F. HALTER-KOCH

WŁADYSŁAW NARKIEWICZ

Polynomial mappings defined by forms with a common factor

<http://www.numdam.org/item?id=JTNB_1992__4_2_187_0>

Polynomial mappings defined
by forms with a common factor

par F. Halter-Koch and W. Narkiewicz

1. Introduction and Preliminaries

For a field K, we denote by $\widehat{K}$ a fixed algebraic closure of K. For a point $\mathbf{x} = (x_1, \ldots, x_n) \in \widehat{K}^n$, we set

$$d_K(\mathbf{x}) = \max \left\{ [K(x_i) : K] \mid i = 1, \ldots, n \right\}.$$

A subset $V \subset \widehat{K}^n$ is called *K-homogeneous* if there exist homogeneous polynomials $H_1, \ldots, H_r \in K[X_1, \ldots, X_n]$ such that

$$V = \{\mathbf{x} \in \widehat{K}^n \mid H_1(\mathbf{x}) = \cdots = H_r(\mathbf{x}) = 0\}.$$

For m, $n \in \mathbb{N}$ and polynomials $F_1, \ldots, F_m \in K[X_1, \ldots, X_n]$, we define the polynomial mapping

$$F = (F_1, \ldots, F_m) : \widehat{K}^n \to \widehat{K}^m \text{ by } F(\mathbf{x}) = (F_1(\mathbf{x}), \ldots F_m(\mathbf{x})),$$

and we set

$$\deg_*(F) = \min \{\deg (F_j) \mid j = 1, \ldots, m\},$$
$$\deg^*(F) = \max \{\deg (F_j) \mid j = 1, \ldots, m\}.$$

We consider the following finiteness property, which a field K may have or not:

(P0)

> Given m, $n \in \mathbb{N}$, a K-homogeneous set $V \subset \hat{K}^n$, a subset $\mathfrak{X} \subset V$ and polynomial mappings
> $$F = (F_1, \ldots, F_m), \ G = (G_1, \ldots, G_m) : \hat{K}^n \to \hat{K}^m$$
> defined by homogeneous polynomials F_j, $G_j \in K[X_1, \ldots, X_n]$, with the following properties:
> (1) There exist homogeneous polynomials $F_0, F_1', \ldots, F_m'$ in $K[X_1, \ldots, X_n]$ such that $F_j = F_0 F_j'$ for all $j \in \{1, \ldots, m\}$, and $F_1', \ldots, F_m'$ do not have a common non-trivial zero in V;
> (2) $\deg_*(F) > \deg^*(G) + 2\deg(F_0)$;
> (3) $\{d_K(\mathbf{x}) \mid \mathbf{x} \in \mathfrak{X}\}$ is bounded;
> (4) $F(\mathfrak{X}) \supset G(\mathfrak{X})$;
> (5) $G|\mathfrak{X}$ is injective.
> Then $\mathfrak{X}$ is a finite set.

An algebraically closed or real closed field clearly never has the property (P0). As the main result of this paper, we shall prove that every finitely generated field has the property (P0) (Theorem 3). This is well known if $F_0 = 1$ and additionally either $m = n = 1$ or K is a global field, cf. [5], [6].

In [1] we also dealt with the case $F_0 = 1$; there we considered a property which is somewhat weaker than (P0) and proved it for a larger class of fields. We also discussed the various hypotheses stated in (P0), we outlined the history of the problem, and we gave several further references.

The case $F_0 \neq 1$ was up to now only considered in a rather special case, cf. [7]. The following example (due to the referee) shows that the inequality in (2) cannot be weakened in a trivial way: let K be an infinite field, $m = n = 3$, $V = \hat{K}^3$, $F = (X_1^2 X_3, X_2^2 X_3, X_1 X_2^2 X_3)$ (hence $F_0 = X_3$), $G = (X_1, X_2, X_3)$ and $\mathfrak{X} = \{(x, y, z) \mid x, z \in K^\times\} \subset V$; then (1), (3) and (5) are obviously satisfied, and (4) holds since $(x, y, z) = F(x^{-1}z, x^{-1}z, x^3 z^{-2})$; however, $\deg_*(F) = \deg^*(G) + 2\deg(F_0)$, and $\mathfrak{X}$ is infinite.

The main tools in proving that a field has the property (P0) are the theory of height functions (to be explained in § 2), the theory of places of algebraic function fields (§ 3) and the following set-theoretical Lemma, already proved in [1].

