_1^{(2)}, \boldsymbol{\sigma}_2^{(2)}) = (\mathbf{f}_B^{(2)})' B^{-1} \qquad \text{where} \qquad \boldsymbol{\sigma}_1^{(1)}$$

contains the first m_0 components of $\boldsymbol{\sigma}^{(1)}$ and $\boldsymbol{\sigma}_1^{(2)}$ contains the last r components of $\boldsymbol{\sigma}^{(1)}$. Similarly $\boldsymbol{\sigma}^{(2)}$ is partitioned. Then

$$\Delta_{kj} = [V_2(f_{kj}^{(1)} - Z_{kj}^{(1)}) - V_1(f_{kj}^{(2)} - Z_{kj}^{(2)})] \tag{8}$$

but as in [7]

$$(f_{kj}^{(1)} - Z_{kj}^{(1)}) = (\mathbf{C}_j' - \boldsymbol{\sigma}_1^{(1)}A_j)\mathbf{x}_{kj}^0 - \sigma_{2j}^{(1)}$$

where $\sigma_{2j}^{(1)}$ is the j^{th} component of $\boldsymbol{\sigma}_2^{(1)}$. Similarly

$$(f_{kj}^{(2)} - Z_{kj}^{(2)}) = (\mathbf{d}_j' - \boldsymbol{\sigma}_1^{(2)}A_j)\mathbf{x}_{kj}^0 - \sigma_{2j}^{(2)}$$

Hence from (8) :

$$\Delta_{kj} = [V_2(\mathbf{C}_j' - \boldsymbol{\sigma}_1^{(1)}A_j) - V_1(\mathbf{d}_j' - \boldsymbol{\sigma}_1^{(2)}A_j)]\mathbf{x}_{kj}^0 + [V_1\sigma_{2j}^{(2)} - V_2\sigma_{2j}^{(1)}] \tag{9}$$

where V_1 and V_2 denote respectively the values of numerator and denominator of the objective function of problem (B) at the basic feasible solution $\boldsymbol{\gamma}_B = B^{-1}b$.

Now $\underset{\text{all } k \text{ \& } j}{\text{Max}} \Delta_{kj} \leqslant 0$, then the solution under consideration is optimal, otherwise more iterations are required, but

$$\underset{k \text{ \& } j}{\text{Max}}\ \Delta_{kj} = \text{Max}\ [\underset{k}{\text{Max}}\ \Delta_{k1}, \ldots, \underset{k}{\text{Max}}\ \Delta_{kr}] \tag{10}$$

From (9), for a given j, $\underset{k}{\text{Max}}\ \Delta_{kj}$ occurs at an extreme point of S_j. Let $Z_j = [V_2(\mathbf{C}_j' - \boldsymbol{\sigma}_1^{(1)}A_j) - V_1(\mathbf{d}_j' - \boldsymbol{\sigma}_1^{(2)}A_j)]\mathbf{x}_j$.

Since each extreme point $\mathbf{x}_{kj}^0$ is a basic feasible solution to $B_j\mathbf{x}_j = \mathbf{b}_j$ we have :

$$\underset{k}{\text{Max}}\ \Delta_{kj} = [V_1\sigma_{2j}^{(2)} - V_2\sigma_{2j}^{(1)}] + Z_j^0$$

where Z_j^0 is the optimal value of the j^{th} ($j = 1, 2, \ldots, r$) problem :

Maximize Z_j

Subject to

$$\begin{aligned} B_j\mathbf{x}_j &= \mathbf{b}_j \\ \mathbf{x}_j &\geqslant 0 \end{aligned} \tag{11}$$

Moreover, an optimal basic solution to (11) gives an extreme point $\mathbf{x}_{kj}^0$ for which the corresponding Δk_j is maximum over k. Knowing this $\mathbf{x}_{kj}^0$ the corresponding $\mathbf{p}_{kj}$, $f_{kj}^{(1)}$ & $f_{kj}^{(2)}$ can be generated accordingly. Now

$$\begin{aligned} \underset{\text{all } k \text{ \& } j}{\text{Max}}\ \Delta_{kj} &= \underset{j}{\text{Max}}\ [Z_j^0 + V_1\sigma_{2j}^{(2)} - V_2\sigma_{2j}^{(1)}] \\ &= Z_s^0 + V_1\sigma_{2s}^{(2)} - V_2\sigma_{2s}^{(1)} \end{aligned} \tag{12}$$

where the max is taken, say, for $j = S$.

If $\mathbf{x}_{rs}^0$ is an optimal extreme point of (11) for $j = S$, and

$$\mathbf{p}_{rs} = A_s\mathbf{x}_{rs}^0,\ \mathbf{q}_{rs} = (\mathbf{p}_{rs}, \mathbf{e}_s)$$

then $\mathbf{q}_{rs}$ enters the basis at the next iteration. The vector to leave the basis is determined in the usual way.

Next the problem (B) is transformed to get the new values of B^{-1}, $\boldsymbol{\sigma}^{(1)}, \boldsymbol{\sigma}^{(2)}$ and $\boldsymbol{\gamma}_B$. As before the new values of $\boldsymbol{\sigma}^{(1)}$ and $\boldsymbol{\sigma}^{(2)}$ are used to determine a new set of r linear programming problems. By solving theses r new linear programming problems, we determine the vector to enter the basis as before. In the absence of degeneracy (of course degeneracy can be handled in the usual way), the above process terminates in a finite number of steps by the theory of simplex method for L.F.F.P.

SECTION III

As in case of linear programming [6], [8] the technique developed in Section II, can be applied to the following L.F.F.P. Problems :

I. — Generalized L.F.F.P. Problem

$$\text{Max } Z = \frac{\left[\sum_{j=1}^{r} c_j x_j + c_0\right]}{\left[\sum_{j=1}^{r} d_j x_j + d_0\right]} \tag{12}$$

Subject to

$$\sum_{j=1}^{r} x_j \mathbf{a}_j = \mathbf{b}$$
$$x_j \geqslant 0$$

where the vectors $\mathbf{a}_j$ are not specified but they are restricted such that $B_j \mathbf{a}_j \leqslant \mathbf{b}_j$ $(j = 1, 2, ..., r)$. It is assumed that these inequalities determine a convex polyhedron ([6]).

II. — L.F.F.P. with upper bonds on the variables

Maximize

$$Z = \frac{[\mathbf{c}_1'\mathbf{x}_1 + \mathbf{c}_2'\mathbf{x}_2 + u]}{[\mathbf{d}_1'\mathbf{x}_1 + \mathbf{d}_2'\mathbf{x}_2 + v]}$$

Subject to

$$A_1\mathbf{x}_1 + A_2\mathbf{x}_2 = b$$
$$I\mathbf{x}_1 \leqslant \mathbf{h}$$
$$\mathbf{x}_1, \mathbf{x}_2 \geqslant 0 \tag{13}$$

where A_1, A_2 are matrices of order $(m \times n_1)$ and $(m \times n_2)$ respectively and I is a $(n_1 \times n_1)$ identity matrix. This problem can be solved by using the technique of Section II, as in [8].

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