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## More on inhomogeneous Diophantine approximation

par CHRISTOPHER G. PINNER

RÉSUMÉ. Pour un nombre irrationnel  $\alpha$  et un nombre réel  $\gamma$ , on considère la constante d'approximation non-homogène

$$M(\alpha, \gamma) := \liminf_{|n| \rightarrow \infty} |n| |n\alpha - \gamma|$$

en rapport avec le développement en fraction continue négatif semi-régulier de  $\alpha$

$$\alpha = \frac{1}{a_1 - \frac{1}{a_2 - \frac{1}{a_3 - \dots}}}$$

et un  $\alpha$ -développement adéquat de  $\gamma$ . Nous donnons une majoration de

$$\rho(\alpha) := \sup_{\gamma \notin \mathbf{Z} + \alpha\mathbf{Z}} M(\alpha, \gamma),$$

dans le cas où  $\alpha$  est mal approximé, qui s'avère fine lorsque les quotients partiels  $a_i$  sont presque tous pairs et supérieurs ou égaux à 4. Lorsque le développement de  $\alpha$  est de période 1, on décrit entièrement le spectre des valeurs prises par

$$\mathbf{L}(\alpha) := \{M(\alpha, \gamma) : \gamma \notin \mathbf{Z} + \alpha\mathbf{Z}\},$$

au-dessus du premier point d'accumulation.

ABSTRACT. For an irrational real number  $\alpha$  and real number  $\gamma$  we consider the inhomogeneous approximation constant

$$M(\alpha, \gamma) := \liminf_{|n| \rightarrow \infty} |n| |n\alpha - \gamma|$$

via the semi-regular *negative continued fraction expansion* of  $\alpha$

$$\alpha = \frac{1}{a_1 - \frac{1}{a_2 - \frac{1}{a_3 - \dots}}}$$

and an appropriate alpha-expansion of  $\gamma$ . We give an upper bound on the case of worst inhomogeneous approximation,

$$\rho(\alpha) := \sup_{\gamma \notin \mathbf{Z} + \alpha\mathbf{Z}} M(\alpha, \gamma),$$

which is sharp when the partial quotients  $a_i$  are almost all even and at least four. When the negative expansion has period one we give a complete description of the spectrum of values

$$\mathbf{L}(\alpha) := \{M(\alpha, \gamma) : \gamma \notin \mathbf{Z} + \alpha\mathbf{Z}\},$$

above the first limit point.

## 1. Introduction

For a fixed irrational, real number  $\alpha$  and real  $\gamma$  in  $[0, 1)$  one defines the *two-sided inhomogeneous approximation constant*

$$M(\alpha, \gamma) := \liminf_{|n| \rightarrow \infty} |n| \|n\alpha - \gamma\|,$$

where  $\|x\|$  denotes the distance from  $x$  to the nearest integer. The homogeneous case  $\gamma = 0$  is of course classical. Here we shall think of  $\alpha$  as fixed and  $\gamma$  varying to obtain an inhomogeneous spectrum of values for  $\alpha$

$$\mathbf{L}(\alpha) := \{M(\alpha, \gamma) : \gamma \notin \mathbf{Z} + \alpha\mathbf{Z}\}.$$

We shall say that  $\gamma$  and  $\gamma'$  are equivalent (with respect to  $\alpha$ ), denoted  $\gamma \sim \gamma'$ , if  $\gamma = \gamma' + n\alpha + m$  for some integers  $n, m$ , where clearly  $\gamma \sim \pm \gamma'$  implies that  $M(\alpha, \gamma) = M(\alpha, \gamma')$ . Historically there has been most interest in the case of worst inhomogeneous approximation

$$\rho(\alpha) := \sup_{\gamma \neq 0} M(\alpha, \gamma),$$

particularly for quadratic  $\alpha$ . It is conjectured that for quadratic  $\alpha$  the value of  $\rho(\alpha)$  should always be isolated (this would follow from a quadratic forms conjecture of Barnes–Swinnerton-Dyer [1], and may well be equivalent to it). In our previous paper [5] we approached the computation of  $M(\alpha, \gamma)$  via the regular continued fraction expansion of  $\alpha$ , verifying the isolation of  $\rho(\alpha)$  when the regular expansion had period one or two, or the period all even partial quotients. We show here how to alternatively use the *negative continued fraction expansion*. The formulae and bounds obtained this way are similar but simpler to work with (the absence of a sign alternation making the expressions more symmetric). We are thus able to show the isolation of  $\rho(\alpha)$  for additional classes of quadratic  $\alpha$  having straightforward negative expansions. For example when the partial quotients are all even and at least four we explicitly give the  $\gamma$  achieving  $\rho(\alpha)$  (see Theorem 2). In Section 2 we give a complete description of the spectrum above the first limit point when the negative expansion of  $\alpha$  has period one (the structure is similar to that of the traditional Lagrange spectrum). As an added advantage the use of the negative expansion leads naturally to a separate

consideration of the positive and negative integers, and hence to formulae for the one-sided approximation constants;

$$M_+(\alpha, \gamma) := \liminf_{n \rightarrow \infty} n \|n\alpha - \gamma\|, \quad M_-(\alpha, \gamma) := \liminf_{n \rightarrow -\infty} |n| \|n\alpha - \gamma\|.$$

By the negative expansion we mean that

$$\alpha = \frac{1}{a_1 - \frac{1}{a_2 - \frac{1}{a_3 - \dots}}} =: [0; a_1, a_2, a_3, \dots]^-,$$

where the integers  $a_i \geq 2$  are generated by rounding up rather than rounding down in the continued fraction algorithm:

$$\alpha_0 := \{\alpha\}, \quad a_{n+1} := \left\lceil \frac{1}{\alpha_n} \right\rceil, \quad \alpha_{n+1} := \left\lceil \frac{1}{\alpha_n} \right\rceil - \frac{1}{\alpha_n},$$

with corresponding convergents  $p_n/q_n = [0; a_1, \dots, a_n]^-$  given by

$$p_{n+1} := a_{n+1}p_n - p_{n-1}, \quad p_0 = 0, \quad p_{-1} = -1, \\ q_{n+1} := a_{n+1}q_n - q_{n-1}, \quad q_0 = 1, \quad q_{-1} = 0.$$

The negative expansion  $[0; a_1, a_2, a_3, \dots]^-$  can of course be thought of as a regular expansion where the partial quotients are alternately positive and negative integers. Using van der Poorten style identities for dealing with illegal partial quotients,

$$[\dots, a, b, c, d, \dots] = [\dots, a, 0, -1, 1, -1, 0, -b, -c, \dots] \\ = [\dots, a, 0, 1, -1, 1, 0, -b, -c, \dots],$$

and  $[\dots, a, 0, b, \dots] = [\dots, a + b, \dots]$ , to write

$$[\dots, a, b, c, d, e, \dots] = [\dots, a + 1, -1, -b + 1, -c, -d, -e, \dots] \\ = [\dots, a + 1, \underbrace{-2, 2, \dots, (-2)^{b-1}}_{b-1}, (-1)^b(c + 1), (-1)^b d, (-1)^b e, \dots], \text{ etc.},$$

it is straightforward to switch between regular and negative expansions:

$$[0; a'_1, a'_2, a'_3, a'_4, a'_5, a'_6, a'_7, \dots] \\ = [0; a'_1 + 1, \underbrace{2, \dots, 2}_{a'_2-1}, a'_3 + 2, \underbrace{2, \dots, 2}_{a'_4-1}, a'_5 + 2, \underbrace{2, \dots, 2}_{a'_6-1}, a'_7 + 2, \dots]^-.$$

Writing

$$\bar{\alpha}_i := [0; a_i, a_{i-1}, \dots, a_1]^- , \quad \alpha_i = [0; a_{i+1}, a_{i+2}, \dots]^- ,$$

it is readily seen that

$$D_i := q_i \alpha - p_i = \alpha_0 \cdots \alpha_i, \quad q_i = (\bar{\alpha}_1 \cdots \bar{\alpha}_i)^{-1}.$$

For a real number  $\gamma < 1$  we generate the coefficients  $b_i$  in the *alpha-expansion of  $\gamma$*  by taking

$$\gamma_0 := \{\gamma\}, \quad b_{n+1} := \left\lfloor \frac{\gamma_n}{\alpha_n} \right\rfloor, \quad \gamma_{n+1} := \left\{ \frac{\gamma_n}{\alpha_n} \right\},$$

so that

$$(1.1) \quad \{\gamma\} = \sum_{i=1}^n b_i D_{i-1} + \gamma_n D_{n-1} = \sum_{i=1}^{\infty} b_i D_{i-1}$$

gives the unique expansion of  $\gamma$  of the form  $\sum_{i=1}^{\infty} b_i D_{i-1}$  such that

- (i)  $0 \leq b_i \leq a_i - 1$ ,
- (ii) the sequence of  $b_i$  does not contain a block of the form  $b_t = a_t - 1$ , with  $b_j = a_j - 2$  for all  $j > t$  or with  $b_k = a_k - 1$  for some  $k > t$  and  $b_j = a_j - 2$  for any  $t < j < k$ .

