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NILS-PETER SKORUPPA

Résumé

Nous présentons une interprétation géometrique d'une loi arithmétique pour deduire des formules explicites pour les coefficients des formes modulaires elliptiques et des formes de Jacobi. Nous discutons des applications de ces formules etcomme exemple nous dérivons de manière algorithmique un critère analogue au critère de Tunnell concernant des nombres congruentes.

Abstract

We give a geometric interpretation of an arithmetic rule to generate explicit formulas for the Fourier coefficients of elliptic modular forms and their associated Jacobi forms. We discuss applications of these formulas and derive as an example a criterion similar to Tunnel's criterion for a number to be a congruent number.

1. Introduction

This report is intended to interpret and to make more explicit a result concerning effective, closed formulas for the Fourier coefficients and periods of modular forms of arbitrarily given level. These formulas may be considered as a new contribution to one of the "Grundprobleme der Theorie der elliptischen Modulfunktionen ... : Die Konstruktion der Integrale 1-ter Gattung der N-ten Stufe und die Bestimmung ihrer Perioden." (cf. [H, p.461]).

So far only two methods have been known to produce explicitly a basis for the space $M_k(m)$ of elliptic modular forms of (even) weight k on the group $\Gamma_0(m)$: theta series or the trace formula (and variations of these). The advantage of the new method in comparison with the method of theta series is that it is valid without any restriction on the level m. The new method yields an arithmetic rule to build the Fourier coefficients of the

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modular forms in $M_k(m)$ for any given m and k without any cumbersome requirements. It is more explicit than the trace formula method since it provides formulas for the Fourier coefficients of modular forms which can simply be written down and do not have to be generated by a slightly unpredictable process.. Recall that the trace formula method to generate a basis for $M_k(m)$ consists in applying sufficiently many Hecke operators to that modular form whose n-th Fourier coefficient is the trace of the n-th Hecke operator acting on $M_k(m)$. Explicit formu

However, probably the most interesting property of the new method is the fact that it does not only produce for arbitrarily given level and weight the Fourier coefficients of a basis of the corresponding space of modular forms, but that it produces as well, via the Shimura lift, the associated Jacobi forms. Jacobi forms serve as a mediator for the two important arithmetical data attached to a modular form f: its Fourier coefficients $a_f(n)$ and its periods $L(f|M,l)$ $(0 < l < k,\ M \in \mathrm{GL}^+(2,\mathbb{Q}))$. In particular, combining (the generalization to Jacobi forms of) Waldspurger's theorem and the new method one can produce in an algorithmic way theorems similar to Tunnell's theorem on congruence numbers.

To give a flavour of the kind of explicit formulas that the new method produces one may apply it to literally the same situation as originally considered by Tunnell [T] in connection with the congruent number problem. Here the new formulas lead to the following (see sec. 6. for details):

THEOREM. *If a positive fundamental discriminant $D \equiv 1 \mod 8$ is a congruent number, i.e. the area of a right triangle with rational sides, then $\nu_+(D,r) = \nu_-(D,r)$, where r denotes any solution to $r^2 \equiv D \mod 128$ and*

$$\nu_\pm(D,r) = \Big\{(a,b,c) \in \mathbb{Z}^3 \mid b^2 - 4ac = D,\ b^2 < D,\ \pm a > 0,\\ a \equiv 3\frac{b+r}{2} \mod 32,\ 3c \equiv \frac{b-r}{2} \mod 32\Big\}.$$

From the point of view of explicit calculations our formulas seem to be not a priori computationally less expensive than, say, the trace formula method. But because of their simple structure they are at least very easy to implement on a computer.

It is an open problem to construct infinite series of Hecke eigenforms, aside from those which are obtained by the process of twisting. The new method does not do this either, at least not in an obvious way. However, as we shall indicate below, it is not quite hopeless to try the new method to generate explicitly Hecke eigenforms.

In the following we shall restrict throughout to the case of weight 2. We do this to save notation and technicalities. All of the following results hold true, mutatis mutandis, for arbitrary even weight (cf. [S2]).