MAIN LEMMA. *Let $\mathfrak{X}$, $\mathfrak{Y}$ be sets and F, $G : \mathfrak{X} \to \mathfrak{Y}$ mappings such that $F(\mathfrak{X}) \supset G(\mathfrak{X})$. Let $f : \mathfrak{X} \to \mathbb{R}$ be a mapping such that $f(\mathfrak{X}) \subset \mathbb{R}$ is discrete, and let $C > 0$ be a real constant with the property that x, $y \in \mathfrak{X}$, $f(x) \geq C$ and $F(x) = G(y)$ implies $f(y) > f(x)$. If*

$$\mathfrak{X}_C = \{x \in \mathfrak{X} \mid f(x) < C\},$$

then $F(\mathfrak{X}_C) \supset G(\mathfrak{X}_C)$. If moreover $\mathfrak{X}_C$ is finite and G is injective, then $\mathfrak{X} = \mathfrak{X}_C$.

2. Heights

In this section, we collate the necessary facts from the theory of height functions, following [8].

Let K be a field, equipped with a family of absolute values

$$\mathbf{M} = (|\cdot|_v)_{v \in M_K}$$

satisfying the product formula. We regard M_K as a set of places of K such that, for every $v \in M_K$, $|\cdot|_v$ is an absolute value defining v; we denote by K_v the completion of K at v.

For a finite extension field L of K, let M_L be the set of all places of L extending those of M_K. If $v \in M_K$, $w \in M_L$ and $w \mid v$, then

$$L \underset{K}{\otimes} K_v = \prod_{w|v} \widetilde{L_w} \,,$$

where $\widetilde{L_w}$ is a finite-dimensional local K_v-algebra with residue field L_w. We define an absolute value $|\cdot|_w$ of L by

$$|x|_w = \left| \mathcal{N}_{L_w/K_v}(x) \right|_v^{n_w} , \quad \text{where} \quad n_w = \frac{[\widetilde{L_w} : K_v]}{[L : K] \cdot [L_w : K_v]} \,.$$

Then $(|\cdot|_w)_{w \in M_L}$ satisfies the product formula.

The family $\mathbf{M} = (|\cdot|_v)_{v \in M_K}$ gives rise to a height function $\mathbf{H} : \hat{K}^n \to \mathbb{R}$, defined for every $n \in \mathbb{N}$, as follows: for $\mathbf{x} = (x_1, \ldots, x_n) \in \hat{K}^n$, let L be a finite extension of K such that $\mathbf{x} \in L^n$, and set

$$\mathbf{H}(\mathbf{x}) = \mathbf{H}(x_1, \ldots, x_n) = \prod_{w \in M_L} \max\{1, |x_1|_w, \ldots, |x_n|_w\} \,.$$

Thanks to the normalization of $|\cdot|_w$, this definition does not depend on the choice of the field L. We call $\mathbf{H} = \mathbf{H_M}$ the *height function associated with* $\mathbf{M}$.

For a monic polynomial $f = X^d + a_{d-1}X^{d-1} + \cdots + a_1 X + a_0 \in \hat{K}[X]$ of degree $d \in \mathbb{N}$, we set

$$\mathbf{H}(f) = \mathbf{H}(a_0, a_1, \ldots, a_{d-1}) \,.$$

LEMMA 1. *Let K be a field, equipped with a family of absolute values satisfying the product formula, and let $\mathbf{H}$ be the associated height function. If $n \in \mathbb{N}$, $\mathbf{x} \in \hat{K}^n$ and $z \in \hat{K}$, then we have*

$$\mathbf{H}(z)\,\mathbf{H}(z\mathbf{x}) \geq \mathbf{H}(\mathbf{x})\,.$$

Proof. We set $\mathbf{x} = (x_1, \ldots, x_n)$ and $L = K(x_1, \ldots, x_n, z)$. For $w \in M_L$, we choose $i_w \in \{1, \ldots, n\}$ such that $|x_{i_w}|_w = \max\{|x_1|_w, \ldots, |x_n|_w\}$; then the assertion is equivalent with

$$\prod_{w \in M_L} \max\{1, |z|_w\} \cdot \prod_{w \in M_L} \max\{1, |x_{i_w} z|_w\} \geq \prod_{w \in M_L} \max\{1, |x_{i_w}|_w\}\,,$$

and hence also with

$$\prod_{\substack{w \in M_L \\ |z|_w > 1}} |z|_w \cdot \prod_{\substack{w \in M_L \\ |z|_w < 1}} \frac{\max\{1, |x_{i_w} z|_w\}}{\max\{1, |x_{i_w}|_w\}} \cdot \prod_{\substack{w \in M_L \\ |z|_w > 1}} \frac{\max\{1, |x_{i_w} z|_w\}}{\max\{1, |x_{i_w}|_w\}} \geq 1\,.$$