We define the integers  $Q_k = Q_k(\gamma, \alpha)$  by

$$Q_k := \sum_{i=1}^k b_i q_{i-1},$$

and parameters  $\xi_k := Q_k/q_k$  so that

$$Q_k = \xi_k q_k, \quad \|Q_k \alpha - \gamma\| = \gamma_k D_{k-1},$$

with

$$0 \leq \xi_k, \gamma_k \leq 1.$$

We set

$$\lambda(n) = \lambda(n; \alpha, \gamma) := |n| \|n\alpha - \gamma\|.$$

In evaluating  $M_+(\alpha, \gamma)$  we shall frequently encounter

$$\begin{aligned} \lambda(Q_k) &= \xi_k \gamma_k q_k D_{k-1}, \\ \lambda(Q_k + q_{k-1}) &= (\xi_k + \bar{\alpha}_k)(1 - \gamma_k) q_k D_{k-1}, \end{aligned}$$

and for  $M_-(\alpha, \gamma)$

$$\begin{aligned} \lambda(Q_k - (q_k - q_{k-1})) &= |1 - \bar{\alpha}_k - \xi_k| |1 - \alpha_k - \gamma_k| q_k D_{k-1}, \\ \lambda(Q_k - q_k) &= (1 - \xi_k)(\alpha_k + \gamma_k) q_k D_{k-1}. \end{aligned}$$

To obtain more symmetrical expressions for these four functions it is often convenient to replace the  $b_k$  by the sequence of integers  $t_k$ , where

$$b_k = \frac{1}{2}(a_k - 2 + t_k),$$

and to define

$$d_k^- := \sum_{1 \leq j \leq k} t_j (q_{j-1}/q_k) = t_k \bar{\alpha}_k + t_{k-1} \bar{\alpha}_k \bar{\alpha}_{k-1} + t_{k-2} \bar{\alpha}_k \bar{\alpha}_{k-1} \bar{\alpha}_{k-2} + \dots,$$

$$d_k^+ := \sum_{j > k} t_j \left( \frac{D_{j-1}}{D_{k-1}} \right) = t_{k+1} \alpha_k + t_{k+2} \alpha_k \alpha_{k+1} + t_{k+3} \alpha_k \alpha_{k+1} \alpha_{k+2} + \dots,$$

(use of the negative expansion avoiding a sign alternation in  $d_k^-$ ). Hence

$$\xi_k = \frac{1}{2} \left( 1 - \bar{\alpha}_k + d_k^- - \frac{1}{q_k} \right), \quad \gamma_k = \frac{1}{2} (1 - \alpha_k + d_k^+),$$

and as  $k \rightarrow \infty$  (and  $D_{k-1} \rightarrow 0$ ) we can replace  $\lambda(Q_k)$ ,  $\lambda(Q_k + q_{k-1})$ ,  $\lambda(Q_k - (q_k - q_{k-1}))$  and  $\lambda(Q_k - q_k)$  by

$$\begin{aligned} s_1(k) &:= \frac{1}{4} (1 - \bar{\alpha}_k + d_k^-) (1 - \alpha_k + d_k^+) q_k D_{k-1}, \\ s_2(k) &:= \frac{1}{4} (1 + \bar{\alpha}_k + d_k^-) (1 + \alpha_k - d_k^+) q_k D_{k-1}, \\ s_3(k) &:= \frac{1}{4} |1 - \bar{\alpha}_k - d_k^-| |1 - \alpha_k - d_k^+| q_k D_{k-1}, \\ s_4(k) &:= \frac{1}{4} (1 + \bar{\alpha}_k - d_k^-) (1 + \alpha_k + d_k^+) q_k D_{k-1}, \end{aligned}$$

where

$$q_k D_{k-1} = \frac{1}{1 - \alpha_k \bar{\alpha}_k}.$$

Of course the  $t_k$  are integers with the same parity as  $a_k$  and  $-(a_k - 2) \leq t_k \leq a_k$ . We observe that

$$-(1 - \bar{\alpha}_k) \leq d_k^- \leq (1 + \bar{\alpha}_k), \quad -(1 - \alpha_k) \leq d_k^+ \leq (1 + \alpha_k),$$

with  $d_k^- \geq 1 - \bar{\alpha}_k$  (respectively  $d_k^+ \geq 1 - \alpha_k$ ) iff the sequence  $t_k, t_{k-1}, \dots$  (respectively  $t_{k+1}, t_{k+2}, \dots$ ) takes the form  $t_i = a_i$  with  $t_j = a_j - 2$  for any preceding  $t_j$ . Notice that if  $t_i \neq a_i$  then the expansion of  $1 - \alpha - \gamma$  is obtained by simply changing the signs of the  $t_i$ , where  $M_-(\alpha, \gamma) = M_+(\alpha, 1 - \alpha - \gamma)$ , the sign change merely interchanging  $s_1(k), s_2(k)$  with  $s_3(k), s_4(k)$ .

**Theorem 1.** For  $\gamma \neq 0$

$$\begin{aligned} M_+(\alpha, \gamma) &= \liminf_{k \rightarrow \infty} \min\{\lambda(Q_k), \lambda(Q_k + q_{k-1})\} \\ &= \liminf_{k \rightarrow \infty} \min\{s_1(k), s_2(k)\}. \end{aligned}$$

If the alpha-expansion of  $\gamma$  has  $b_i = a_i - 1$  at most finitely many times then,

$$\begin{aligned} M_-(\alpha, \gamma) &= \liminf_{k \rightarrow \infty} \min\{\lambda(Q_k - (q_k - q_{k-1})), \lambda(Q_k - q_k)\} \\ &= \liminf_{k \rightarrow \infty} \min\{s_3(k), s_4(k)\}, \end{aligned}$$

and

$$(1.2) \quad M(\alpha, \gamma) = \liminf_{k \rightarrow \infty} \min\{s_1(k), s_2(k), s_3(k), s_4(k)\}.$$

We readily deduce the following bound on  $\rho(\alpha)$ ;

**Corollary 1.** *For  $\gamma \neq 0$*

$$M(\alpha, \gamma) \leq \rho^*(\alpha) := \frac{1}{4} \liminf_{k \rightarrow \infty} \frac{\max\{\alpha_k, \bar{\alpha}_k, (1 - \alpha_k)(1 - \bar{\alpha}_k)\}}{(1 - \alpha_k \bar{\alpha}_k)}.$$

*In particular if  $\liminf_{i \rightarrow \infty} a_i = R \geq 3$ , then*

$$(1.3) \quad M(\alpha, \gamma) \leq \frac{1}{4} \left(1 - \frac{1}{R}\right).$$

If the  $a_i \geq 3$  for almost all  $i$  then, since  $\alpha_k, \bar{\alpha}_k \leq [0; \bar{3}]^- + o(1)$ ,

$$(1.4) \quad \rho^*(\alpha) = \frac{1}{4} \liminf_{k \rightarrow \infty} \frac{(1 - \alpha_k)(1 - \bar{\alpha}_k)}{(1 - \alpha_k \bar{\alpha}_k)}.$$

When the  $a_i$  are all even and at least four we can achieve this bound by simply taking the  $t_i = 0$ :

**Theorem 2.** *Suppose that the negative expansion of  $\alpha$  has  $a_i$  even for  $i \geq N$ . Then*

$$\gamma^* = \sum_{i=N}^{\infty} \frac{1}{2}(a_i - 2)D_{i-1} = \frac{1}{2}(D_{N-2} - D_{N-1})$$

has

$$M(\alpha, \gamma^*) = \frac{1}{4} \liminf_{k \rightarrow \infty} \frac{(1 - \alpha_k)(1 - \bar{\alpha}_k)}{(1 - \alpha_k \bar{\alpha}_k)}.$$

*In particular if the  $a_i \geq 4$  we have  $\rho(\alpha) = M(\alpha, \gamma^*) = \rho^*(\alpha)$ . Moreover if  $\alpha$  is also quadratic,*

$$(1.5) \quad \alpha = [0; a_1, \dots, a_{N-1}, \overline{a_N, \dots, a_{N+r-1}}]^- , \quad 4 \leq a_{N+i} \leq A,$$

*then the value of  $\rho(\alpha)$  is isolated with*

$$M(\alpha, \gamma) \leq \left(1 - A^{-2 \lceil \frac{r+1}{2} \rceil}\right)^{1/2} \rho(\alpha)$$

*for  $\gamma \neq 0, \gamma^*$ .*

We note that the simplified bound (1.4) need not hold when  $a_i = 2$  infinitely often (so that the condition  $a_i \geq 4$  is needed here). For example if for  $i \geq 0$  the  $a_{N+2i} = 2$  with  $a_{N+2i+1} \geq 4$  even, then

$$(1.6) \quad \rho(\alpha) = M(\alpha, \gamma^{**}) = \frac{1}{4} \liminf_{i \rightarrow \infty} \frac{1}{2 - \bar{\alpha}_{N+2i-1} - \alpha_{N+2i}} = \rho^*(\alpha)$$

is larger than  $M(\alpha, \gamma^*)$ , where  $\gamma^{**} := D_{N-2} - \frac{1}{2}D_{N-1}$  corresponds to taking  $t_{N+2i} = a_{N+2i}$ ,  $t_{N+2i+1} = -2$ . Theorem 2 also shows that bound (1.3) can not be improved when  $R$  is even (consider period  $R, 2A$  with  $A \rightarrow \infty$ ).

