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# The fractional part of $n\theta + \phi$ and Beatty sequences

par TAKAO KOMATSU

## 1. Introduction

Let $\theta$ be real and $q_n$ be the denominator of the $n$-th convergent of the continued fraction expansion of $\theta$. Thus when $\theta$ is rational, $\theta = p_N/q_N$ for some integer $N$. Denote the fractional part of $\theta$ by $\{\theta\}$, the floor, that is integer part, of $\theta$ by $\lfloor\theta\rfloor$, and its distance from the nearest integer by $\|\theta\|$.

It is well known that the continued fraction convergents are best approximations to $\theta$ in the sense that

$$\min_{0<x<q_n} \|x\theta\| = \|q_{n-1}\theta\| \qquad \text{for } n \geq 1\,.$$

The corresponding results for one-sided best approximations are readily derived, see for example van Ravenstein [9], and are

$$\min_{0<x<q_n} \{x\theta\} = \begin{cases} \{q_{n-1}\theta\} & \text{if } n \text{ is odd;} \\ \{(q_n - q_{n-1})\theta\} & \text{if } n \text{ is even,} \end{cases} \quad \text{for } n \geq 2\,,$$

and

$$\max_{0<x<q_n} \{x\theta\} = \begin{cases} \{(q_n - q_{n-1})\theta\} & \text{if } n \text{ is odd;} \\ \{q_{n-1}\theta\} & \text{if } n \text{ is even,} \end{cases} \quad \text{for } n \geq 2\,.$$

The purpose of this paper is to give similar results in the inhomogeneous case $x\theta + \phi$, where $\theta$ and $\phi$ are real numbers, not necessarily irrational, and to apply these results to find the characteristic word

$$f(n;\theta,\phi) = \lfloor (n+1)\theta + \phi\rfloor - \lfloor n\theta + \phi\rfloor \qquad \text{for } n = 1, 2, \cdots .$$

of the Beatty sequence. A Beatty sequence is a sequence of the form $\lfloor n\theta + \phi\rfloor$ for fixed real numbers $\theta$ and $\phi$. The homogeneous case $\phi = 0$ has been dealt

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with extensively (see for example [3], [4], [5], [8], [10], [11], [12], and [2] for additional references). The inhomogeneous case $\phi \neq 0$ is also discussed from several viewpoints (see [5], [7], [8]), but these sources do not necessarily make it easy to get the actual Beatty sequences. The paper [5] is complete on the theory, but only the limited periodic case yields Beatty sequences. The paper [8] provides a very effective description of Beatty sequences, but regrettably, it is now known that the results do not match the facts (see the corrected version, [6]).

## 2. The minimum of the fractional part of $x\theta + \phi$

We first consider how to find an integer $\kappa$ satisfying

$$\min_{x \in \mathbb{Z}} \{x\theta + \phi\} = \{\kappa\theta + \phi\}.$$

When both $\theta$ and $\phi$ are rational, the problem is not too difficult. But how should we treat the irrational case? The answer is given in the following theorem. Using this result, we can find $\kappa$ directly in every case — though two values must be compared when $\theta$ is irrational.

THEOREM 1. *Let $\theta$, $\phi$ be real numbers, and let $p_n/q_n$ denote the $n$-th convergent of $\theta$. Then for $n \geq 2$,*

$$\min_{0 \leq x < q_n} \{x\theta + \phi\} = \{\kappa_n\theta + \phi\} < \frac{2}{q_n},$$

*where $\kappa_n \equiv (-1)^n q_{n-1}(\lfloor q_n\phi \rfloor + t) \mod q_n$ and $t = 0$ or $(-1)^n$.*

*When $\theta = p_n/q_n \quad (n \geq 2)$, $\{x\theta + \phi\}$ has period $q_n$ in $x$ and*

$$\min_{x \in \mathbb{Z}} \{x\theta + \phi\} = \min_{0 \leq x < q_n} \{x\theta + \phi\} = \{\kappa_n\theta + \phi\} < \frac{1}{q_n};$$

*and $t = 0$.*

*Proof.* Set $\alpha_n = \lfloor q_n\phi \rfloor$. Consider the case $\theta \neq p_n/q_n$. We recall that

$$\frac{p_2}{q_2} < \frac{p_4}{q_4} < \cdots < \theta < \cdots < \frac{p_3}{q_3} < \frac{p_1}{q_1} \quad \text{and} \quad \left|\theta - \frac{p_n}{q_n}\right| < \frac{1}{q_n q_{n+1}} < \frac{1}{q_n^2}.$$

When $n$ is odd,

$$\frac{p_n x + \alpha_n - 1}{q_n} < -\frac{x}{q_n^2} + x\frac{p_n}{q_n} + \phi < x\theta + \phi < \frac{p_n x + \alpha_n + 1}{q_n}.$$

To minimize $\{x\theta + \phi\}$, we need

$$\begin{aligned} \frac{xp_n + \alpha_n}{q_n} &\in \mathbb{Z} \qquad \text{if} \quad x\theta + \phi \geq \frac{xp_n + \alpha_n}{q_n}, \\ \frac{xp_n + \alpha_n - 1}{q_n} &\in \mathbb{Z} \qquad \text{if} \quad x\theta + \phi < \frac{xp_n + \alpha_n}{q_n}. \end{aligned}$$

And this is possible since $(p_n, q_n) = 1$ and $0 \leq x < q_n$.