Since obviously

$$\prod_{\substack{w \in M_L \\ |z|_w > 1}} \frac{\max\{1, |x_{i_w} z|_w\}}{\max\{1, |x_{i_w}|_w\}} \geq 1\,,$$

and (by the product formula)

$$\prod_{\substack{w \in M_L \\ |z|_w > 1}} |z|_w = \prod_{\substack{w \in M_L \\ |z|_w < 1}} |z|_w^{-1}\,,$$

it is sufficient to observe that $0 < a < 1$ and $b > 0$ implies $\max\{1, ab\} \geq a \max\{1, b\}$. $\square$

The next Lemma gives information about the behaviour of heights under polynomial mappings.

LEMMA 2. *Let K be a field, equipped with a family of absolute values satisfying the product formula, and let $\mathbf{H}$ be the associated height function.*

Let $F = (F_1, \ldots, F_m) : \hat{K}^n \to \hat{K}^m$ *be a polynomial mapping, defined by homogeneous polynomials* $F_1, \ldots, F_m \in K[X_1, \ldots, X_n]$, $d_* = \deg_*(F)$ *and* $d^* = \deg^*(F)$.

i) *There exists a constant $C_1 \in \mathbb{R}_{>0}$ such that, for all $\mathbf{x} \in \hat{K}^n$,*

$$\mathbf{H}(F(\mathbf{x})) \leq C_1 \mathbf{H}(\mathbf{x})^{d^*}.$$

ii) *Let $V \subset \hat{K}^n$ be a K-homogeneous set such that $F_1, \ldots, F_m$ do not have a common non-trivial zero in V. Then there exists a constant $C_2 \in \mathbb{R}_{>0}$ such that, for all $\mathbf{x} \in V$,*

$$\mathbf{H}(F(\mathbf{x})) \geq C_2 \mathbf{H}(\mathbf{x})^{d_*}.$$

Proof.

i) follows almost immediately from the definition of the height function, cf. [8, 2.3].

ii) is proved in [5, Proposition 2]. □

PROPOSITION 1. *Let K be a field, equipped with a family of absolute values satisfying the product formula, and let $\mathbf{H}$ be the associated height function.*

For m, $n \in \mathbb{N}$, let $F = (F_1, \ldots, F_m)$, $G = (G_1, \ldots, G_m) : \hat{K}^n \to \hat{K}^m$ be polynomial mappings defined by homogeneous polynomials F_j, $G_j \in K[X_1, \ldots, X_n]$. Let $F_0 \in K[X_1, \ldots, X_n]$ be a common factor of $F_1, \ldots, F_m$, say $F_j = F_0 F_j'$ where $F_j' \in K[X_1, \ldots, X_n]$, and suppose that $\deg_*(F) > \deg^*(G) + 2\deg(F_0)$. Let $V \subset \hat{K}^n$ be a K-homogeneous set such that $F_1', \ldots, F_m'$ do not have a common non-trivial zero in V.

i) *There exists a constant $C_0 \in \mathbb{R}_{>0}$ such that $\mathbf{x}, \mathbf{y} \in V$, $\mathbf{H}(\mathbf{x}) \geq C_0$ and $F(\mathbf{x}) = G(\mathbf{y})$ implies $\mathbf{H}(\mathbf{y}) > \mathbf{H}(\mathbf{x})$.*

ii) *Let C_0 be a constant satisfying i), $\mathfrak{X} \subset V$ a subset such that $F(\mathfrak{X}) \supset G(\mathfrak{X})$, $G|\mathfrak{X}$ is injective, and $\mathfrak{X}_0 = \{\mathbf{x} \in \mathfrak{X} \mid \mathbf{H}(\mathbf{x}) \leq C_0\}$ is finite. Then $\mathfrak{X}$ is finite.*

Proof.