Remark on notations

If E is a set we use $\mathbb{C}[E]$ for the vector space of all formal linear combinations $\sum n_x(x)$ where x runs through E and the n_x are complex numbers which are zero for all but finitely many x. By deg we always mean the linear map $\mathbb{C}[E] \to \mathbb{C}$ which takes each element of E to 1, and by $\mathbb{C}[E]^0$ we always mean the subspace of $\mathbb{C}[E]$ of elements of degree 0, i.e. the kernel of deg. If a group G acts on E then we always view $\mathbb{C}[E]$ as a representation space for G by extending the action of G on E by linearity to $\mathbb{C}[E]$. If G acts on the complex vector space V then $\mathrm{H}_0(G, V)$ is the space of G-co-invariants of V, i.e the quotient V/V' where V' denotes the subspace generated by all elements $g \cdot v - v$ with g and v running through G and V, respectively.

Throughout m denotes a fixed positive integer, and we use the abbreviations

$$\Gamma_1 := \mathrm{SL}(2,\mathbb{Z}), \quad \Gamma := \Gamma_0(m) = \Gamma_1 \cap \begin{pmatrix} \mathbb{Z} & \mathbb{Z} \\ m\mathbb{Z} & \mathbb{Z} \end{pmatrix}.$$

A pair of integers Δ, ρ is called *m-admissible* if $\Delta \equiv r^2 \mod 4m$.

2. Heegner cycles and intersection numbers

We describe first of all the arithmetic rule mentioned in the introduction for generating the Fourier coefficients of elliptic modular forms or Jacobi forms. This rule is essentially given by counting intersection numbers of Heegner cycles.

Let $\mathbb{H}$ denote the Poincaré upper half plane, i.e. the set of all complex numbers τ with positive imaginary part. Let $\mathbb{H}^* = \mathbb{H} \cup \mathbb{P}^1(\mathbb{Q})$ be the extended upper half plane obtained by adjoining the *cusps* $\mathbb{P}^1(\mathbb{Q}) = \mathbb{Q} \cup \{\infty\}$. The group $\mathrm{GL}^+(2,\mathbb{R})$ of 2×2-matrices with real entries and positive determinant acts on $\mathbb{H}$ by

$$(A, \tau) \mapsto A\tau = \frac{a\tau + b}{c\tau + d} \qquad (A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}).$$

The action of $\mathrm{GL}^+(2,\mathbb{Q})$, the subgroup of matrices in $\mathrm{GL}^+(2,\mathbb{R})$ with rational entries, extends to an action on $\mathbb{H}^*$ which preserves the cusps.

By a (hyperbolic) line in $\mathbb{H}$ we understand either a semicircle with center on the real axis or a vertical line perpendicular to the real axis. In order to deal with intersection numbers we need oriented hyperbolic lines. We orient the hyperbolic lines by indexing them by real binary quadratic forms with positive discriminant. To any such form

$$Q = [a, b, c] = aX^2 + bXY + cY^2 \qquad (a, b, c \in \mathbb{R}, \Delta := b^2 - 4ac > 0)$$

we associate the hyperbolic line C_Q given by the equation

$$a|\tau|^2 + b\Re(\tau) + c = 0.$$

C_Q is oriented from $\frac{-b+\sqrt{\Delta}}{2a}$ to $\frac{-b-\sqrt{\Delta}}{2a}$ if $a \neq 0$, and if $a = 0$, from $-\frac{c}{b}$ to ∞ for positive b, and from ∞ to $-\frac{c}{b}$ for negative b. The group $\mathrm{GL}^+(2, \mathbb{R})$ acts on these oriented hyperbolic lines C_Q by the action induced from the action on $\mathbb{H}$, and it acts from the right on the set of real binary quadratic forms with positive discriminant by

$$(Q, A) \mapsto Q \circ A := Q(aX + bY, cX + dY) \qquad \left(A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\right).$$

The orientation of C_Q is chosen in such a way that one has

$$A^{-1} C_Q = C_{Q \circ A}$$

for all Q and A. Let $\mathbb{L}$ denote the set of all oriented hyperbolic lines C_Q.