Now, since $(q_{n-1}, p_{n-1})$ is a solution of

$$p_n x - q_n y = (-1)^{n-1},$$

$x = (-1)^n q_{n-1}\alpha_n$, $y = (-1)^n p_{n-1}\alpha_n$ is a solution of the diophantine equation

$$p_n x - q_n y = -\alpha_n .$$

Thus, the solutions of this equation are

$$\begin{aligned} x &\equiv (-1)^n q_{n-1} \lfloor q_n \phi \rfloor \mod q_n , \\ y &\equiv (-1)^n p_{n-1} \lfloor q_n \phi \rfloor \mod p_n . \end{aligned}$$

If we find $x$ with $0 \leq x < q_n$, we get

$$\begin{aligned} \kappa_n &= (-1)^n q_{n-1} \lfloor q_n \phi \rfloor - (-1)^n q_n \lfloor q_{n-1} \phi \rfloor \qquad \text{or} \\ \kappa_n &= (-1)^n q_{n-1} \lfloor q_n \phi \rfloor - (-1)^n q_n \lfloor q_{n-1} \phi \rfloor + q_n . \end{aligned}$$

Therefore, when $n$ is odd and $(xp_n + \alpha_n)/q_n \in \mathbb{Z}$,

$$\kappa_n \equiv -\alpha_n q_{n-1} \mod q_n .$$

In like manner, when $n$ is odd and $(xp_n + \alpha_n - 1)/q_n \in \mathbb{Z}$,

$$\kappa_n \equiv -(\alpha_n - 1) q_{n-1} \mod q_n .$$

Similarly, when $n$ is even,

$$\frac{p_n x + \alpha_n}{q_n} < x\theta + \phi < \frac{x}{q_n^2} + x\frac{p_n}{q_n} + \phi < \frac{p_n x + \alpha_n + 2}{q_n} .$$

Thus,

$$\lfloor \kappa_n\theta+\phi \rfloor = \frac{p_n\kappa_n+\alpha_n}{q_n} \quad \text{or} \quad \frac{p_n\kappa_n+\alpha_n+1}{q_n},$$

that is

$$\kappa_n \equiv \alpha_n q_{n-1} \quad \text{or} \quad (\alpha_n+1)q_{n-1} \mod q_n .$$

When $\theta=p_n/q_n$ for some $n\ge 2$,

$$\frac{xp_n+\alpha_n}{q_n} \le x\theta+\phi < \frac{xp_n+\alpha_n}{q_n}+\frac{1}{q_n}.$$

Now

$$\kappa_n \equiv (-1)^n\alpha_n q_{n-1} \mod q_n$$

gives

$$(xp_n+\alpha_n)/q_n \in \mathbb{Z}.$$

*Remark.* If we disallow $x=0$ (in order that van Ravenstein's result [9] is a particular case), Theorem 1 must be modified. In this case

$$\kappa_n = \left\{(-1)^{n-1}\frac{q_{n-1}}{q_n}\right\} q_n$$

when $0\le\{\phi\}<1/q_n$.

Corresponding results for the maximum can be obtained symmetrically: One uses the ceiling instead of the floor in the proof above.

COROLLARY. *Let $\theta$, $\phi$ be any real numbers and let $p_n/q_n$ be the n-th convergent of $\theta$. Then for $n\ge 2$*

$$\max_{0\le x<q_n}\{x\theta+\phi\} = \{\rho_n\theta+\phi\} > 1-\frac{2}{q_n},$$

*where $\rho_n\equiv(-1)^n q_{n-1}(\lfloor q_n\phi\rfloor+t+1) \mod q_n$ and $t=0$ or $(-1)^n$.*

*When $\theta=p_n/q_n$ $(n\ge 2)$, $\{x\theta+\phi\}$ has period $q_n$ in $x$ and*

$$\max_{x\in\mathbb{Z}}\{x\theta+\phi\} = \max_{0\le x<q_n}\{x\theta+\phi\} = \{\rho_n\theta+\phi\} > 1-\frac{1}{q_n};$$

*and $t=0$.*

*Remark.* In the range $0<x<q_n$ in which $x=0$ is disallowed,

$$\rho_n = \left\{(-1)^n\frac{q_{n-1}}{q_n}\right\} q_n$$

when $1-1/q_n\le\{\phi\}<1$.

## 3. The minimum value of $\|x\theta+\phi\|$

Combining the results for minimum and maximum, we get the following theorem:

THEOREM 2. *Let $\theta$, $\phi$ be any real numbers and let $p_n/q_n$ be the $n$-th convergent of $\theta$. Then for $n\geq 2$*

$$\min_{0\leq x<q_n}\|x\theta+\phi\|=\|\xi_n\theta+\phi\|,$$

*where $\xi_n\equiv(-1)^n q_{n-1}(\lfloor q_n\phi\rfloor+t) \mod q_n$ and $t=-1,\ 0$ or $1$ if $n$ is odd; and $t=0,\ 1$ or $2$ if $n$ is even.*

*When $\theta=p_n/q_n\quad(n\geq 2)$, $\|x\theta+\phi\|$ has period $q_n$ in $x$ and*

$$\min_{x\in\mathbb{Z}}\|x\theta+\phi\|=\min_{0\leq x<q_n}\|x\theta+\phi\|=\|\xi_n\theta+\phi\|;$$

*and $t=0$ or $1$.*

*Furthermore, if $a_{n+1}\geq 2$, then for any real $\theta$ and $\phi$ we have $t=0$ or $1$. And for any real $\theta$ and $\phi$ and for infinitely many $n$ with $n\geq 2$ we have $t=0$ or $1$.*

*Proof.* The first two assertions are obvious. It suffices to deal with the case when $n$ is odd. Then from the first assertion there are three cases, say

$$\xi_n^{(0)}=\left\{-\frac{q_{n-1}}{q_n}\alpha_n\right\}q_n,$$
$$\xi_n^{(-1)}=\left\{-\frac{q_{n-1}}{q_n}(\alpha_n-1)\right\}q_n,$$
$$\xi_n^{(1)}=\left\{-\frac{q_{n-1}}{q_n}(\alpha_n+1)\right\}q_n.$$

We will show that $\|\xi_n^{(-1)}\theta+\phi\|\geq\|\xi_n^{(0)}\theta+\phi\|$ if $a_{n+1}\geq 2$. Now

$$\begin{aligned}\|\xi_n^{(-1)}\theta+\phi\|=\{\xi_n^{(-1)}\theta+\phi\}&=(\xi_n^{(-1)}\theta+\phi)-\frac{p_n\xi_n^{(-1)}+\alpha_n-1}{q_n}\\&=-\left(\frac{p_n}{q_n}-\theta\right)\xi_n^{(-1)}+\left(\phi-\frac{\alpha_n}{q_n}\right)+\frac{1}{q_n}.\end{aligned}$$