Finally, the following bound (useful in the explicit computations of Section 2) shows that large  $|t_i|$  produce small values for  $M(\alpha, \gamma)$ .

**Lemma 1.** *Suppose that  $\gamma \not\sim 0$ . If  $t_k = a_k$  infinitely often then*

$$(1.7) \quad M(\alpha, \gamma) \leq \frac{1}{4} \liminf_{\substack{k \rightarrow \infty \\ t_k = a_k}} \frac{\bar{\alpha}_k}{(1 - \alpha_k \bar{\alpha}_k)},$$

otherwise

$$(1.8) \quad M(\alpha, \gamma) \leq \frac{1}{4} \liminf_{k \rightarrow \infty} \frac{(a_k - |t_k|)\bar{\alpha}_k}{(1 - \alpha_k \bar{\alpha}_k)}.$$

### 2. Period One $\alpha$

We suppose that  $\alpha$  has a period one expansion

$$(2.5) \quad \alpha = [0; a_1, \dots, a_N, \bar{a}]^-, \quad a \geq 4,$$

and set

$$\theta := [0; \bar{a}]^- = \frac{1}{2}(a - \sqrt{a^2 - 4}).$$

From Theorem 1 we can write

$$M(\alpha, \gamma) = \frac{1}{4} M^*(\alpha, \gamma) / (1 - \theta^2)$$

and evaluate  $M^*(\alpha, \gamma)$  using the liminf of the slightly simpler functions

$$\begin{aligned} s_1^*(k) &= (1 - \theta + d_k^-)(1 - \theta + d_k^+), \\ s_2^*(k) &= (1 + \theta + d_k^-)(1 + \theta - d_k^+), \\ s_3^*(k) &= |1 - \theta - d_k^-| |1 - \theta - d_k^+|, \\ s_4^*(k) &= (1 + \theta - d_k^-)(1 + \theta + d_k^+), \end{aligned}$$

with

$$d_k^+ := t_{k+1}\theta + t_{k+2}\theta^2 + \dots, \quad d_k^- := t_k\theta + t_{k-1}\theta^2 + \dots.$$

We define sets of  $\gamma = \sum_{i=1}^{\infty} \frac{1}{2}(a_i - 2 + t_i)D_{i-1}$  whose sequences  $t_i$  are eventually periodic:

When  $a$  is odd define

$$\begin{aligned} S_0 &:= \{\gamma : t_i \text{ periodic, period } (-1, 1)\}, \\ S_{-3} &:= \{\gamma : t_i \text{ periodic, period } (-3, 3)\}, \\ S_{-2} &:= \{\gamma : t_i \text{ periodic, period } (-1, -1, 1, 1)\}, \\ S_{-1} &:= \{\gamma : t_i \text{ periodic, period } (-1, 1, -1, 1, -1, 1, 1)\}, \\ S_k &:= \{\gamma : t_i \text{ periodic, period } (-1, 1, -1, -1, 1, 1, 1)^k\}, \quad k \geq 1, \end{aligned}$$

and when  $a$  is even

$$\begin{aligned} S_0 &:= \{\gamma : t_i \text{ periodic, period } 0\}, \\ S_{-1} &:= \{\gamma : t_i \text{ periodic, period } (-2, 2)\}, \\ S_k &:= \{\gamma : t_i \text{ periodic, period } 0, (-2, 2, 1)^k\}, \quad k \geq 1. \end{aligned}$$



When  $a = 4$  (as for  $\alpha = \sqrt{3}$ ) we interestingly obtain a second sequences of  $\gamma$  with values also tending to the first limit point;

$$S_{-2} := \{\gamma : t_i \text{ periodic, period } (a, -2)\},$$

$$S_{-k-2} := \{\gamma : t_i \text{ periodic, period } (a, -2), (2, -2)^k\}$$

We set

$$\delta_k := M(\alpha, \gamma), \quad \gamma \in S_k.$$

**Theorem 3.** *Suppose that  $\alpha = [0; a_1, \dots, a_N, \bar{a}]^-$  with  $a \geq 4$ . Then  $\rho(\alpha) = \delta_0$ .*

*When  $a \geq 5$  is odd the values of  $M(\alpha, \gamma)$ ,  $\gamma \neq 0$ , greater than*

$$\delta_\infty := \frac{1}{4} \frac{(1 - 2\theta + \theta^2)^2 - \frac{\theta^6(1-\theta)^2}{(1+\theta^2)^2}}{(1 - \theta^2)}$$

$$= \frac{1}{4} \frac{(a - 2)}{\sqrt{a^2 - 4}} \left(1 - \frac{1}{a}\right)^2 \left(1 - \frac{8}{(a - 1)(a + \sqrt{a^2 - 4})^2}\right),$$

are given by

$$\delta_0 = \frac{1}{4} \frac{(1 - \theta)^2 - \frac{\theta^2}{(1+\theta)^2}}{(1 - \theta^2)} = \frac{1}{4} \frac{a^2 - 5}{(a + 2)\sqrt{a^2 - 4}},$$

$$\delta_{-3} = \frac{1}{4} \frac{\left(1 - 2\theta + \frac{3\theta^2}{1+\theta}\right)^2}{(1 - \theta^2)} = \frac{1}{4} \frac{(a - 1)^2}{(a + 2)\sqrt{a^2 - 4}}, \quad \text{if } a \geq 7,$$

$$\delta_{-2} = \frac{1}{4} \frac{\left(1 - 2\theta + \frac{\theta^2(1+\theta)}{(1+\theta^2)}\right)^2}{1 - \theta^2} = \frac{1}{4} \frac{(a - 2)(a - 1)^2}{a^2\sqrt{a^2 - 4}},$$

$$\delta_{-1} = \frac{1}{4} \frac{\left(1 - 2\theta + \theta^2 + \frac{\theta^7(1-\theta)(1-\theta^2)}{(1-\theta^8)}\right)^2 - \frac{\theta^6(1-\theta)^2(1+\theta^2)^2}{(1-\theta^8)^2}}{(1 - \theta^2)},$$

and for  $k \geq 1$

$$\delta_k = \frac{1}{4} \frac{\left(1 - 2\theta + \theta^2 + \frac{\theta^{4k+1}(1-\theta)(1-\theta^2)}{(1-\theta^{4k+2})}\right)^2 - \frac{\theta^6(1-\theta)^2}{(1+\theta^2)^2} \left(\frac{1+\theta^{4k-2}}{1-\theta^{4k+2}}\right)^2}{(1 - \theta^2)},$$

with  $\delta_k \searrow \delta_\infty$  as  $k \rightarrow \infty$ , and  $M(\alpha, \gamma) = \delta_k$  iff  $\pm\gamma$  is in  $S_k$ .

If  $a \geq 6$  is even, then the values of  $M(\alpha, \gamma)$ ,  $\gamma \neq 0$ , greater than

$$\delta_\infty := \frac{1}{4} \frac{\left(\frac{1-\theta}{1+\theta}\right)^2 - \theta^2}{(1 - \theta^2)} = \frac{1}{4} \left(\frac{a}{a + 2}\right) \left(1 - \frac{2}{\sqrt{a^2 - 4}}\right),$$

are given by

$$\delta_0 = \frac{1}{4} \frac{(1 - \theta)^2}{(1 - \theta^2)} = \frac{1}{4} \frac{(a - 2)}{\sqrt{a^2 - 4}},$$

$$\delta_{-1} = \frac{1}{4} \frac{(1 - \theta)^2 - \frac{4\theta^2}{(1 + \theta)^2}}{(1 - \theta^2)} = \frac{1}{4} \frac{(a^2 - 8)}{(a + 2)\sqrt{a^2 - 4}},$$

and for  $k \geq 1$

$$\delta_k = \frac{1}{4} \frac{\left(\frac{1 - \theta}{1 + \theta}\right)^2 \left(\frac{1 - \theta^{2k+2}}{1 - \theta^{2k+1}}\right)^2 - \theta^2 \left(\frac{1 - \theta^{2k}}{1 - \theta^{2k+1}}\right)^2}{1 - \theta^2},$$

with  $\delta_k \searrow \delta_\infty$  as  $k \rightarrow \infty$ , and  $M(\alpha, \gamma) = \delta_k$  iff  $\pm\gamma$  is in  $S_k$ .