i) We set $d_0 = \deg(F_0)$, $d^* = \deg^*(G)$, $d_* = \deg_*(F)$, and $F' = (F_1', \ldots, F_m') : \hat{K}^n \to \hat{K}^m$. Then we have $\deg_*(F') = d_* - d_0$, $d_* - 2d_0 > d^*$, and Lemma 1 implies

$$\mathbf{H}(F(\mathbf{x})) = \mathbf{H}(F_0(\mathbf{x})F'(\mathbf{x})) \geq \frac{\mathbf{H}(F'(\mathbf{x}))}{\mathbf{H}(F_0(\mathbf{x}))}$$

for all $\mathbf{x} \in \hat{K}^n$. By Lemma 2, there exist constants C_1, $C_2 \in \mathbb{R}_{>0}$ such that $\mathbf{x}$, $\mathbf{y} \in \mathfrak{X}$ and $F(\mathbf{x}) = G(\mathbf{y})$ implies

$$C_1\mathbf{H}(\mathbf{y})^{d^*} \geq \mathbf{H}(G(\mathbf{y})) = \mathbf{H}(F(\mathbf{x})) \geq \frac{C_2\mathbf{H}(\mathbf{x})^{d_* - d_0}}{C_1\mathbf{H}(\mathbf{x})^{d_0}} = C_2 C_1^{-1}\mathbf{H}(\mathbf{x})^{d_* - 2d_0},$$

and hence $\mathbf{H}(\mathbf{y}) > \mathbf{H}(\mathbf{x})$, provided that $\mathbf{H}(\mathbf{x})$ is large enough.

ii) follows immediately from i) and the Main Lemma, applied for F, G : $\mathfrak{X} \to \hat{K}^m$ and $f = \mathbf{H} : \mathfrak{X} \to \mathbb{R}$. $\square$

THEOREM 1. *Let K be a field, equipped with a family of absolute values satisfying the product formula, and let $\mathbf{H}$ be the associated height function. Suppose that for every $C \in \mathbb{R}_{>0}$ and $n \in \mathbb{N}$, the set*

$$\{\mathbf{x} \in \hat{K}^n \mid \mathbf{H}(\mathbf{x}) + d_K(\mathbf{x}) \leq C\}$$

*is finite. Then K has the property (**P0**). In particular, every global field has the property (**P0**).*

Proof. The first assertion follows immediately from Proposition 1. If K is a global field (i. e., either an algebraic number field or an algebraic function field in one variable over a finite field), then we use the finiteness results from [8, 2.5]. $\square$

We close this section with two further Lemmas concerning heights to be used in § 3. For simplicity, we restrict ourselves to the case of non-archimedean absolute values.

LEMMA 3. *Let K be a field, equipped with a family of non-archimedean absolute values satisfying the product formula, and let $\mathbf{H}$ be the associated height function.*

i) *For monic polynomials f, $g \in K[X]$, we have $\mathbf{H}(fg) \leq \mathbf{H}(f)\mathbf{H}(g)$.*

ii) *If $n \in \mathbb{N}$ and $\mathbf{x}$, $\mathbf{y} \in K^n$, then $\mathbf{H}(\mathbf{x} - \mathbf{y}) \leq \mathbf{H}(\mathbf{x})\mathbf{H}(\mathbf{y})$.*

Proof.

i) is obvious; we even have equality, cf. [2, ch. 3, Prop. 2.4].

ii) Set $\mathbf{x} = (x_1, \ldots, x_n)$, $\mathbf{y} = (y_1, \ldots, y_n)$, and let L be a finite extension of K such that $\mathbf{x}, \mathbf{y} \in L^n$. Then we have

$$
\begin{aligned}
\mathbf{H}(\mathbf{x} - \mathbf{y}) &= \prod_{w \in M_L} \max\{1, |x_1 - y_1|_w, \ldots, |x_n - y_n|_w\} \\
&\leq \prod_{w \in M_L} \max\{1, |x_1|_w, |y_1|_w, \ldots, |x_n|_w, |y_n|_w\} \\
&\leq \prod_{w \in M_L} \max\{1, |x_1|_w, \ldots, |x_n|_w\} \cdot \prod_{w \in M_L} \max\{1, |y_1,|_w, \ldots, |y_n|_w\} \\
&= \mathbf{H}(\mathbf{x})\mathbf{H}(\mathbf{y}). \quad \square
\end{aligned}
$$

LEMMA 4. *Let K be a field, equipped with a family of non-archimedean absolute values satisfying the product formula, and let $\mathbf{H}$ be the associated height function.*