Since

$$-\frac{1}{q_{n+1}}<-\left(\frac{p_n}{q_n}-\theta\right)\xi_n^{(-1)}<0\qquad\text{and}\qquad 0\leq\phi-\frac{\alpha_n}{q_n}<\frac{1}{q_n},$$

it follows that

$$0<\frac{1}{q_n}-\frac{1}{q_{n+1}}<\|\xi_n^{(-1)}\theta+\phi\|<\frac{2}{q_n}. \tag{1}$$

On the other hand

$$\begin{aligned}\|\xi_n^{(0)}\theta+\phi\| &= \min\left[\{\xi_n^{(0)}\theta+\phi\},1-\{\xi_n^{(0)}\theta+\phi\}\right]\\ &= \begin{cases} -(p_n/q_n-\theta)\xi_n^{(0)}+(\phi-\alpha_n/q_n), & \text{if this value is positive,}\\ (p_n/q_n-\theta)\xi_n^{(0)}-(\phi-\alpha_n/q_n), & \text{otherwise.}\end{cases}\end{aligned}$$

Since

$$-\frac{1}{q_n}<-\left(\frac{p_n}{q_n}-\theta\right)\xi_n^{(0)}+\left(\phi-\frac{\alpha_n}{q_n}\right)<\frac{1}{q_{n+1}},$$

we have

$$0<\|\xi_n^{(0)}\theta+\phi\|<\frac{1}{q_{n+1}}. \tag{2}$$

From (1) and (2) the fact $q_{n+1}=a_{n+1}q_n+q_{n-1}\geq 2q_n$ now yields the conclusion.

Using Hurwitz's theorem for infinitely many $p_n/q_n$, that is for infinitely many $n$

$$\frac{1}{q_n}-\frac{1}{\sqrt{5}q_n}<\|\xi_n^{(-1)}\theta+\phi\|<\frac{2}{q_n} \quad \text{and} \quad 0<\|\xi_n^{(0)}\theta+\phi\|<\frac{1}{\sqrt{5}q_n},$$

we obtain the last part of Theorem 2.

*Remark.* In the range $0<x<q_n$, in which $x=0$ is disallowed,

$$\xi_n=\left\{\frac{q_{n-1}}{q_n}\right\}q_n \quad \text{or} \quad \left\{-\frac{q_{n-1}}{q_n}\right\}q_n$$

when $0\leq\{\phi\}<1/q_n$ or $1-1/q_n\leq\{\phi\}<1$.

## 4. The third possibility in Theorem 2

If $\theta$ is irrational and $a_{n+1}=1$ for some $n(\geq 2)$, there may be the third possibility, namely $t=-1$ when $n$ is odd; $t=2$ when $n$ is even. In this section we describe $\theta$ and $\phi$ which give this third possibility.

Once again it suffices to consider just odd $n$. Again set $\alpha_n = \lfloor q_n\phi \rfloor$ and $\xi_n^{(t)} \equiv -q_{n-1}(\alpha_n + t) \mod q_n$, where $t = -1,\ 0,\ 1$. Then, the third possibility occurs if

$$\|\xi_n^{(-1)}\theta + \phi\| < \|\xi_n^{(0)}\theta + \phi\|,\ \|\xi_n^{(1)}\theta + \phi\|. \tag{3}$$

But (3) is equivalent to

$$x\theta + \phi - \frac{p_n x + \alpha_n - 1}{q_n} < \frac{1}{2q_n}$$

or

$$\left(\frac{p_n}{q_n} - \theta\right) x > \frac{1}{2q_n} + \left(\phi - \frac{\alpha_n}{q_n}\right), \tag{4}$$

where $x = \xi_n^{(-1)}$. Since $\phi - \alpha_n/q_n \geq 0$ and $0 \leq x \leq q_n - 1$, $\theta$ must satisfy

$$\frac{p_n}{q_n} - \theta > \frac{c}{2q_n(q_n - 1)}, \tag{5}$$

where $c$ is a constant with $1 < c < 2$, which we determine later. Let $a_{n+1} = 1$, $a_n$ and $a_{n+2}$ be large positive integers, say $L_n$ and $L_{n+2}$, respectively for odd $n$ in the continued fraction expansion of $\theta$. Then

$$p_n - q_n\theta = \frac{1}{\theta_{n+1}q_n + q_{n-1}} = \frac{1}{(1 + 1/(L_{n+2} + \epsilon))q_n + q_{n-1}} > \frac{1}{2(q_n - 1)},$$

where $\theta_{n+1} = [a_{n+1},\ a_{n+2},\ \cdots]$ and $0 < \epsilon < 1$. Thus, this $\theta$ satisfies (5). By comparing with (5), we get

$$1 < c < 2\left(\frac{q_n - 1}{q_{n+1} + q_n/(L_{n+2} + \epsilon)}\right) < 2.$$

Since we may take $L_{n+2}$ as large as we like, the condition on $c$ is

$$1 < c < 2\left(\frac{q_n - 1}{q_{n+1}}\right) < 2.$$

This condition will be sharpened in (8).