For  $a = 4$  we have the additional values

$$\delta_{-2} = \frac{1}{4} \frac{\theta}{1 - \theta^2},$$

$$\delta_{-k-2} = \frac{1}{4} \frac{\theta}{1 - \theta^2} \left(1 - \frac{1}{3}\theta^2 \left(\frac{1 - \theta^{2k}}{1 - \theta^{2k+2}}\right)^2\right),$$

with  $\delta_{-k} \searrow \delta_\infty$ .

Since  $M(\alpha, 0) = \theta/(1 - \theta^2) = 1/\sqrt{a^2 - 4}$  the exclusion of the homogeneous case  $\gamma \sim 0$  is only relevant when  $a \leq 7$ , with  $M(\alpha, 0) \geq \rho(\alpha)$  when  $a \leq 6$  (with equality when  $a = 6$ ). We note that  $\delta_\infty$  is actually a limit point of limit points from below; for example if the expansion  $t_i$  for  $\gamma$  consists of blocks  $0, (-2, 2)^{k_i}$  or  $(-1, 1)(-1, -1, 1, 1)^{k_i}$ , with  $k_i$  not eventually constant and  $k = \liminf k_i$  then  $M(\alpha, \gamma) \nearrow \delta_\infty$  as  $k \rightarrow \infty$  (with  $\delta_\infty$  achieved if  $k = \infty$ , the limit points from taking the  $k_i$  to have period  $\bar{k}, \bar{l}$  with  $l \rightarrow \infty$ , tending to  $\delta_\infty$  as  $k \rightarrow \infty$ ). When  $a = 5$ , the set  $S_{-3}$  has  $\delta_{-3} < \delta_\infty$  and so is not included in the list. When  $a$  is odd the value of  $\delta_{-1}$  actually lies between  $\delta_1$  and  $\delta_2$ , otherwise the values are given in decreasing order. The value of  $\rho(\alpha)$  for odd  $a \geq 5$  together with the optimal  $\gamma$  can be deduced from paper I of Barnes–Swinnerton-Dyer [1] (Theorem 1 for  $a \geq 7$  and Theorem 3 for  $a = 5$ ). Komatsu [4] has also evaluated  $M(\alpha, \gamma)$  for special values of  $\gamma$  (in the regular continued fraction these  $\alpha$  of course have period two,  $\bar{1}, a - 2$ ). The remaining case  $a = 3$  (corresponding to the golden ratio) has been dealt with by Davenport [3], and by Cusick, Rockett and Szűsz [2] who show a similar structure from  $\rho(\alpha) = 1/(4\sqrt{5})$  (achieved with  $t_i$  of period  $(-1, 3, -1)$ ) down to the first limit point  $1/(10 + 2\sqrt{5})$ , the intermediate values corresponding to expansions with period  $(-1, 3, -1)^k(1, -1)$ .

### 3. Proofs for Section 1

#### 3.1. Proof of Theorem 1.

Observe that any positive integer  $n$ ,  $q_{k-1} \leq n < q_k$ , has an expansion

$$n = \sum_{i=1}^k z_i q_{i-1}, \quad z_k \geq 1,$$

so that

$$\gamma' := \{n\alpha\} = \sum_{i=1}^k z_i D_{i-1}$$

gives the  $\alpha$  expansion of  $\{n\alpha\}$ . This expansion amounts to taking  $z_k = \lfloor n/q_{k-1} \rfloor$ , repeating this process for  $n - z_k q_{k-1}$  and so on. We shall assume that  $\|n\alpha - \gamma\| = \pm(\{n\alpha\} - \gamma)$  (since otherwise  $\|n\alpha - \gamma\| = 1 - (\{n\alpha\} - \gamma) > \gamma$  or  $\|n\alpha - \gamma\| = 1 + (\{n\alpha\} - \gamma) > 1 - \gamma$  and  $|n|\|n\alpha - \gamma\|$  is unbounded).

We suppose that  $n \neq Q_k$  so that  $z_s \neq b_s$  for some  $1 \leq s \leq k$  with  $z_j = b_j$  for any  $1 \leq j < s$ .

If  $z_s < b_s$  then  $\|n\alpha - \gamma\| = (b_s - z_s + \gamma_s - \gamma'_s)D_{s-1}$ . Hence if  $s \neq k$

$$n = Q_s + (z_s - b_s)q_{s-1} + \sum_{i=s+1}^k z_i q_{i-1} \geq Q_s + q_{k-1} - (a_s - 1)q_{s-1} > Q_s,$$

and

$$\|n\alpha - \gamma\| = (b_s - z_s - \gamma'_s)D_{s-1} + \|Q_s\alpha - \gamma\| > \|Q_s\alpha - \gamma\|,$$

so that  $\lambda(n) > \lambda(Q_s)$  (with the second inequality implying that  $Q_s \rightarrow \infty$  as  $n \rightarrow \infty$  if  $\lambda(n) \not\rightarrow \infty$ ). Thus it is enough to consider  $s = k$ , in which case

$$\lambda(n) = (z_k q_{k-1} + Q_{k-1})(b_k - z_k + \gamma_k)D_{k-1}.$$

For  $0 \leq z_k \leq b_k$  this is clearly minimised for  $z_k = 0$  or  $z_k = b_k$  so that  $\lambda(n) > \min\{\lambda(Q_k), \lambda(Q_{k-1})\}$ .

So suppose that  $z_s > b_s$  and

$$\|n\alpha - \gamma\| = (z_s - b_s)D_{s-1} + \sum_{i=s+1}^{\infty} (z_i - b_i)D_{i-1}.$$

If  $s \neq k$  or  $s = k$  and  $z_s \geq b_s + 2$  then  $n' = n - q_{k-1}$  has  $\|n'\alpha - \gamma\| = \|n\alpha - \gamma\| - D_{k-1}$  and  $\lambda(n') < \lambda(n)$ . Hence we can assume that  $s = k$ ,  $z_k = b_k + 1$  and  $n = Q_k + q_{k-1}$ .

If the alpha-expansion of  $\gamma$  has  $b_i = a_i - 1$  at most finitely many times then, since  $\sum b_i D_{i-1} + \sum (a_i - 2 - b_i) D_{i-1} = 1 - \alpha$ , we know that  $-\gamma$  is equivalent to a gamma with  $b'_i = (a_i - 2 - b_i)$  for almost all  $i$ . From this one can readily deduce that  $M_-(\alpha, \gamma) = M_+(\alpha, -\gamma) = \liminf_{k \rightarrow \infty} \min\{\lambda(Q_k - (q_k - q_{k-1})), \lambda(Q_k - q_k)\}$ .  $\square$

**3.2. Proof of Corollary 1.**

Defining

$$w_k(\gamma) := \begin{cases} (1 - \alpha_k)(1 - \bar{\alpha}_k), & \text{if } d_k^+ \leq 1 - \alpha_k \text{ and } d_k^- \leq 1 - \bar{\alpha}_k, \\ \alpha_k, & \text{if } d_k^+ > 1 - \alpha_k, \\ \bar{\alpha}_k, & \text{if } d_k^- > 1 - \bar{\alpha}_k, \end{cases}$$

Corollary 1 follows from the more precise bound

$$(3.1) \quad M(\alpha, \gamma) \leq \frac{1}{4} \liminf_{k \rightarrow \infty} w_k(\alpha) q_k D_{k-1}.$$

If  $d_n^+ \leq 1 - \alpha_n$  and  $d_n^- \leq 1 - \bar{\alpha}_n$ , then

$$\begin{aligned} \min\{s_1(n), s_3(n)\} &\leq \sqrt{s_1(n)s_3(n)} \\ &= \frac{1}{4} q_n D_{n-1} \left( (1 - \bar{\alpha}_n)^2 - (d_n^-)^2 \right)^{1/2} \left( (1 - \alpha_n)^2 - (d_n^+)^2 \right)^{1/2} \\ &\leq \frac{1}{4} q_n D_{n-1} (1 - \bar{\alpha}_n)(1 - \alpha_n). \end{aligned}$$

If  $d_n^- \leq 1 - \bar{\alpha}_n$  and  $d_n^+ > 1 - \alpha_n$  then

$$\sqrt{s_3(n)s_2(n)} = \frac{1}{4} q_n D_{n-1} (1 - (\bar{\alpha}_n + d_n^-)^2)^{\frac{1}{2}} (\alpha_n^2 - (1 - d_n^+)^2)^{\frac{1}{2}},$$

and in the same way if  $d_n^- > 1 - \bar{\alpha}_n$  and  $d_n^+ \leq 1 - \alpha_n$  then

$$\sqrt{s_3(n)s_4(n)} = \frac{1}{4} q_n D_{n-1} (\bar{\alpha}_n^2 - (1 - d_n^-)^2)^{\frac{1}{2}} (1 - (\alpha_n + d_n^+)^2)^{\frac{1}{2}}.$$