Let $d, n \in \mathbb{N}$, and let $\mathbf{x} = (x_1, \ldots, x_n) \in \hat{K}^n$ be such that $d_K(\mathbf{x}) \leq d$. For $i \in \{1, \ldots, n\}$, let $f_i \in K[X]$ be the minimal polynomial of x_i, and $f = f_1 \cdot \ldots \cdot f_n \in K[X]$. Then we have

$$
\mathbf{H}(f) \leq \mathbf{H}(\mathbf{x})^{nd}.
$$

Proof. For $i \in \{1, \ldots, n\}$, we have

$$
f_i = \prod_{\nu=1}^{d_i} \left(X - x_i^{(\nu)} \right),
$$

where $x_i^{(\nu)} \in \hat{K}$ are the conjugates of x_i (counted with multiplicity in the inseparable case) and $d_i \leq d$. Since $\mathbf{H}(X - x_i^{(\nu)}) = \mathbf{H}(x_i^{(\nu)}) = \mathbf{H}(x_i)$, we obtain, using Lemma 3,

$$
\mathbf{H}(f) = \mathbf{H}\left(\prod_{i=1}^{n} \prod_{\nu=1}^{d_i} \left(X - x_i^{(\nu)} \right) \right) \leq \prod_{i=1}^{n} \mathbf{H}(x_i)^{d_i} \leq \prod_{i=1}^{n} \mathbf{H}(\mathbf{x})^{d_i} \leq \mathbf{H}(\mathbf{x})^{nd}. \quad \square
$$

3. Algebraic function fields

We start with some preliminaries on height functions and places of algebraic function fields; for the general theory of places cf. [3, ch. I].

Let $K_1 = K_0(t)$ be a rational function field over an arbitrary field K_0, and let $\mathcal{P}$ be the set of all monic irreducible polynomials in $K_0[t]$. For $p \in \mathcal{P}$, we define an absolute value $|\cdot|_p$ as follows: If

$$x = c \prod_{p \in \mathcal{P}} p^{\nu_p} \,,$$

where $c \in K_0^\times$, $\nu_p \in \mathbb{Z}$ and $\nu_p = 0$ for almost all $p \in \mathcal{P}$, then

$$|x|_p = e^{-\nu_p \deg(p)} \,;$$

for $x = \alpha\beta^{-1}$, where $\alpha\,,\,\beta \in K_0[t]$, we set

$$|x|_\infty = e^{\deg(\alpha) - \deg(\beta)} \,.$$

Then $(|\cdot|_v)_{v \in \mathcal{P} \cup \{\infty\}}$ is a family of non-archimedean absolute values satisfying the product formula. The corresponding height function will be denoted by $\mathbf{H}_\partial$; we call it the *canonical height function* for $K_0(t)$.

If $\mathbf{x} \in K_1^n$, then $\mathbf{x} = (\alpha_1\beta^{-1}, \ldots, \alpha_n\beta^{-1})$, where $\alpha_1, \ldots, \alpha_n\,,\,\beta \in K_0[t]$, β is monic and $\gcd(\alpha_1, \ldots, \alpha_n\,,\,\beta) = 1$; this implies

$$\max\{|\alpha_1|_p, \ldots, |\alpha_n|_p, |\beta|_p\} = 1, \ \forall p \in \mathcal{P}$$

and hence

$$\mathbf{H}_\partial(x) = e^{\partial(\mathbf{x})} \,,$$

where

$$\partial(\mathbf{x}) = \max\{\deg(\alpha_1), \ldots, \deg(\alpha_n), \deg(\beta)\} \,.$$

Every $\tau \in \hat{K}_0$ defines a place $\phi_\tau : K_1 \to \hat{K}_0 \cup \{\infty\}$ by $\phi_\tau(h) = h(\tau)$. A place $\phi : \hat{K}_1 \to \hat{K}_0 \cup \{\infty\}$ is called a K_0- *place* if $\phi|K_1 = \phi_\tau$ for some $\tau \in \hat{K}_0$. Obviously, every ϕ_τ $(\tau \in \hat{K}_0)$ extends to a K_0-place of $\hat{K}_1$. For a K_0-place ϕ of $\hat{K}_1$, we denote by $R_\phi = \phi^{-1}(\hat{K}_0) \subset \hat{K}_1$ the local ring associated with ϕ. We extend the ring homomorphism $\phi : R_\phi \to \hat{K}_0$ to homomorphisms of the polynomial rings $\phi : R_\phi[X_1, \ldots, X_n] \to \hat{K}_0[X_1, \ldots, X_n]$ (by acting on the coefficients) and to mappings $\phi : R_\phi^n \to \hat{K}_0^n$ (by acting on the components). If $C \in R_\phi[X_1, \ldots, X_n]$ and $\mathbf{x} \in R_\phi^n$, then $(\phi C)(\phi(\mathbf{x})) = \phi(C(\mathbf{x}))$.