Let C_Q and C_R be two oriented hyperbolic lines. We define their intersection number $C_Q \cdot C_R$ by the formula

$$C_Q \cdot C_R := \frac{1}{2} \left(\operatorname{sign} Q(\lambda_{R,0}) - \operatorname{sign} Q(\lambda_{R,1})\right).$$

Here, for any R, the numbers $\lambda_{R,i}$ denote those elements of $\mathbb{P}^1(\mathbb{R})$ such that C_R is the line running from $\lambda_{R,0}$ to $\lambda_{R,1}$. For any $\lambda \in \mathbb{P}^1(\mathbb{R})$ we use $Q(\lambda)$ to denote the number $Q(\lambda, 1)$ if λ is real, and the number $Q(1, 0)$ if $\lambda = \infty$. By $\operatorname{sign} \lambda$ we denote the sign of the real number λ, the sign of 0 being 0.

It is easy to verify that $C_Q \cdot C_R = -C_R \cdot C_Q$, in particular $C_Q \cdot C_Q = 0$. Two given different lines C_Q and C_R may intersect or they may not. If they intersect then they do so in exactly one point, and then $C_Q \cdot C_R$ equals the usual intersection number at that point, i.e. the sign of $\Im(t_R/t_Q)$, where t_Q and t_R are the tangent vectors to C_Q and C_R (expressed as complex numbers) at their common point, respectively. If they do not intersect then they may have a common boundary point in $\mathbb{P}^1(\mathbb{R})$ or they may not. If they have no such common boundary point then their intersection number is 0. If they have a common boundary point then the usual argument of infinitesimally shifting suggests for the intersection number either 0 or a fixed non-zero value accordingly to which side one moves C_Q. The above intersection number is just the arithmetic mean of these two possible assignments.

As agreed above, let $\mathbb{C}[\mathbb{L}]$ and $\mathbb{C}[\mathbb{P}^1(\mathbb{R})]$ be the complex vector spaces of all finite formal linear combinations $\sum_{C\in\mathbb{L}} n_C(C)$ and $\sum_{\lambda\in\mathbb{P}^1(\mathbb{R})} n_\lambda(\lambda)$ with complex numbers n_C and n_λ which vanish for all but finitely many C and λ. We extend the action of $\mathrm{GL}^+(2,\mathbb{R})$ on $\mathbb{L}$ and $\mathbb{P}^1(\mathbb{R})$ to $\mathbb{C}[\mathbb{L}]$ and $\mathbb{C}[\mathbb{P}^1(\mathbb{R})]$, respectively, and we extend by linearity the intersection number to an antisymmetric bilinear form on $\mathbb{C}[\mathbb{L}]$. Let

$$\partial\colon \mathbb{C}[\mathbb{L}] \to \mathbb{C}[\mathbb{P}^1(\mathbb{R})]$$

be the linear map taking (C_Q) to $(\lambda_{Q,1}) - (\lambda_{Q,0})$. We have

Proposition 1. *Let C_1, C_2 be elements of $\mathbb{C}[\mathbb{L}]$ and $A \in \mathrm{GL}^+(2,\mathbb{R})$, then one has*

(1) $C_1 \cdot C_2 = 0$ *if C_2 is in the kernel of ∂,*
(2) $(AC_1)\cdot(AC_2) = C_1 \cdot C_2$.

Proof. This is immediate from the definitions.

Thus the intersection pairing on $\mathbb{C}[\mathbb{L}]$ induces an intersection pairing on $\mathbb{C}[\mathbb{L}]/\ker(\partial)$. The latter space may be identified with $\partial(\mathbb{C}[\mathbb{L}])$, which equals the subspace $\mathbb{C}[\mathbb{P}^1(\mathbb{R})]^0$ of $\mathbb{C}[\mathbb{P}^1(\mathbb{R})]$ consisting of those elements $\sum n_\lambda(\lambda)$ with degree $\sum n_\lambda$ equal to zero. Thus, we have an antisymmetric pairing "$\cdot$" on $\mathbb{C}[\mathbb{P}^1(\mathbb{R})]^0$ which is invariant under the action of $\mathrm{GL}^+(2,\mathbb{R})$.