Bound (1.3) is immediate from (1.4) and the observation that  $(1 - \alpha_i) < (1 - \alpha_i \bar{\alpha}_i)$  with  $\bar{\alpha}_i = 1/(R - \bar{\alpha}_{i-1}) \geq 1/R$  infinitely often.  $\square$

**3.3. Proof of Theorem 2 and (1.6).**

Assume that  $a_i$  is even for  $i \geq N$ . For  $\gamma = \gamma^*$  or  $\gamma^{**}$  we have  $\gamma \sim -\gamma$  and  $M(\alpha, \gamma) = \liminf_{k \rightarrow \infty} \min\{s_1(k), s_2(k)\}$ . For  $\gamma^*$  we have  $t_{N+i} = 0$  giving  $d_k^+, d_k^- \rightarrow 0$  and

$$\begin{aligned} M(\alpha, \gamma^*) &= \frac{1}{4} \liminf_{k \rightarrow \infty} \min \left\{ \frac{(1 - \bar{\alpha}_k)(1 - \alpha_k)}{(1 - \alpha_k \bar{\alpha}_k)}, \frac{(1 + \bar{\alpha}_k)(1 + \alpha_k)}{(1 - \alpha_k \bar{\alpha}_k)} \right\} \\ &= \frac{1}{4} \liminf_{k \rightarrow \infty} \frac{(1 - \bar{\alpha}_k)(1 - \alpha_k)}{(1 - \alpha_k \bar{\alpha}_k)}. \end{aligned}$$

Suppose now that  $\alpha$  is also of the form (1.5). Notice that  $a_i \geq 4$  for almost all  $i$  gives  $\alpha_i, \bar{\alpha}_i \leq [0; \bar{4}]^- + o(1) = (2 - \sqrt{3} + o(1))$ . Hence if  $\gamma$  has  $t_i = a_i$  infinitely often then (1.7) gives  $M(\alpha, \gamma) \leq \frac{1}{4} \limsup_{i \rightarrow \infty} \frac{\bar{\alpha}_i}{1 - \alpha_i \bar{\alpha}_i} \leq \frac{1}{8\sqrt{3}}$ , while  $M(\alpha, \gamma^*) = \frac{1}{4} \liminf_{i \rightarrow \infty} \frac{(1 - \alpha_i)(1 - \bar{\alpha}_i)}{1 - \alpha_i \bar{\alpha}_i} > \frac{1}{4\sqrt{3}}$ , and  $M(\alpha, \gamma) \leq \frac{1}{2} M(\alpha, \gamma^*)$ . Hence we can assume that  $t_i = a_i$  at most finitely many times. Set  $l := \lfloor \frac{r}{2} \rfloor$ . Suppose that  $\gamma \not\sim \pm \gamma^*, 0$ . Then, for each  $i = 0, \dots, r-1$ , there will be infinitely  $n \equiv i \pmod{r}$  with  $t_m \neq 0$  for some  $m$  with  $n-l \leq m \leq n$  or  $n+1 \leq m \leq n+1+l$  (and  $t_j = 0$  for any  $j$  closer to  $n$  or  $n+1$  than  $m$ ). Now if  $m \leq n$  then  $|d_n^-|q_n \geq 2q_{m-1} - |d_{m-1}^-|q_{m-1} \geq q_{m-1}$  and

$|d_n^-| \geq \bar{\alpha}_n \cdots \bar{\alpha}_m \geq A^{-(l+1)}$  (likewise if  $m > n$  the  $|d_n^+| \geq A^{-(l+1)}$ ). Hence, as in the proof of Corollary 1,

$$\begin{aligned} \min\{s_1(n), s_3(n)\} &\leq \frac{1}{4}q_n D_{n-1}(1 - \bar{\alpha}_n)(1 - \alpha_n)(1 - (d_k^+)^2)^{\frac{1}{2}}(1 - (d_k^-)^2)^{\frac{1}{2}} \\ &\leq \frac{1}{4}q_n D_{n-1}(1 - \bar{\alpha}_n)(1 - \alpha_n)(1 - A^{-2(l+1)})^{\frac{1}{2}}, \end{aligned}$$

and

$$\begin{aligned} M(\alpha, \gamma) &\leq \frac{1}{4} \left(1 - A^{-2\lceil \frac{r+1}{2} \rceil}\right)^{\frac{1}{2}} \min_{i=0, \dots, r-1} \lim_{\substack{q \rightarrow \infty \\ n=qr+i}} (1 - \bar{\alpha}_n)(1 - \alpha_n)q_n D_{n-1} \\ &= \left(1 - A^{-2\lceil \frac{r+1}{2} \rceil}\right)^{\frac{1}{2}} \rho(\alpha). \end{aligned}$$

Suppose now that  $a_{N+2i} = 2$  and  $a_{N+2i+1} \geq 4$  for all  $i \geq 0$ . For  $\gamma^{**}$  we have  $d_{N+2i}^- \rightarrow 1$ ,  $d_{N+2i-1}^+ \rightarrow 1$ ,  $d_{N+2i-1}^- \rightarrow -\bar{\alpha}_{N+2i-1}$ ,  $d_{N+2i}^+ \rightarrow -\alpha_{N+2i}$  with  $\alpha_{N+2i}, \bar{\alpha}_{N+2i-1} \leq [0; \frac{4}{2}]^- + o(1) = \frac{1}{2}(2 - \sqrt{2}) + o(1)$ , and writing  $\mu_i := \frac{1}{4}q_{N+2i-1}D_{N+2i-1}$ ,

$$s_2(N + 2i - 1) \rightarrow \mu_i,$$

$$s_2(N + 2i) \geq s_1(N + 2i) \rightarrow \mu_i(3 - 2\bar{\alpha}_{N+2i-1})(1 - 2\alpha_{N+2i}) \geq \mu_i + o(1),$$

$$s_1(N + 2i - 1) \rightarrow \mu_i(3 - 2\alpha_{N+2i})(1 - 2\bar{\alpha}_{N+2i-1}) \geq \mu_i + o(1),$$

and  $M(\alpha, \gamma^{**}) = \liminf_{i \rightarrow \infty} \mu_i = \rho^*(\alpha)$ . □

### 3.4. Proof of Lemma 1.

Bound (1.7) follows at once from bound (3.1). Bound (1.8) follows on observing (for  $\pm\gamma$ ) that the minimum of

$$\begin{aligned} \lambda(Q_k) &= q_{k-1}D_{k-1}(b_k + \xi_{k-1})\gamma_k, \\ \lambda(Q_{k-1}) &= q_{k-1}D_{k-1}\xi_{k-1}(b_k + \gamma_k), \\ \lambda(Q_{k-1} - q_{k-1}) &= q_{k-1}D_{k-1}(b_k + 1 + \gamma_k)(1 - \xi_{k-1}), \\ \lambda(Q_k + q_{k-1}) &= q_{k-1}D_{k-1}(1 + b_k + \xi_{k-1})(1 - \gamma_k), \end{aligned}$$

is certainly no more than

$$\frac{1}{4}(\lambda(Q_k) + \lambda(Q_{k-1}) + \lambda(Q_{k-1} - q_{k-1}) + \lambda(Q_k + q_{k-1})) = \frac{1}{4}(a_k + t_k)q_{k-1}D_{k-1}.$$

□

## 4. Proof of Theorem 3

### 4.1. Evaluating the $\delta_k$ .

We evaluate  $\delta_k^* = 4(1 - \theta^2)\delta_k$ . Apart from the  $\delta_{-k}$ ,  $k \geq 2$  when  $a = 4$  (which have some  $t_i = a$ ) we can assume that

$$\delta_k^* = \liminf_{n \rightarrow \infty} \min\{s_1^*(n), s_2^*(n), s_3^*(n), s_4^*(n)\}$$

with

$$s_1^*(n), s_3^*(n) = (1 - \theta \pm d_n^-)(1 - \theta \pm d_n^+), \quad s_2^*(n), s_4^*(n) = (1 + \theta \pm d_n^-)(1 + \theta \mp d_n^+).$$

Except for  $\delta_{-3}$ ,  $a$  odd, we have  $|d_n^-|, |d_n^+| < 2\theta$  so that  $s_2^*(n), s_4^*(n) > (1 - \theta)^2$  and we need only evaluate the  $s_1^*(n), s_3^*(n)$ . For  $\gamma$  in  $S_0$  with  $a$  even the  $d_n^+, d_n^- \sim 0$  and plainly  $\delta_0^* = (1 - \theta)^2$  (the largest possible value). For  $\gamma$  having  $t_i$  of period  $(t, -t)$ ,  $t = 1, 2, 3$  we have  $\{d_n^+, d_n^-\} \sim \pm t\theta/(1 + \theta)$  and  $s_1^*(n), s_3^*(n) \sim (1 - \theta)^2 - (t\theta/(1 + \theta))^2$ , giving the value of  $\delta_0^*$ ,  $a$  odd and  $\delta_{-1}^*$ ,  $a$  even (and  $\delta_{-3}^* < \delta_\infty^*$  if  $a = 5$ ). When  $t = 3$ ,  $a \geq 7$  odd,  $\min\{s_2^*(n), s_4^*(n)\} \sim (1 + \theta - t\theta/(1 + \theta))^2$  is smaller and gives  $\delta_{-3}^*$ .