We still have to make $x \equiv -q_{n-1}(\alpha_n - 1) \mod q_n$ large enough and have to make $\phi - \alpha_n/q_n \geq 0$ small enough to satisfy (4). First, we select $x$, and then may choose $\alpha_n = k_n p_n + 1$, $x = q_n - k_n$. To satisfy the conditions of (4), the value of $k_n$ must be in the range

$$k_n < q_n - \frac{(1 + 2\{q_n\phi\})(q_n - 1)}{c} \leq q_n - \frac{q_n - 1}{c}.$$

Together with the condition for $c$ we get

$$1 \le k_n < q_n - \frac{q_{n+1}}{2} = \frac{q_n - q_{n-1}}{2} < \frac{q_n}{2}.$$

This condition will be sharpened in (7). We conclude that $x$ is as large as

$$\frac{q_{n+1}}{2} = \frac{q_n + q_{n-1}}{2} < x \le q_n - 1.$$

Next, we select $\phi$. We can say that the condition (3) is

$$\xi_n^{(-1)}\theta + \phi - \frac{p_n\xi_n^{(-1)} + \alpha_n - 1}{q_n} < \frac{p_n\xi_n^{(0)} + \alpha_n}{q_n} - (\xi_n^{(0)}\theta + \phi),$$

where the two fractions are integers. When $\xi_n^{(-1)} = q_n - k_n$ we have $\xi_n^{(0)} = q_n - q_{n-1} - k_n$ since $q_n > L_n q_{n-1}$ and given the conditions on $k_n$. Recalling the definition of $\alpha_n$, we get

$$\frac{\alpha_n}{q_n} < \phi < \frac{q_{n-1}\theta - p_{n-1}}{2} - (q_n - k_n)\theta + p_n.$$

This is equivalent to

$$\begin{aligned}\left\{\frac{\alpha_n}{q_n}\right\} < \{\phi\} &< \left\{\frac{q_{n-1}\theta - p_{n-1}}{2} - (q_n - k_n)\theta\right\} \\ &= \frac{q_{n-1}\theta - p_{n-1}}{2} + (p_n - q_n\theta) + \{k_n\theta\},\end{aligned}$$

whence

$$\{k_n\theta\} < 1 - \frac{q_{n-1}\theta - p_{n-1}}{2} - (p_n - q_n\theta)\left(< 1 - \frac{1}{2q_{n+1}} - \frac{1}{q_{n+1} + q_n}\right). \tag{6}$$

To see that, we note

$$\frac{1}{q_n} \le k_n\left(\frac{p_n}{q_n} - \theta\right) + \frac{1}{q_n} < \frac{q_n - q_{n+1}/2}{q_n q_{n+1}} + \frac{1}{q_n} < \frac{3}{2q_n},$$

whence we have

$$\frac{q_{n-1}\theta - p_{n-1}}{2} + (p_n - q_n\theta) + \{k_n\theta\} = \left\{\frac{q_{n-1}\theta - p_{n-1}}{2} - (q_n - k_n)\theta + p_n\right\}$$

and

$$\left\{\frac{\alpha_n}{q_n}\right\} = k_n\left(\frac{p_n}{q_n} - \theta\right) + \frac{1}{q_n} + \{k_n\theta\}.$$

Among these $k_n$ satisfying (6) we choose $k_n$ such that the left hand side of the inequality defining the range of $\phi$ is less than the right hand side, so

$$k_n\left(\frac{p_n}{q_n} - \theta\right) + \frac{1}{q_n} < \frac{q_{n-1}\theta - p_{n-1}}{2} + (p_n - q_n\theta),$$

that is,

$$1 \leq k_n < q_n - \frac{q_{n-1}}{2} - \frac{1}{2(p_n - q_n\theta)} = \left(1 - \frac{\theta_{n+1}}{2}\right)q_n - q_{n-1} < \frac{q_n}{2} - q_{n-1}. \tag{7}$$

When $n \geq 3$, it follows from (5) that such $k_n$ exist if we take $c$ as

$$1 < \frac{2(q_n - 1)}{2q_n - q_{n-1}} < c < \frac{2(q_n - 1)}{q_{n+1}} < 2. \tag{8}$$

We remark that $2(q_n - 1)/(2q_n - q_{n-1})$ is near to 1 and $2(q_n - 1)/q_{n+1}$ is near to 2 because $L_n$ is large.

When there is a $k_n$ satisfying (7) and

$$\left\{\frac{k_n p_n + 1}{q_n}\right\} < \left\{\frac{q_{n-1}\theta - p_{n-1}}{2} - (q_n - k_n)\theta\right\},$$

we can select $\phi$ as

$$\left\{\frac{k_n p_n + 1}{q_n}\right\} < \{\phi\} < \left\{\frac{q_{n-1}\theta - p_{n-1}}{2} - (q_n - k_n)\theta\right\}. \tag{9}$$

The $\theta$ and $\phi$ so selected give the third possibility for odd $n$. When $n$ is even, instead of (9) we can choose $\phi$ as

$$\left\{\frac{q_{n-1}\theta - p_{n-1}}{2} - (q_n - k_n)\theta\right\} < \{\phi\} < \left\{\frac{k_n p_n - 1}{q_n}\right\}, \tag{9'}$$

By these steps, we can always find $\theta$ and $\phi$ to give the third possibility for any odd $n$, or any even $n$ when $a_{n+1} = 1$, by taking $L_n$ and $L_{n+2}$ large enough so that there is an integer $k_n$ satisfying the conditions (7) and (9) (or (9′)). Moreover, it is possible to obtain $\phi$ giving the third possibility for infinitely many odd (or even) $n$'s.

We now show the way to find the next interval such that $I_{n+2} \subset I_n$:

Take an increasing sequence of constants $\{\delta_n\}$ with $0 < \delta_n < 1/3$ for all positive integers $n$. For each odd $n$ ($\geq 3$) we shall always choose

$$L_{n+2} \geq L_n \geq \max\left(L, \frac{1}{\delta_{n+1} - \delta_n}, \frac{1}{\delta_n - \delta_{n-1}}\right) \qquad \text{and} \qquad k_n < \delta_n q_n ,$$

where $L$ is an integer with $L \geq 8$ and $k_n$ satisfying condition (7). Put

$$\alpha_n = \left\{\frac{k_n p_n}{q_n}\right\} + \frac{1}{q_n} \qquad \text{and} \qquad \beta_n = \alpha_n + \frac{1}{m q_n}, \tag{10}$$

where using the notation of the ceiling

$$m = \lceil \frac{6L^2 + 8L + 2}{L^2 - 7L - 2} \rceil ,$$

and we assume that

$$\alpha \geq 2/q_n \qquad \text{and} \qquad \beta_n \leq 1 - 1/2q_n . \tag{11}$$

Then, comparing with (7),

$$\left\{\frac{k_n p_n + 1}{q_n}\right\} = \alpha_n < \beta_n < \left\{\frac{q_{n-1}\theta - p_{n-1}}{2} - (q_n - k_n)\theta\right\}$$

and $\beta_n \leq 1 - 1/2q_n$ yields (6). Let $I_n$ be the interval of (10). Note that if $I_{n+2} \subset I_n$ then $I_{n+2}$ will automatically satisfy condition (11).