To obtain something more arithmetical we restrict to *Heegner cycles*. These are those oriented hyperbolic lines C_Q where the quadratic forms Q have rational coefficients. Special Heegner cycles are *split Heegner cycles*, i.e. those C_Q where the discriminant of Q is a square of a rational number. The set of all split Heegner cycles coincides with the set of all oriented hyperbolic lines joining two cusps.

We choose now once and for all a positive integer m.

Let $\mathcal{Q}(m)$ denote the set of all integral quadratic forms $[a,b,c]$ such that m divides a. We recall that a pair Δ,ρ of integers is *m-admissible* if $\Delta \equiv \rho^2 \bmod 4m$. To each such pair Δ,ρ we can associate the subset

$$\mathcal{Q}(\Delta,\rho) = \left\{[a,b,c] \in \mathcal{Q}(m) \mid b^2 - 4ac = \Delta,\ b \equiv \rho \mod 2m\right\}.$$

The set $\mathcal{Q}(m)$ is the union of all these $\mathcal{Q}(\Delta,\rho)$. Note that $\mathcal{Q}(m)$ as well as its individual subsets $\mathcal{Q}(\Delta,\rho)$ are preserved under the action of the subgroup $\Gamma=\Gamma_0(m)$ of $\mathrm{GL}^+(2,\mathbb{Q})$.

To each fundamental discriminant D which is a square modulo $4m$ there corresponds a generalized genus character

$$\chi_D\colon \mathcal{Q}(m)\to\{0,\pm1\}.$$

If, for a given Q, the discriminant Δ of Q is divisible by D such that Δ/D is a square modulo $4m$, and if there are integers x,y and a positive divisor m' of m such that $n:=Q(x/\sqrt{m'},y\sqrt{m'})$ is relatively prime to D, then one has $\chi_D(Q)=\left(\frac{D}{n}\right)$, where we use $\left(\frac{D}{*}\right)$ for the Dirichlet character modulo D with $\left(\frac{D}{p}\right)$ = Legendre symbol for odd primes p; otherwise one has $\chi_D(Q)=0$ (cf. [GKZ], where this genus character was first introduced). Note that χ_D is invariant under Γ, i.e. $\chi_D(Q\circ A)=\chi_D(Q)$ for all Q and all $A\in\Gamma$.

Fix now an m-admissible pair D,r, such that D is a fundamental discriminant, and a split Heegner cycle C. For each m-admissible pair of integers Δ,ρ with $D\Delta>0$ we can then consider the number

$$I_{C,D,r}(\Delta,\rho)=\sum_{Q\in\mathcal{Q}(D\Delta,r\rho)}\chi_D(Q)\,C_Q\cdot C.$$

If $D\Delta$ is not a perfect square then this sum is actually finite, i.e. one has $C_Q\cdot C=0$ for all but finitely many Q. In fact, since C is a split Heegner cycle it is running from a cusp p to a cusp q, and if we choose a matrix A in $\mathrm{GL}^+(2,\mathbb{Q})$ which maps 0 and ∞ to p and q, respectively, then $C=AC_{XY}$. By Proposition 1 the intersection numbers are invariant under the action of $\mathrm{GL}^+(2,\mathbb{Q})$ and hence $C_Q\cdot C=C_{Q\circ A}\cdot C_{XY}$. Let $Q\circ A=[a,b,c]$ and assume $C_{Q\circ A}\cdot C_{XY}\neq0$. Then it is immediate from the definition of the intersection numbers that a and c have opposite signs, so $\det(A)^2D\Delta=b^2+4|ac|$. Since b and $\det(A)$ are rational numbers and since $D\Delta$ is not a perfect square we deduce that $ac\neq0$, whence the inequalities:

$$0<|ac|<\frac{D\Delta\det(A)^2}{4},\quad |b|<\sqrt{D\Delta}\det(A).$$

The a,b,c are rational numbers whose denominator is bounded by the square of the common denominators of the entries of A. But there are only finitely many rational numbers which have this property and satisfy the listed inequalities. Note that this argumentation yields as a side result an effective upper bound for the number of forms Q with C_Q intersecting C.

In particular, we recognize that $I_{C,D,r}(\Delta,\rho)$ grows in Δ, for fixed C, D, r, at most like $|\Delta|^{\frac{3}{2}}$, and that this is also the order of magnitude of steps needed to compute this number.