Now if the  $t_i$  have period  $0, (-2, 2, )^k$  the smallest pair  $\{d_n^-, d_n^+\}$  (and smallest  $\{-d_n^-, -d_n^+\}$ ) are asymptotically

$$\left\{ -\frac{2\theta(1 - \theta^{2k})}{(1 + \theta)(1 - \theta^{2k+1})}, \frac{2\theta^2(1 - \theta^{2k})}{(1 + \theta)(1 - \theta^{2k+1})} \right\}$$

occurring when  $t_n = 0$  (or  $t_{n+1} = 0$ ) giving the smallest  $s_1^*(n)$  (or  $s_3^*(n)$ ) and the value claimed for  $\delta_k^*$ ,  $k \geq 1$ , when  $a$  is even.

For  $a \geq 5$  odd and  $\delta_{-1}^*, \delta_{-2}^*, \delta_k^*$ ,  $k \geq 1$  we note that the values claimed are certainly less than  $(1 - 2\theta + \theta^2 + \theta^3)^2 \leq 1 - 4\theta + 6\theta^2$ . Now if  $t_n, t_{n+1} = 1, -1$  (or vice versa) then  $\theta + \theta^2 \geq d_n^-, -d_n^+ \geq \theta - \theta^2 - \theta^3$  producing  $s_1^*(n), s_3^*(n) \geq (1 - \theta^2 - \theta^3)(1 - 2\theta - \theta^2) > 1 - 2\theta - 2\theta^2$ . Hence it is enough to consider  $s_1^*(n)$  when  $t_n, t_{n+1} = -1, -1$  (for both  $\gamma$  and its negative). For  $\delta_{-2}^*$  these  $n$  have  $d_n^+, d_n^- \sim (-\theta + \theta^2)/(1 + \theta^2)$  and  $s_1^*(n)$  gives the value claimed. For  $\pm\gamma$  in  $S_{-1}$  these  $n$  have  $\{d_n^-, d_n^+\}$  asymptotic to

$$\left\{ \frac{-\theta + \theta^2 + \theta^3 - \theta^4 + \theta^5 - \theta^6 + \theta^7 - \theta^8}{1 - \theta^8}, \frac{-\theta + \theta^2 - \theta^3 + \theta^4 - \theta^5 + \theta^6 + \theta^7 - \theta^8}{1 - \theta^8} \right\}$$

with  $s_1^*(n)$  giving the value claimed for  $\delta_{-1}^*$ . For the  $\delta_k$ ,  $k \geq 1$  when the  $-1, -1$  occurs in a block  $-1, 1, -1, -1, 1, 1$  or  $1, 1, -1, -1, 1, -1$  we have  $\{d_n^-, d_n^+\}$  tending to

$$\left\{ \frac{-\theta(1 - \theta) - \frac{\theta^3(1 - \theta)(1 - \theta^{4k})}{1 + \theta^2}}{1 - \theta^{4k+2}}, \frac{-\frac{\theta(1 - \theta)(1 - \theta^{4k})}{1 + \theta^2} + \theta^{4k+1}(1 - \theta)}{1 - \theta^{4k+2}} \right\}$$

with  $s_1^*(n)$  asymptotically giving the value claimed for  $\delta_k^*$ , with this certainly less than  $(1 - 2\theta + \theta^2 + \theta^3)^2$ . When the  $-1, -1$  occur inside blocks  $1, 1, -1, -1, 1, 1$  we have  $d_n^-, d_n^+ \geq -\theta + \theta^2 + \theta^3 - \theta^4 - \theta^5$  giving a larger and hence irrelevant  $s_1^*(n) \geq (1 - 2\theta + \theta^2 + \theta^3 - \theta^4 - \theta^5)^2$ .

Finally we deal with the  $\delta_{-k}$ ,  $k \geq 2$  when  $a = 4$ . In this case we need check  $s_1^*(n), s_2^*(n)$  for both  $\gamma$  and its negative. If  $t_n = a$  then  $d_n^- \geq a\theta - 2\theta^2$ ,

$d_n^+ \geq -2\theta$  and  $s_1^*(n) \geq (2 - \theta - \theta^2)(1 - 3\theta) > \theta$  (likewise if  $t_{n+1} = a$ , this merely reversing the roles of  $d_n^+$  and  $d_n^-$ ). Moreover when  $t_n = a$  we have  $d_n^- > 0$ ,  $d_n^+ < 0$  and  $s_2^*(n) > 1$  and when  $t_{n+1} = a$

$$s_2^*(n) = \theta(1 + (\theta + d_n^-))(1 - (\theta + d_{n+1}^+)).$$

Hence we can ignore the  $t_n = a$  and when  $t_{n+1} = a$  merely check  $s_2^*(n)$ . For  $\gamma$  in  $S_{-2}$  and  $t_{n+1} = a$  we have  $d_n^-, d_{n+1}^+ \sim (-2\theta + a\theta^2)/(1 - \theta^2) = -\theta$  giving  $\delta_{-2}^* = \theta$ . For  $\gamma$  with period  $2, -2, a, -2, (2, -2, )^{k-1}$  and  $t_{n+1} = a$  we have

$$d_n^-, d_{n+1}^+ \sim -\frac{2\theta}{(1 + \theta)} + \frac{(a - 2)\theta^{2k+2}}{(1 - \theta^{2k+2})}$$

and  $s_2^*(n) \rightarrow \theta \left( 1 - \left( \theta - \frac{2\theta^2}{1+\theta} - \frac{2\theta^{2k+2}}{1-\theta^{2k+2}} \right)^2 \right)$ , the value claimed for  $\delta_{-k-2}^*$ .

For the negative of this  $-2, 0, a, 0, (-2, 2, )^{k-1}$  we have  $d_n^-, d_{n+1}^+ \sim -2\theta^2/(1 + \theta) - 2\theta^{2k+2}/(1 - \theta^{2k+2})$  asymptotically giving the same value. For the remaining  $n$  with  $t_n, t_{n+1} \neq a$  we have  $|d_n^-|, |d_n^+| < 2\theta$  and again we need only consider  $s_1^*(n)$ . Now if  $t_n = -2$ ,  $t_{n+1} \neq a$  (or vice versa) the bounds  $d_n^- \geq -2\theta + a\theta^3 - 2\theta^5 \geq -2\theta + \theta^2$ ,  $d_n^+ \geq a\theta^2 - 2\theta^4 > \theta$  give  $s_1^*(n) > 1 - 3\theta + \theta^2 = \theta$  so these do not affect the value of  $\delta_{-k-2}^*$ .  $\square$

**4.2. Proof of the Theorem when  $a$  is odd.**

Writing  $\delta_\infty^* = 4(1 - \theta^2)\delta_\infty$  we suppose that  $\gamma$  has  $M^*(\alpha, \gamma) > \delta_\infty^*$ . We note the rough bounds

$$\delta_\infty^* = (1 - 2\theta + \theta^2)^2 - \frac{\theta^6(1 - \theta)^2}{(1 + \theta^2)^2} > (1 - 2\theta + \theta^2)^2 - \theta^6 > (1 - 4\theta + 5\theta^2)$$

and  $\delta_\infty^* > (1 - 2\theta + \theta^2 - \frac{1}{2}\theta^5)^2$ . Now if  $\gamma$  has  $t_i = a$  infinitely often then, from (1.7),

$$M^*(\alpha, \gamma) \leq \theta < \delta_\infty^* - (1 - 5\theta + 5\theta^2) \leq \delta_\infty^* - 4\theta^2,$$

and if  $|t_i| \geq 5$  infinitely often, then from (1.8)

$$M^*(\alpha, \gamma) \leq (a - 5)\theta = 1 - 5\theta + \theta^2 < \delta_\infty^* - \theta.$$

Hence it is enough to consider  $\gamma$  with  $t_i = \pm 1, \pm 3$  for all  $i$  (if we were only interested in  $\rho(\alpha)$  we could similarly rule out  $t_i = \pm 3$  infinitely often).