PROPOSITION 2. *Let K_0 be a field which is neither algebraically closed nor real closed, let $K_1 = K_0(t)$ be a rational function field over K_0 and $\mathbf{H}_\partial$ the canonical height function.*

For every $M \in \mathbb{N}$, there exists a K_0-place $\phi : \hat{K}_1 \to \hat{K}_0 \cup \{\infty\}$ with the following property:

If $m \in \mathbb{N}$ and $\mathcal{Z} \subset \hat{K}^m$ is a subset satisfying

$$\mathbf{H}_\partial(\mathbf{z}) + d_K(\mathbf{z}) + m \leq M \quad \text{for all} \quad \mathbf{z} \in \mathcal{Z} \,,$$

then we have $\mathcal{Z} \subset R_\phi^m$, and $\phi | \mathcal{Z} : \mathcal{Z} \to \hat{K}_0^m$ is injective.

Proof. Let $\tau \in \hat{K}_0$ be an element satisfying

$$N = [K_0(\tau) : K_0] > 2M^3 \log M \,,$$

and let $\phi : \hat{K}_1 \to \hat{K}_0 \cup \{\infty\}$ a place extending ϕ_τ. We assert that ϕ has the required property.

Let $m \in \mathbb{N}$ and $\mathcal{Z} \subset \hat{K}^m$ be a subset such that $\mathbf{H}_\partial(\mathbf{z}) + d_K(\mathbf{z}) + m \leq M$ for all $\mathbf{z} \in \mathcal{Z}$. Putting

$$\mathcal{Z}^* = \mathcal{Z} \cup \{\mathbf{z} - \mathbf{z}' \mid \mathbf{z}, \mathbf{z}' \in \mathcal{Z}\} \,,$$

it is sufficient to prove that $\mathcal{Z}^* \subset R_\phi^m$, and $\phi(\mathbf{x}) \neq 0$ for all $0 \neq \mathbf{x} \in \mathcal{Z}^*$.

For $\mathbf{x} \in \mathcal{Z}^*$, we have $d_K(\mathbf{x}) \leq M^2$, and $\mathbf{H}_\partial(\mathbf{x}) \leq M^2$ by Lemma 3. For $\mathbf{x} = (x_1, \ldots, x_m)$, we denote by $f_i \in K[X]$ the minimal polynomial of x_i, and we set

$$f = f_1 \cdot \ldots \cdot f_m = X^d + \frac{\alpha_{d-1}}{\beta} X^{d-1} + \cdots + \frac{\alpha_1}{\beta} X + \frac{\alpha_0}{\beta} \in K_1[X] \,,$$

where $\alpha_0, \ldots, \alpha_{d-1}, \beta \in K_0[t]$, β is monic and $\gcd\{\alpha_0, \ldots, \alpha_{d-1}, \beta\} = 1$. Lemma 4 implies

$$\mathbf{H}_\partial(f) \leq \mathbf{H}_\partial(\mathbf{x})^{mM^2} \leq M^{2M^3} \,,$$

and therefore

$$\max\{\deg(\alpha_0), \ldots, \deg(\alpha_{d-1}), \deg(\beta)\} = \partial(\alpha_0 \beta^{-1}, \ldots, \alpha_{d-1} \beta^{-1})$$
$$= \log \mathbf{H}_\partial(f) \leq 2M^3 \log M < N \,.$$

Hence we have $\beta(\tau) \neq 0$, and $\alpha_i \neq 0$ implies $\alpha_i(\tau) \neq 0$ for all $i \in \{0, 1, \ldots, d-1\}$.

Since $\phi(\beta) = \phi_\tau(\beta) = \beta(\tau) \neq 0$, we infer $f \in R_\phi[X]$, $x_1, \ldots, x_m$ are integral over R_ϕ, and since R_ϕ is integrally closed, we obtain $\mathbf{x} \in R_\phi^m$.

If $\mathbf{x} \neq 0$, then we have $x_j \neq 0$ for some $j \in \{1, \ldots, m\}$ and $\alpha_i \neq 0$ for some $i \in \{0, \ldots, d-1\}$. Let $l \in \{0, \ldots, d-1\}$ be minimal such that $\alpha_l \neq 0$; then we obtain

$$0 = f(x_j) = x_j^l \left(x_j^{d-l} + \frac{\alpha_{l-1}}{\beta} x_j^{d-l-1} + \cdots + \frac{\alpha_l}{\beta} \right).$$

Dividing by x_j^l and multiplying with β yields an equation

$$\alpha_l = x_j \gamma, \quad \text{where} \quad \gamma \in R_\phi.$$

Since $\phi(\alpha_l) = \phi_\tau(\alpha_l) = \alpha_l(\tau) \neq 0$, we obtain $\phi(x_j) \neq 0$ and hence $\phi(\mathbf{x}) \neq 0$. $\square$

THEOREM 2. *Let K_0 be a field, and suppose that every finite extension field of K_0 has the property* (**P0**). *Then every finitely generated extension field of K_0 has the property* (**P0**).