We examine the fractional parts of various integer multiples of $p_{n+2}/q_{n+2}$ with an eye to combining them to find an integer $k_{n+2}$ such that

$$\left\{k_{n+2}\frac{p_{n+2}}{q_{n+2}}\right\} + \frac{1}{q_{n+2}}$$

is in a subinterval of $I_n$. First,

$$0 < \frac{p_n}{q_n} - \frac{p_{n+2}}{q_{n+2}} < \frac{p_n}{q_n} - \frac{p_{n+1}}{q_{n+1}} = \frac{1}{q_n q_{n+1}},$$

so

$$0 < \left\{k_n \frac{p_n}{q_n}\right\} - \left\{k_n \frac{p_{n+2}}{q_{n+2}}\right\} < \frac{\delta_n}{q_{n+1}} . \tag{12}$$

We also have

$$\left\{q_{n+1}\frac{p_{n+2}}{q_{n+2}}\right\} = q_{n+1}\frac{p_{n+2}}{q_{n+2}} - p_{n+1} = \frac{1}{q_{n+2}}, \tag{13}$$

$$\begin{aligned}\left\{-q_n\frac{p_{n+2}}{q_{n+2}}\right\} &= \left\{\left(-q_n\frac{p_{n+2}}{q_{n+2}} + q_n\frac{p_{n+1}}{q_{n+1}}\right) - \left(q_n\frac{p_{n+2}}{q_{n+2}} - p_n\right)\right\}\\ &= -\frac{q_n}{q_{n+1}q_{n+2}} + \frac{1}{q_{n+1}},\end{aligned}$$

and

$$\left\{q_{n-1}\frac{p_{n+2}}{q_{n+2}}\right\} = \frac{1}{q_{n+1}} + \frac{1}{q_{n+2}} - \frac{q_n}{q_{n+1}q_{n+2}}, \tag{14}$$

where the third equality can be obtained by adding the first two. From (12) and (14) we have

$$\begin{aligned}-\frac{1}{q_{n+2}} < \left\{k_n\frac{p_n}{q_n}\right\} + \frac{1}{q_n} - \left(\left\{(k_n + q_{n-1})\frac{p_{n+2}}{q_{n+2}}\right\} + \frac{1}{q_{n+2}}\right)\\ < \frac{\delta_n}{q_{n+1}} + \frac{1}{q_n} - \frac{1}{q_{n+1}} < \frac{\delta_{n+1}}{q_{n+1}}.\end{aligned}$$

Now let $l$ be the non-negative integer satisfying

$$\frac{l}{q_{n+2}} < \alpha_n - \left\{(k_n + q_{n-1})\frac{p_{n+2}}{q_{n+2}}\right\} < \frac{l+1}{q_{n+2}}.$$

Using (13) we get

$$\begin{aligned}\alpha_n &< \left\{(k_n + q_{n-1})\frac{p_{n+2}}{q_{n+2}}\right\} + \frac{l}{q_{n+2}} + \frac{1}{q_{n+2}}\\ &= \left\{(k_n + q_{n-1})\frac{p_{n+2}}{q_{n+2}}\right\} + \left\{lq_{n+1}\frac{p_{n+2}}{q_{n+2}}\right\} + \frac{1}{q_{n+2}}\\ &= \left\{k_{n+2}\frac{p_{n+2}}{q_{n+2}}\right\} + \frac{1}{q_{n+2}}\end{aligned} \tag{15}$$

and

$$\begin{aligned}&\left\{k_{n+2}\frac{p_{n+2}}{q_{n+2}}\right\} + \frac{1}{q_{n+2}} + \frac{1}{mq_{n+2}}\\ &\quad= \left\{(k_n + q_{n-1})\frac{p_{n+2}}{q_{n+2}}\right\} + \frac{l}{q_{n+2}} + \frac{1}{q_{n+2}} + \frac{1}{mq_{n+2}}\\ &\quad< \alpha_n + \frac{1}{q_{n+2}} + \frac{1}{mq_{n+2}} < \beta_n,\end{aligned} \tag{16}$$

where $k_{n+2} = k_n + q_{n-1} + lq_{n+1}$ for some $l$ with

$$0 \le l \le \frac{\delta_{n+1}q_{n+2}}{q_{n+1}}.$$

Now

$$k_{n+2} < (l+1)q_{n+1} \le \delta_{n+2}q_{n+2}.$$

Having chosen $L_{n+2}$ and $k_{n+2}$ we define $\alpha_{n+2}$ and $\beta_{n+2}$ analogously to $\alpha_n$ and $\beta_n$ in (10). In view of (15) and (16) we have $I_{n+2} \subset I_n$.

Therefore,

$$\theta = [0,\ 1,\ 1,\ L_3,\ 1,\ L_5,\ \cdots,\ 1,\ L_{2i-1},\ 1,\ L_{2i+1},\ \ldots]$$
$$\phi \in \cdots I_{2i+1} \subset I_{2i-1} \subset \cdots \subset I_5 \subset I_3 \subset [0,\ 1]$$

give the third possibility for each odd $n = 3,\ 5,\ \cdots,\ 2i-1,\ 2i+1,\ \cdots$.