Also note that the argumentation in the last paragraph does not hold for non-split Heegner cycles C. Indeed, the fact that the stabilizer in Γ of a non-split C is infinite causes infinitely many Q in $\mathcal{Q}(\Delta,\rho)$ to have non-vanishing intersection with C. If Δ is a perfect square, then the sum defining $I_C(\Delta, r)$ is not finite either. There is an infinite contribution arising from those Q such that $Q\circ A$ has a or c equal to 0. There is a sensible way to fix the latter problem (cf. [S2]). However, here we restrict to the case that $D\Delta$ is not a perfect square.

Before we discuss the arithmetic nature of the numbers $I_{C,D,r}(\Delta,\rho)$, we introduce some more formalism in order to state some of their formal properties. By linearity we can extend the definition of the $I_{C,D,r}(\Delta,\rho)$ to all C from the subspace of $\mathbb{C}[\mathbb{L}]$ generated by split Heegner cycles. It is then immediate from their definition and Proposition 1 that these numbers depend only on C modulo the subspace $\ker\partial$, i.e. they depend only on $\partial(C)$. The image under ∂ of the subspace generated by all split Heegner cycles equals $\mathbb{C}[\mathbb{P}^1(\mathbb{Q})]^0$, i.e. it is the space of all formal linear combinations $\sum n_p(p)$, where p runs through the cusps $\mathbb{P}^1(\mathbb{Q})$, where n_p is zero for all but finitely many p and where $\sum n_p = 0$. Secondly, we note that the numbers $I_{C,D,r}(\Delta,\rho)$, for fixed D, r, Δ, ρ, depend only on the Γ-orbit of C. This is immediate from the original definition using $C_Q\cdot(AC) = C_{Q\circ A}\cdot C$ and the fact that $\mathcal{Q}(D\Delta, r\rho)$ and χ_D are invariant under A for any A in Γ. Thus, let $\mathrm{Co}(m)$ denote the space of Γ-co-invariants of $\mathbb{C}[\mathbb{P}^1(\mathbb{Q})]^0$, i.e. set

$$\mathrm{Co}(m) = \mathrm{H}_0(\Gamma, \mathbb{C}[\mathbb{P}^1(\mathbb{Q})]^0) \qquad (\Gamma = \Gamma_0(m)).$$

Then we can state

Proposition 2. *The number $I_{C,D,r}(\Delta,\rho)$ depends only on the Γ-coinvariance class $[\delta(C)]$ of $\delta(C)$ in $\mathrm{Co}(m)$.*

The space $\mathrm{Co}(m)$ is finite dimensional and it is possible to pick a rather natural set of generators of this vector space. More precisely, one has

Proposition 3. *The Γ-homomorphism $\mathbb{C}[\Gamma_1] \to \mathbb{C}[\mathbb{P}^1(\mathbb{Q})]^0$ taking an element $A\in\Gamma_1$ to $(A\infty)-(A0)$ defines by passage to quotients an exact sequence*

$$0 \to \left(\mathbb{C}[\Gamma\backslash\Gamma_1]^S + \mathbb{C}[\Gamma\backslash\Gamma_1]^{ST}\right) \to \mathbb{C}[\Gamma\backslash\Gamma_1] \to \mathrm{Co}(m) \to 0$$

(where we are tacitly identifying the space of co-invariants $\mathrm{H}_0(\Gamma, \mathbb{C}[\Gamma_1])$ *and* $\mathbb{C}[\Gamma\backslash\Gamma_1]$ *via the obvious isomorphism). Here*

$$S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \qquad T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix},$$

and $\mathbb{C}[\Gamma\backslash\Gamma_1]^A$*, for any* A *in* Γ_1*, denotes the subspace of* A*-invariants in* $\mathbb{C}[\Gamma\backslash\Gamma_1]$ *with respect to right multiplication by* A*.*

Proof. Clearly $\mathrm{Co}(m)$ is generated by co-invariance classes of the form $[(c)-(\infty)]$ with $c \in \mathbb{Q}$. Thus, to show that the third map is onto it suffices to exhibit a pre-image for any such class. Let $c_{-1} = \infty$ and let $c_0 = [c], c_1, \ldots, c_r = c$ be the convergents of any of the two continued fraction decompositions of c. Then

$$(c) - (\infty) = \sum_{i=0}^{r} \left((c_i) - (c_{i-1})\right),$$

and by well-known facts from the theory of continued fractions, there exists for each i a matrix A_i in Γ_1 which takes ∞ and 0 to c_i and c_{i-1}, respectively.