Proof. It is sufficient to prove that every finitely generated extension field of transcendence degree 1 over K_0 has the property (**P0**); the general case follows by an obvious induction on the transcendence degree.

Let $K_1 = K_0(t)$ be a rational function field over K_0, and let K be a finite extension of K_1. We shall prove that K has the property (**P0**); obviously, we have $\hat{K} = \hat{K}_1$. Let $\mathbf{H}_\partial$ be the canonical height function associated with $K_0(t)$.

Let $F = (F_1, \ldots, F_m)$, $G = (G_1, \ldots, G_m) : \hat{K}^n \to \hat{K}^m$ be polynomial mappings defined by homogeneous polynomials $F_j, G_j \in K[X_1, \ldots, X_n]$. Let $F_0 \in K[X_1, \ldots, X_n]$ be a common factor of $F_1, \ldots, F_m$, say $F_j = F_0 F_j'$ where $F_j' \in K[X_1, \ldots, X_n]$, and suppose that $\deg_*(F) > \deg^*(G) + 2\deg(F_0)$. Let $V \subset \hat{K}^n$ be a K-homogeneous set such that $F_1', \ldots, F_m'$ do not have a common non-trivial zero in V, and let $\mathfrak{X} \subset V$ be a subset such that $F(\mathfrak{X}) \supset G(\mathfrak{X})$ and $G|\mathfrak{X}$ is injective. By Proposition 1, there exists a constant $C_0 \in \mathbb{R}_{>0}$ such that $\mathbf{x}, \mathbf{y} \in \mathfrak{X}$, $\mathbf{H}_\partial(\mathbf{x}) \geq C_0$ and $F(\mathbf{x}) = G(\mathbf{y})$ implies $\mathbf{H}_\partial(\mathbf{y}) > \mathbf{H}_\partial(\mathbf{x})$. Putting

$$\mathfrak{X}_0 = \{\mathbf{x} \in \mathfrak{X} \mid \mathbf{H}_\partial(\mathbf{x}) \leq C_0\},$$

the Main Lemma yields $F(\mathfrak{X}_0) \supset G(\mathfrak{X}_0)$, and according to Proposition 1 it is sufficient to prove that $\mathfrak{X}_0$ is finite. By Lemma 2, there exists a constant $C_1 \in \mathbb{R}_{>0}$ such that

$$\mathbf{H}_\partial(G(\mathbf{x})) \le C_1 H_\partial(\mathbf{x})^{\deg^*(G)} \le C_1 C_0^{\deg^*(G)}$$

for all $\mathbf{x} \in \mathfrak{X}_0$.

Let $H_1, \ldots, H_r \in K[X_1, \ldots, X_n]$ be homogeneous polynomials such that $V = \{\mathbf{x} \in \hat{K}^n \mid H_1(\mathbf{x}) = \cdots = H_r(\mathbf{x}) = 0\}$. Since $F_1', \ldots, F_m', H_1, \ldots, H_r$ do not have a common non-trivial zero in $\hat{K}^n$, it follows from Hilbert's Nullstellensatz [9, §130] that there exist polynomials $F_{j\nu}$, $H_{j\nu} \in K[X_1, \ldots, X_n]$ such that

$$X_\nu^q = \sum_{j=1}^m F_{j\nu} F_j' + \sum_{j=1}^r H_{j\nu} H_j$$

for some exponent $q \in \mathbb{N}$ and all $\nu \in \{1, \ldots, n\}$.

Let D be the (finite) set of all coefficients of the polynomials F_0, F_j', G_j, $H_j, F_{j\nu}$ and $H_{j\nu}$. Since K_0 has the property (P0), it is neither algebraically closed nor real closed, and Proposition 2 applies: there exists a K_0-place $\phi : \hat{K} \to \hat{K}_0 \cup \{\infty\}$ such that $D \subset R_\phi$, $\mathfrak{X}_0 \subset R_\phi^n$, $\phi | D \cup \{0\}$ is injective and $\phi | G(\mathfrak{X}_0)$ is injective.