EXAMPLES.

$$\theta = \frac{\sqrt{35}+4}{19} = [0,\ 1,\ 1,\ 10,\ 1,\ 10,\ \cdots] \quad (= 0.52189893\cdots)$$
$$\phi = 0.573777\ldots$$

give the third possibility for $n = 3,\ 5,\ 7,\ 9,\ \cdots$.

$$\theta = \sqrt{35} - 5 = [0,\ 1,\ 10,\ 1,\ 10,\ \cdots]\ (= 5.91607978\cdots)$$
$$\phi = 0.81678\cdots$$

give the third possibility for $n = 2,\ 4,\ 6,\ 8,\ \cdots$.

## 5. The sorting of $\{x\theta + \phi\}$

We can solve the sorting problem by using the former results. When $\theta$ is irrational, let $q_n$ be the denominator of the $n$-th convergent of $\theta$, as above. When $\theta$ is rational, let $\theta = p_n/q_n$, where $p_n$ and $q_n$ ($> 0$) are coprime. Let $(\{u_j\theta + \phi\})$, $j = 1,\ 2,\ \cdots,\ N$, be the ordered sequence of fractional parts. That is, $\{u_1,\ u_2,\ \cdots,\ u_N\} = \{0,\ 1,\ 2,\ \cdots,\ N-1\}$, and $\{u_j\theta + \phi\} < \{u_{j+1}\theta + \phi\}$.

When $\theta$ is irrational and $\phi = 0$, the following result is well known:

LEMMA. *When $N = q_n$, for $j = 1, 2, \ldots, q_n$*

$$u_j = \left\{ \frac{(j-1)u_2}{q_n} \right\} q_n ,$$

*where $u_2 = q_{n-1}$ if $n$ is odd, $u_2 = q_n - q_{n-1}$ if $n$ is even.*

*Proof.* This follows immediately from Lemma 2.1 and Theorem 3.3 of [9].

We will sort the fractional part of $x\theta+\phi$, which includes the result above.

THEOREM 3. *When $N = q_n$, for $j = 1, 2, \cdots, q_n$*

$$u_j = \left\{ (-1)^n \frac{q_{n-1}}{q_n} (\lfloor q_n \phi \rfloor + t - j + 1) \right\} q_n ,$$

*where $t = 0$ or $(-1)^n$ and is independent of $j$.*

*Proof.* Set $\alpha_n = \lfloor q_n \phi \rfloor$. When $x\theta + \phi$ is not integer, for each $x$ there exists an integer $t$ satisfying

$$\begin{aligned} \cdots < \frac{xp_n + \alpha_n + t - 1}{q_n} < \frac{xp_n + \alpha_n + t}{q_n} < x\theta + \phi \\ < \frac{xp_n + \alpha_n + t + 1}{q_n} < \frac{xp_n + \alpha_n + t + 2}{q_n} < \cdots . \end{aligned}$$

Let $x^{(t)} = x_n^{(t)}$ with $0 \leq x^{(t)} < q_n$ satisfy

$$x^{(t)} p_n + \alpha_n + t \equiv 0 \mod q_n .$$

Then

$$\{x^{(t)}\theta + \phi\} < \{x^{(t-1)}\theta + \phi\} < \{x^{(t-2)}\theta + \phi\} < \cdots .$$

From Theorem 1 $t$ must be 0 or $(-1)^n$.

When $x\theta + \phi$ is integer, there exists $x$ satisfying

$$\frac{xp_n + \alpha_n}{q_n} = x\theta + \phi .$$

In particular, $t = 0$ when $\theta$ is rational, or if $0 \leq \phi < 1/q_n$.

## 6. Application to the inhomogeneous Beatty sequence

In this section, we introduce the easiest method for getting the inhomogeneous Beatty sequences in every case. Indeed, one easily obtain the sequence by the use our theorems once one knows the corresponding homogeneous Beatty sequences; see the examples in §7.

Consider the sequence of differences, that is the characteristic sequence

$$f(n;\theta,\phi) = \lfloor (n+1)\theta+\phi \rfloor - \lfloor n\theta+\phi \rfloor \qquad n = 1,\ 2,\ \dots\ .$$

First, let $\theta$ be rational. Then, the following theorem is essential.

THEOREM 4. *Let $\theta$ be a rational number, $\phi$ a real number, and let $\kappa$ be an integer satisfying*

$$\min_{x\in\mathbb{Z}}\{x\theta+\phi\} = \{\kappa\theta+\phi\}\,.$$

*Then for all integer $n$ we have $f(n+\kappa;\theta,\phi) = f(n;\theta,0)$.*

*Proof.* Plainly

$$\lfloor (n+\kappa)\theta+\phi \rfloor + \{(n+\kappa)\theta+\phi\} = (n+\kappa)\theta+\phi = n\theta + \lfloor \kappa\theta+\phi \rfloor + \{\kappa\theta+\phi\}\,.$$

Because $0 \le \{(n+\kappa)\theta+\phi\} - \{\kappa\theta+\phi\} < 1$, we have

$$\lfloor n\theta \rfloor = \lfloor (n+\kappa)\theta+\phi \rfloor - \lfloor \kappa\theta+\phi \rfloor\,.$$

Since also $\lfloor (n+1)\theta \rfloor = \lfloor (n+\kappa+1)\theta+\phi \rfloor - \lfloor \kappa\theta+\phi \rfloor$, the result follows.

Next, we consider the case $\theta$ irrational. The main theorem is Theorem 5, which will follow from Proposition 1 and Proposition 2. The method of proof is similar to that of the rational case. As seen in the assertions, we need not necessarily take an integer $x$ which minimizes the fractional part of $x\theta+\phi$, but it is surely the best way to use such an integer.