Now if  $t_n = -3$  and  $t_{n+1} \leq -1$  (or vice versa) infinitely often, then

$$s_1^*(n) \leq \left( 1 - \theta - \theta + \frac{3\theta^2}{1 - \theta} \right) \left( 1 - \theta - 3\theta + \frac{3\theta^2}{1 - \theta} \right)$$

and

$$M^*(\alpha, \gamma) \leq \left( 1 - 3\theta + \frac{3\theta^2}{1 - \theta} \right)^2 - \theta^2 < \delta_\infty^* - \theta^2.$$

Similarly we can dismiss  $t_i = 3, t_{i\pm 1} \geq 1$  (by considering the negative  $(1 - \alpha - \gamma)$ ). So if  $\{t_n, t_{n+1}\} = \{-3, 1\}$  then

$$s_1^*(n) \leq \left(1 - \theta + \frac{\theta}{1 - \theta}\right) (1 - \theta - 3\theta + 3\theta^2) < \delta_\infty^* - \theta^2.$$

Likewise if  $\{t_n, t_{n+1}\} = \{3, -1\}$ . Hence apart from the period  $-3, 3$  elements of  $S_{-3}$  we can assume that  $t_i = \pm 1$  for all  $i$ . We assume that  $\gamma \notin S_0$  so that  $\gamma$  (or its negative) has infinitely many blocks  $t_n, t_{n+1} = -1, -1$  with  $s_1^*(n) = (1 - 2\theta + \theta d_{n-1}^-)(1 - 2\theta + \theta d_{n+1}^+)$ . Now we can rule out blocks  $-1, -1, -1$  and  $-1, 1, -1, -1, 1, -1$  since  $t_{n+2} = -1$  would give  $d_{n+1}^+ < 0, d_{n-1}^- \leq \theta/(1 - \theta)$  and  $s_1^*(n) \leq (1 - 2\theta)^2 + \theta^2 < \delta_\infty^*$ , and  $t_{n-1}, t_{n-2} = t_{n+2}, t_{n+3} = 1, -1$  would give  $d_n^+, d_n^- \leq \theta - \theta^2 + \theta^3/(1 - \theta)$  and  $s_1^*(n) \leq (1 - 2\theta + \theta^2 - \theta^3 + \theta^4/(1 - \theta))^2 < \delta_\infty^*$ . Hence we can assume the blocks  $-1, -1$  and  $1, 1$  occur inside blocks  $-1, 1, 1, -1, -1, 1$  or  $1, -1, -1, 1, 1, -1$ . Suppose that  $t_{n-1}, \dots, t_{n+4} = 1, -1, -1, 1, 1, -1$  then

$$s_1^*(n) = (1 - 2\theta + \theta^2 + \theta^2 d_{n-2}^-)(1 - 2\theta + \theta^2 + \theta^3 - \theta^4 - \theta^4(-d_{n+4}^+))$$

$$s_3^*(n + 2) = (1 - 2\theta + \theta^2 + \theta^2(-d_{n+4}^+))(1 - 2\theta + \theta^2 + \theta^3 - \theta^4 - \theta^4 d_{n-2}^-).$$

Now we can rule out blocks  $t_{n+1}, t_n, t_{n-1}, \dots = -1, -1, 1, -1, -1, \dots$  or

$$-1, -1, 1, -1, 1, -1, -1, \dots \text{ or } -1, -1, 1, -1, 1, -1, 1, -1, \dots$$

(or  $t_{n+2}, t_{n+3}, t_{n+4}, \dots$  having the negative of these) since these would give  $d = \min\{d_{n-2}^-, -d_{n+4}^+\} \leq -\theta + \theta^2 - \theta^3 + \theta^4 - \theta^5 + \theta^6/(1 - \theta)$ , and

$$\min\{s_1^*(n), s_3^*(n + 2)\} \leq (1 - 2\theta + \theta^2 + \theta^2 d)(1 - 2\theta + \theta^2 + \theta^3 - \theta^4 - \theta^4 d)$$

$$\leq (1 - 2\theta + \theta^2 + \theta^9)^2 - (\theta^3 - \theta^4 + \theta^5 - \theta^6)^2 < \delta_\infty^*.$$

Hence we can assume that the sequence  $t_{n+1}, t_n, t_{n-1}, \dots$  takes the form

$$-1, -1, 1, 1, \dots \text{ or } -1, -1, 1, -1, 1, 1, -1, -1, \dots$$

or

$$-1, -1, 1, -1, 1, -1, 1, 1, -1, -1, \dots$$

(and the  $t_{n+2}, t_{n+3}, \dots$  the form  $1, 1, -1, -1, \dots$  or  $1, 1, -1, 1, -1, -1, 1, 1, \dots$  or  $1, 1, -1, 1, -1, 1, -1, -1, 1, 1, \dots$ ). Moreover if

$$t_{n+1}, t_n, \dots = -1, -1, 1, -1, 1, -1, \dots$$

and  $t_{n+2}, t_{n+3}, \dots \neq 1, 1, -1, 1, -1, 1, \dots$  then

$$d_{n-2}^- \leq -\theta + \theta^2 - \theta^3 + \theta^4 + \theta^5 - \theta^6 - \theta^7 + \theta^8/(1 - \theta), \quad d_{n+4}^+ \leq \theta - \theta^2 - \theta^3 + \theta^4 + \theta^5,$$

$$\text{and } s_1^*(n) \leq (1 - 2\theta + \theta^2 + \theta^{10}/(1 - \theta))^2 - (\theta^3 - \theta^4 + \theta^5 - \theta^6 - \theta^7)^2 < \delta_\infty^*.$$

Hence excluding the  $\gamma$  in  $S_{-1}$  with period  $-1, -1, 1, -1, 1, -1, 1, 1$  or its negative, and the  $\gamma$  in  $S_{-2}$  with period  $-1, -1, 1, 1$ , we can assume that we have infinitely many blocks  $-1, 1, -1$  or  $1, -1, 1$  with these contained



in blocks  $1, 1, -1, -1, 1, -1, 1, 1, -1, -1$  or  $-1, -1, 1, 1, -1, 1, -1, -1, 1, 1$ . Suppose that  $t_n, \dots, t_{n+5} = -1, -1, 1, -1, 1, 1$  then

$$s_1^*(n) = (1 - 2\theta + \theta d_{n-1}^-)(1 - 2\theta + \theta^2 - \theta^3 + \theta^4 + \theta^5 - \theta^5(-d_{n+5}^+))$$

$$s_3^*(n+4) = (1 - 2\theta + \theta(-d_{n+5}^+))(1 - 2\theta + \theta^2 - \theta^3 + \theta^4 + \theta^5 - \theta^5 d_{n-1}^-).$$

Now we can rule out  $d = \min\{d_{n-1}^-, -d_{n+5}^+\} \leq (\theta + \theta^2)/(1 + \theta^2)$  else  $\min\{s_1^*(n), s_3^*(n+4)\} \leq (1 - 2\theta + \theta d)(1 - 2\theta + \theta^2 - \theta^3 + \theta^4 + \theta^5 - \theta^5 d) \leq \delta_\infty^*$ .

Hence we can assume that the sequence consists solely of blocks

$$1, -1, (1, 1, -1, -1, )^{k_i}$$

with  $k_i \geq 1$  (or solely of their negatives  $-1, 1, (-1, -1, 1, 1, )^{k_i}$ ). Now if we have a block  $\dots t_{n-1} | t_n \dots$  of the form

$\dots, 1, -1, 1, 1, (-1, -1, 1, 1)^l | -1, -1, 1, -1, 1, 1, (-1, -1, 1, 1)^k -1, -1, 1, -1, \dots$   
with  $l > k \geq 0$  then

$$d_{n-1}^- \leq \frac{\theta + \theta^2}{1 + \theta^2} + 2\theta^{4+4l}(1 - \theta - \theta^2 + \theta^3 + \theta^4),$$

$$d_{n+5}^+ \leq -\frac{\theta + \theta^2}{1 + \theta^2} - 2\theta^{4+4k}(1 - \theta - \theta^2),$$

and

$$s_1^*(n) \leq \left(1 - 2\theta + \theta^2 + \frac{\theta^3(1 - \theta)}{(1 + \theta^2)} + 2\theta^{9+4k}(1 - \theta - \theta^2) + 2\theta^{12+4k}(1 + \theta)\right)$$

$$\left(1 - 2\theta + \theta^2 - \frac{\theta^3(1 - \theta)}{(1 + \theta^2)} - 2\theta^{9+4k}(1 - \theta - \theta^2)\right)$$

$$\leq \delta_\infty^* - 2\theta^{12+4k} \left(2(1 - \theta - \theta^2) \frac{(1 - \theta)}{(1 + \theta^2)} - (1 + \theta)(1 - 2\theta + \theta^2)\right).$$