There exist elements $y_1, \ldots, y_k \in R_\phi$ such that $K = K_0(t, y_1, \ldots, y_k)$, and consequently $\phi(K) = L \cup \{\infty\}$, where $L = K_0(\phi(t), \phi(y_1), \ldots, \phi(y_k))$. L is a finite extension of K_0, thus it has the property (P0). We use the place ϕ to shift the whole situation from K to L. We have $\hat{L} = \hat{K}_0$, the polynomials F_0, F_j', G_j, H_j, $F_{j\nu}$ and $H_{j\nu}$ have coefficients in $K \cap R_\phi$, and consequently their ϕ-images have coefficients in L.

We consider the polynomial mappings $\phi(F) = (\phi(F_1), \ldots, \phi(F_m))$ and $\phi(G) = (\phi(G_1), \ldots, \phi(G_m)) : \hat{K}_0^n \to \hat{K}_0^m$; by construction one has: $\phi(F_j) = \phi(F_0)\phi(F_j')$, $\forall j \in \{1, \ldots, m\}$, and $\deg_*(\phi(F)) > \deg^*(\phi(G)) + 2\deg(\phi(F_0))$. We set

$$V_\phi = \left\{ \mathbf{z} \in \hat{K}_0^n \mid \phi(H_1)(\mathbf{z}) = \cdots = \phi(H_r)(\mathbf{z}) = 0 \right\};$$

then $V_\phi \subset \hat{K}_0^n$ is a K-homogeneous set, and since

$$X_\nu^q = \sum_{j=1}^m \phi(F_{j\nu})\phi(F_j') + \sum_{j=1}^r \phi(H_{j\nu})\phi(H_j),$$

the polynomials $\phi(F_1'), \ldots, \phi(F_m')$ do not have a common non-trivial zero in V_ϕ.

Obviously, we have $\phi(\mathfrak{X}_0) \subset V_\phi$, $\phi(F)(\phi(\mathfrak{X}_0)) \supset \phi(G)(\phi(\mathfrak{X}))$, and we contend that

$$\phi(G) \circ \phi \mid \mathfrak{X}_0 : \mathfrak{X}_0 \to \hat{K}_0^m$$

is injective, i. e., $\phi|\mathfrak{X}_0$ and $\phi(G)|\phi(\mathfrak{X}_0)$ are both injective. Indeed, if $\mathbf{x}$, $\mathbf{y} \in \mathfrak{X}_0$ and $\phi(G) \circ \phi(\mathbf{x}) = \phi(G) \circ \phi(\mathbf{y})$, then $\phi(G(\mathbf{x})) = \phi(G(\mathbf{y}))$ and hence $\mathbf{x} = \mathbf{y}$, since $\phi|G(\mathfrak{X}_0)$ and $G|\mathfrak{X}_0$ are both injective.

Since L has the property (**P0**), we conclude that $\phi(\mathfrak{X}_0)$ is finite, and hence $\mathfrak{X}_0$ is also finite. $\square$

THEOREM 3. *Every finitely generated field has the property* (**P0**).

Proof. By Theorem 2, it suffices to prove that every finite extension of a prime field has the property (**P0**). For finite fields, this is obvious; for algebraic number fields, it follows from Theorem 1. $\square$

REFERENCES

[1] F. Halter-Koch, W. Narkiewicz, *Finiteness properties of polynomial mappings*, to appear.

[2] S. Lang, *Fundamentals of Diophantine Geometry*, Springer 1983.

[3] S. Lang, *Introduction to algebraic geometry*, Interscience Publ. 1958.

[4] D. J. Lewis, *Invariant sets of morphisms in projective and affine number spaces*, J. Algebra **20** (1972), 419–434.

[5] P. Liardet, *Sur les transformations polynomiales et rationnelles*, Sém. Th. Nomb. Bordeaux exp. n° 29, 1971–1972.

[6] P. Liardet, *Sur une conjecture de W. Narkiewicz*, C. R. Acad. Sc. Paris **274** (1972), 1836–1838.

[7] W. Narkiewicz, *On transformations by polynomials in two variables*, II, Coll. Math. **13** (1964), 101–106.

[8] J.-P. Serre, *Lectures on the Mordell-Weil-Theorem*, Aspects of Mathematics, Braunschweig 1989.

[9] B. L. van der Waerden, *Algebra, 2. Teil*, Springer 1967.

Franz Halter-Koch   Władysław Narkiewicz
Institut für Mathematik   Instytut Matematyczny
Karl-Franzens-Universität   Uniwersytet Wrocławski
Heinrichstraße 36/IV   Plac Grunwaldzki 2/4
A-8010 Graz, Österreich   PL-50-384 Wrocław, Polen