PROPOSITION 1. *Let $\theta$ be an irrational number with continued fraction expansion*

$$\theta = [a_0, a_1, a_2, \cdots]\,,$$

*and let $\phi$ be a real number. For $i = 0,1,2,\dots$ denote the convergents of $\theta$ by $p_i/q_i$.*

*If $a_{i+1}/q_{i+1} \ge \{k\theta+\phi\}$ for a non-negative integer $k$, then for all integer $n$ with $-q_i+1 \le n \le q_i-2$ we have $f(n+k;\theta,\phi) = f(n;\theta,0)$.*

This proof depends on the following lemmata:

LEMMA 1. *Let $\theta$ be an irrational number and $\phi$ any real number. If $a_{i+1}/q_{i+1} \geq \{k\theta+\phi\}$ for a non-negative integer $k$, then for every positive odd integer $i$*

$$\{(q_i-q_{i-1})\theta+(k\theta+\phi)\} \geq \{(q_i-q_{i-1})\theta\},$$

$$\{q_{i-1}\theta-(k\theta+\phi)\} \leq \{q_{i-1}\theta\};$$

*and for every positive even integer $i$*

$$\{q_{i-1}\theta+(k\theta+\phi)\} \geq \{q_{i-1}\theta\},$$

$$\{(q_i-q_{i-1})\theta-(k\theta+\phi)\} \leq \{(q_i-q_{i-1})\theta\}.$$

*Proof.* When $i$ is odd,

$$(q_i-q_{i-1})\frac{p_i}{q_i}=p_i-p_{i-1}-\frac{1}{q_i} \quad \text{and} \quad 0<\frac{p_i}{q_i}-\theta<\frac{1}{q_iq_{i+1}}.$$

Then,

$$\{(q_i-q_{i-1})\theta+(k\theta+\phi)\}=\left\{-(q_i-q_{i-1})\left(\frac{p_i}{q_i}-\theta\right)-\frac{1}{q_i}+(k\theta+\phi)\right\},$$

$$\{(q_i-q_{i-1})\theta\}=\left\{-(q_i-q_{i-1})\left(\frac{p_i}{q_i}-\theta\right)-\frac{1}{q_i}\right\}.$$

Since

$$1-\frac{1}{q_i}>\{(q_i-q_{i-1})\theta\}>1-\frac{q_i-q_{i-1}}{q_iq_{i+1}}-\frac{1}{q_i},$$

together with

$$(q_i-q_{i-1})\left(\frac{p_i}{q_i}-\theta\right)+\frac{1}{q_i}>\frac{1}{q_i}\frac{a_{i+1}}{q_{i+1}}\geq\{k\theta+\phi\}$$

the first inequality follows.

When $i$ is even,

$$q_{i-1}\frac{p_i}{q_i}=p_{i-1}-\frac{1}{q_i} \quad \text{and} \quad 0<\theta-\frac{p_i}{q_i}<\frac{1}{q_iq_{i+1}}.$$

Then,

$$\{q_{i-1}\theta+(k\theta+\phi)\}=\left\{q_{i-1}\left(\theta-\frac{p_i}{q_i}\right)-\frac{1}{q_i}+(k\theta+\phi)\right\},$$

$$\{q_{i-1}\theta\} = \left\{q_{i-1}\left(\theta - \frac{p_i}{q_i}\right) - \frac{i}{q_i}\right\}.$$

Since

$$1 - \frac{1}{q_i} < \{q_{i-1}\theta\} < 1 + \frac{q_{i-1}}{q_i q_{i+1}} - \frac{1}{q_i} < 1,$$

together with

$$\frac{1}{q_i} = \frac{a_{i+1}}{q_{i+1}} + \frac{q_{i-1}}{q_i q_{i+1}}\{k\theta + \phi\} + q_{i-1}\left(\theta - \frac{p_i}{q_i}\right),$$

the third inequality is proved.

The second and the fourth inequalities are proved similarly.

LEMMA 2. *If $a_{i+1}/q_{i+1} \geq \{k\theta + \phi\}$ for a non-negative integer $k$, then for every integer $n$ with $|n| < q_i$,*

$$\{n\theta + (k\theta + \phi)\} \geq \{n\theta\}.$$

*Proof.* First let $n > 0$. By the result of [9] and the first and the third inequalities in Lemma 1, Lemma 2 holds for $n$ which maximizes the fractional part when $0 < n < q_i$ for integers $i \geq 2$, whence together with the condition $a_{i+1}/q_{i+1} \geq \{\kappa\theta + \phi\}$ the lemma holds for the other positive $n$.

When $n = 0$, the result is trivial.

Finally, if $n < 0$, by [9] and the second and the fourth inequalities in Lemma 1, for $0 < n' < q_i$,

$$\{n'\theta - (k\theta + \phi)\} \leq \{n'\theta\}.$$

Hence, together with

$$\{-n\theta + (k\theta + \phi)\} = 1 - \{n\theta - (k\theta + \phi)\} \quad \text{and} \quad \{-n\theta\} = 1 - \{n\theta\},$$

we have the claim.

*Proof of Proposition* 1. If $\{k\theta + \phi\} = 0$, then for any integer $n$

$$f(n + k; \theta, \phi) = \lfloor (n+1)\theta + \{k\theta + \phi\} \rfloor - \lfloor n\theta + \{k\theta + \phi\} \rfloor = f(n; \theta, 0).$$

By Lemma 2, $\{n\theta + (k\theta + \phi)\} = \{n\theta + \{k\theta + \phi\}\}$ is never located on the the number line to the left of $\{n\theta\}$. Therefore, if $\{k\theta + \phi\} \neq 0$, for each integer $n$ with $|n| < q_i$

$$\{n\theta + (k\theta + \phi)\} - \{n\theta\} = \{k\theta + \phi\}.$$

This gives

$$f(n+k;\theta,\phi)=f(n;\theta,0)\,.$$

As the inequality of Lemma 2 may not hold for $n=q_i$, there is the possibility that

$$f(q_i-1+k;\theta,\phi)\neq f(q_i-1;\theta,0)\,.$$

PROPOSITION 2. *Let $\theta$ be an irrational number with continued fraction expansion*

$$\theta=[a_0,a_1,a_2,\dots]\,,$$

*and let $\phi$ be a real number. Denote by $p_i/q_i$ $(i=1,2,\cdots)$ the $i$-th convergent of the continued fraction expansion of $\theta$. Let $\kappa_l$ be a non-negative integer satisfying*

$$\min_{0\le x<q_l}\{x\theta+\phi\}=\{\kappa_l\theta+\phi\}\,.$$

*If $l\ge i$ for an integer $i$ with $i\ge 2$, then for all integers $n$ with*

$$-\kappa_l\le n\le q_i-\kappa_l-2$$

*we have $f(n+\kappa_l;\theta,\phi)=f(n;\theta,0)$.*

*Proof.* This is similar to the proof of Theorem 4.