This leaves only the periodic expansions of elements in  $S_k$ . □

**4.3. Proof of the Theorem when  $a$  is even.**

Suppose that  $M^*(\alpha, \gamma) > \delta_\infty^*$  where

$$\delta_\infty^* = \left(1 - \theta - \frac{2\theta}{1 + \theta}\right) \left(1 - \theta + \frac{2\theta^2}{1 + \theta}\right)$$

$$= 1 - 4\theta + 5\theta^2 + \frac{2\theta^3(a - 4)}{1 + \theta} - \frac{2\theta^3(1 + \theta^2)}{(1 + \theta)^2}.$$

We suppose first that all the  $t_i = 0, \pm 2$ . We can certainly assume this when  $a \geq 6$  since if  $t_i = a$  infinitely often then  $M^*(\alpha, \gamma) < \theta$  and if  $|t_i| \geq 4$  infinitely often  $M^*(\alpha, \gamma) \leq (a - 4)\theta = 1 - 4\theta + \theta^2 < \delta_\infty^*$  (when  $a = 4$  we consider separately the  $\gamma$  with  $t_i = a$  infinitely often). We can rule out

infinitely blocks  $(t_n, t_{n+1}) = (-2, -2)$  (or their negative  $(2, 2)$ ) since these give

$$(4.1) \quad s_1^*(n) \leq \left(1 - \theta - 2\theta + \frac{2\theta^2}{1 - \theta}\right)^2 < (1 - 2\theta)^2 < \delta_\infty^*.$$

Also when  $t_n = 0$  we must have  $|d_{n-1}^-|, |d_n^+| < 2\theta/(1 + \theta)$  ruling out blocks  $0, (-2, 2)^k - 2, 0$  or  $0, (2, -2)^k 2, 0$ , since if  $d = \min\{\pm d_{n-1}^-, \pm d_n^+\} < -2\theta/(1 + \theta)$  then the minimum of

$$\begin{aligned} s_1^*(n) &= (1 - \theta - \theta(-d_{n-1}^-))(1 - \theta + d_n^+), \\ s_3^*(n - 1) &= (1 - \theta - \theta d_n^+)(1 - \theta + (-d_{n-1}^-)), \\ s_3^*(n) &= (1 - \theta - \theta d_{n-1}^-)(1 - \theta + (-d_n^+)), \\ s_1^*(n - 1) &= (1 - \theta - \theta(-d_n^+))(1 - \theta + d_{n-1}^-), \end{aligned}$$

is certainly at most

$$(4.2) \quad (1 - \theta - \theta d)(1 - \theta + d) < \delta_\infty^*.$$

Moreover if  $\gamma$  does not have period 0 and  $t_n = 0$  then  $d_n^+$  and  $d_{n-1}^-$  must be of opposite signs, since if for example  $t_n, t_{n+1} = 0, -2$  with  $d_{n-1}^- < 0$  then (using that  $d_{n+3}^+ \leq 2\theta/(1 + \theta)$  if  $t_{n+3} = 0$ )

$$\begin{aligned} s_1^*(n) &\leq (1 - \theta) \left(1 - \theta - 2\theta + 2\theta^2 + \frac{2\theta^4}{1 + \theta}\right) \\ &= 1 - 4\theta + 5\theta^2 - \frac{2\theta^3(1 + \theta^2)}{1 + \theta} < \delta_\infty^*. \end{aligned}$$

Hence we can assume that the sequence of  $t_i$  has period 0 or  $(-2, 2)$  or consists only of blocks  $0, (-2, 2)^{l_i}, l_i \geq 0$  (or only of its negative  $0, (2, -2)^{l_i}$ ). Now if we have a block  $\dots, t_{n-1}, t_n, |t_{n+1}, \dots = \dots, 0, (-2, 2)^{l_0}, |(-2, 2)^k 0, \dots$  with  $0 \leq l < k$  then

$$d_n^+ \leq -\frac{2\theta}{1 + \theta} + \frac{2\theta^{2k+1}}{1 + \theta}, \quad d_{n-1}^- < \frac{2\theta}{1 + \theta} - 2\theta^{2l+1} + \frac{4\theta^{2l+2}}{1 + \theta},$$

and

$$\begin{aligned} s_1^*(n) &\leq \left(1 - \theta + \frac{2\theta^2}{1 + \theta} - 2(1 - \theta)\frac{\theta^{2k}}{1 + \theta}\right) \left(1 - \theta - \frac{2\theta}{1 + \theta} + \frac{2\theta^{2k+1}}{1 + \theta}\right) \\ &= \delta_\infty^* - \frac{2\theta^{2k}}{(1 + \theta)^2} \left(1 - 4\theta + \theta^2 + 2(1 - \theta)\theta^{2k+1}\right) < \delta_\infty^*. \end{aligned}$$

This leaves only the periodic elements of  $S_k$ .

It remains to check the case  $a = 4$  when  $\gamma$  has  $t_i = a$  infinitely often. Observe that if  $t_n = a$  then

$$\begin{aligned} s_4^*(n) &= \theta(1 + (\theta + d_n^+))(1 - (\theta + d_{n-1}^-)), \\ s_2^*(n - 1) &= \theta(1 + (\theta + d_{n-1}^-))(1 - (\theta + d_n^+)), \end{aligned}$$

and we can assume that

$$(4.3) \quad d = \max\{|\theta + d_n^+|, |\theta + d_{n-1}^-|\} \leq \theta - \frac{2\theta^2}{1 + \theta},$$

since otherwise the minimum of these is certainly no more than

$$\theta(1 - d^2) < \theta \left( 1 - \left( \theta - \frac{2\theta^2}{1 + \theta} \right)^2 \right) = \delta_\infty^*.$$

Hence  $d_{n\pm 1}, d_{n\pm 2}, \dots$  must take the form  $0, -2, \dots$  or  $-2, 2, \dots$  or  $-2, a, \dots$  (since  $t_{n+1} = 2$ , or  $t_{n+1} = 0, t_{n+2} \geq 0$  would give  $d_n^+ \geq -2\theta^3/(1 - \theta)$  and  $t_{n+1} = -2, t_{n+2} \leq 0$ , would give  $d_n^+ \leq -2\theta + a\theta^3$ ). We can rule out blocks  $(-2, -2)$  just as in (4.1). Hence condition (4.3) forces the  $(a, 0, -2)$  to lie inside blocks  $t_n, \dots, t_k = a, 0, -2, (2, -2, )^l 0$ . But if  $t_k = 0$  and  $d_k^+, d_{k-1}^- < 1 - \theta$  then  $|d_k^+|, |d_{k-1}^-| \leq 2\theta/(1 + \theta)$  just as in (4.2). So the  $a, 0, -2$  lie inside blocks  $a, 0, -2, (2, -2, )^l 0, a$ . Now if we have  $d_n = a$  and  $d_{n+1}, d_{n+2}, \dots = 0, -2, \dots$  then  $d_{n-1}, d_{n-2}, \dots = 0, -2, \dots$  (and vice versa) since  $d_{n-1} = -2$  would give  $d_n^+ \geq -2\theta^2$ ,  $d_{n-1}^- \leq -2\theta + a\theta^2$  and  $s_2^*(n - 1) < \delta_\infty^*$ . Hence, since going from  $\gamma$  to  $1 - \gamma$  interchanges the blocks  $a, 0, -2$  and  $a, -2, 2$  (and fixes the  $a, -2, a$ ) we can assume that either  $\gamma$  has period  $a, -2$  or consists entirely of blocks  $a, 0, -2, (2, -2, )^{l_i} 0, l_i \geq 0$  (or its negative composed entirely of blocks  $a, -2, 2, (-2, 2, )^{l_i} - 2$ ). Now if we had a block  $\dots t_{n-1}, |t_n, \dots$  of the form

$$\dots, a, 0, (-2, 2, )^k - 2, 0, |a, 0, -2, (2, -2, )^l 0, a, \dots$$

with  $l < k$  then

$$\begin{aligned} d_{n-1}^- &\geq -\frac{2\theta^2}{1 + \theta} - \theta^{3+2k} \frac{2 - (a + 2)\theta + 2\theta^2}{1 - \theta^4}, \\ d_n^+ &\leq -\frac{2\theta^2}{1 + \theta} - \theta^{3+2l} (2 - (a + 2)\theta + 2\theta^2), \end{aligned}$$

and

$$\begin{aligned} s_4^*(n) &\leq \theta \left( 1 + \theta - \frac{2\theta^2}{1 + \theta} - 2\theta^{4+2l} \right) \left( 1 - \theta + \frac{2\theta^2}{1 + \theta} + \frac{2\theta^{6+2l}}{1 - \theta^4} \right) \\ &\leq \delta_\infty^* - 2\theta^{5+2l} (1 - \theta - \theta^2(1 + \theta)) < \delta_\infty^*. \end{aligned}$$

This leaves only the periodic elements of  $S_{-(k+2)}$ . □

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