Now, we combine Propositions 1 and 2 to get the main theorem for an irrational $\theta$.

THEOREM 5. *Let $\theta$ be an irrational number with continued fraction expansion*

$$\theta=[a_0,a_1,a_2,\dots]\,,$$

*and $\phi$ a real number. Denote by $p_i/q_i$ $(i=1,2,\cdots)$ the $i$-th convergent of the continued fraction expansion of $\theta$. Let $\kappa_l$ be a non-negative integer satisfying*

$$\min_{0\le x<q_l}\{x\theta+\phi\}=\{\kappa_l\theta+\phi\}\,.$$

*If $a_{i+1}/q_{i+1}\ge\{\kappa_l\theta+\phi\}$ for $l\ge i\ge 2$, then for all integers $n$ with $0\le n\le q_i-2+\kappa_l$,*

$$f(n;\theta,\phi)=f(n-\kappa_l;\theta,0)\,.$$

*Remark.* We don't have to care for the mysterious condition $a_{i+1}/q_{i+1} \geq \{\kappa_l\theta + \phi\}$ as long as $a_{i+1} \geq 2$ $(i = l-1, l-2)$ or $i \leq l-3$. For, always $a_{i+1}/q_{i+1} \geq 2/q_l > \{\kappa_l\theta + \phi\}$ in these cases.

## 7. Examples

Let $\theta = \sqrt{2}$, $\phi = \sqrt{3}$. Since

$$\sqrt{2} = [1, 2, 2, \ldots] \qquad \text{and} \qquad q_1, q_2, \ldots = 2, 5, 12, 29, 70, 169, \ldots\ ,$$

by Theorem 1

$$\kappa_4 = \left\{(-1)^4 \frac{12}{29} \lfloor 29\sqrt{3} \rfloor\right\} \cdot 29 = 20 \qquad \text{or}$$
$$\kappa_4 = \left\{(-1)^4 \frac{12}{29} \left(\lfloor 29\sqrt{3} \rfloor + (-1)^4\right)\right\} \cdot 29 = 3\,.$$

From

$$\{20 \cdot \sqrt{2} + \sqrt{3}\} = 0.016322 \cdots \qquad \text{and} \qquad \{3 \cdot \sqrt{2} + \sqrt{3}\} = 0.9746914 \cdots\ ,$$

we get $\kappa_4 = 20$.

Next, since by Theorem 5

$$\frac{2}{70} = 0.0285714 \cdots > \left\{20 \cdot \sqrt{2} + \sqrt{3}\right\}\,,$$

we have

$$f(n; \sqrt{2}, \sqrt{3}) = f(n - 20; \sqrt{2}, 0) \quad \text{for } 0 \leq n \leq 47\,.$$

For example, according to the method of Fraenkel et. al. [4], as

$$f(1; \sqrt{2}, 0) = 1, \quad f(2; \sqrt{2}, 0) = 2 \qquad \text{and} \qquad T = (2, 5, 12, 29, 70, \ldots)\,,$$

we get

$$\begin{aligned}
T_1 &= (12)^\infty\,, \\
T_2 &= (T_1, 5)^\infty = (12121)^\infty\,, \\
T_3 &= (T_1, 12)^\infty = (121211212112)^\infty\,, \\
T_4 &= (T_1, 29)^\infty = (12121121211212121121211212121)^\infty\,.
\end{aligned}$$

Furthermore,

$$f(0;\sqrt{2},0) = 1, \quad f(-1;\sqrt{2},0) = 2 \qquad \text{and} \qquad f(-n;\theta,0) = f(n-1;\theta,0)\,.$$

Thus, $f(n;\sqrt{2},\sqrt{3})$ for $0 \le n \le 47$, that is $f(n;\sqrt{2},0)$ for $-20 \le n \le 27$, is

$$21121212112121121212\underline{1}121211212112121211212112121\,,$$

where the underline indicates the place $f(0;\sqrt{2},0)$.

Let $\theta = 36/25$, $\phi = \sqrt{5}$. Since

$$36/25 = [1,2,3,1,2] \qquad \text{and} \qquad q_1, q_2, q_3, q_4 = 2, 7, 9, 25\,,$$

by Theorem 1

$$\kappa_4 = \left\{(-1)^4 \frac{9}{25} \lfloor 25\sqrt{5} \rfloor \right\} \cdot 25 = 20\,.$$

Next, by Theorem 4 we have

$$f(n;36/25,\sqrt{5}) = f(n-20;36/25,0) \quad \text{for all integer } n\,.$$

For example, according to the method of Fraenkel et. al. [4], as

$$f(1;36/25,\sqrt{5}) = 1, \quad f(2;36/25,\sqrt{5}) = 2 \qquad \text{and} \qquad T = (2,7,9,25)\,,$$

we get

$$\begin{aligned}
T_1 &= (12)^\infty\,,\\
T_2 &= (T_1,7)^\infty = (1212121)^\infty\,,\\
T_3 &= (T_1,9)^\infty = (121212112)^\infty\,,\\
T_4 &= (T_1,25)^\infty = (1212121121212121121212121)^\infty\,.
\end{aligned}$$

Therefore, $f(n;36/25,\sqrt{5})$ for $n \in \mathbb{Z}$ is

$$\cdots 1212\underline{\underline{1}}2112121212112121212\underline{1}1212121121212121121212121\ldots\,,$$

where the underline indicates the place $f(0;36/25,0)$ and the double-underline indicates the place $f(0;36/25,\sqrt{5})$.

Of course, we can use the form $36/25 = [1,2,3,1,1]$ to have the same characteristic sequence.

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