To prove exactness at the middle space let K denote the kernel of the third map and let k be the second space in the short sequence. Note that K contains k. In fact, if $D \in \mathbb{C}[\Gamma\backslash\Gamma_1]$ is fixed by ST, then $D = \frac{1}{3}(D + D \cdot ST + D \cdot (ST)^2)$ and the right side of this is obviously mapped to the zero class in $\mathrm{Co}(m)$; if D is fixed by S then $D = \frac{1}{2}(D + D \cdot S)$ and again it is obvious from the right side that this is mapped to 0.

To prove equality we compute dimensions. First of all one verifies by elementary group theory that, for any subgroup H of Γ_1, one has

$$\mathbb{C}[\Gamma\backslash\Gamma_1]^H = \#\Gamma\backslash\Gamma_1/H,$$

i.e. the dimension of the subspace of H-invariants equals the number H-orbits of $\Gamma\backslash\Gamma_1$, which is the number of double cosets above. Secondly, it is well-known that Γ_1 is generated by S and ST, and hence the intersection of the subspaces of S and ST invariants in $\mathbb{C}[\Gamma\backslash\Gamma_1]$ is the subspace of Γ_1-invariants which, by the last equation, is one-dimensional. Combining these formulas we find a formula for $\dim k$, and using the fact that K contains k we find

$$\dim \mathbb{C}[\Gamma\backslash\Gamma_1]/K = \dim \mathrm{Co}(m) \leq$$
$$[\Gamma_1 : \Gamma] + 1 - \#\Gamma\backslash\Gamma_1/\langle S\rangle - \#\Gamma\backslash\Gamma_1/\langle ST\rangle$$

($\langle A\rangle$, for any A in Γ_1, denoting the subgroup generated by A). In the next section we shall obtain as a by-product the inverse inequality (cf. the proof of Theorem 2) which thus proves the Proposition.

To explain the arithmetic nature of the intersection numbers I we shall consider generating functions of a sort for the sequences $\{(I_{C,D,r}(\cdot,\cdot)\}$. It will turn out that these functions belong to a very distinguished class.

3. Jacobi forms, modular forms, periods

Before we can make the latter statement precise we have to review the notions appearing in the title of this section and how these objects are connected. By definition, a Jacobi form (of weight 2), index m and sign ϵ ($= \pm 1$) is a smooth function $\phi(\tau, z)$ on $\mathbb{H} \times \mathbb{C}$, which has the following three properties:

(1) The function ϕ is periodic in each variable with period 1 and its Fourier expansion is of the form

$$\phi = \sum_{\substack{\Delta, r \in \mathbb{Z} \\ \Delta \equiv \rho^2 \mod 4m}} c_\phi(\Delta, \rho)\, \mathrm{e}_{\Delta,\rho},$$

where $\mathrm{e}_{\Delta,\rho}$ denotes the function

$$\mathrm{e}_{\Delta,\rho}(\tau, z) = \exp\left(2\pi i \left\{\frac{\rho^2 - \Delta}{4m} u + \frac{\rho^2 + |\Delta|}{4m} iv + \rho z\right\}\right) \quad (\tau = u + iv).$$

(2) The Fourier coefficients $c_\phi(\Delta, \rho)$ depend on ρ only modulo $2m$, i.e. $c_\phi(\Delta, \rho) = c_\phi(\Delta, \rho + 2m)$ for all Δ, ρ, and they vanish for $\epsilon\Delta < 0$.

(3) The function ϕ satisfies the transformation law

$$\phi(\frac{-1}{\tau}, \frac{\tau}{z}) = j_\epsilon(\tau) \exp\left(2\pi i m \frac{z^2}{\tau}\right) \phi(\tau, z),$$

where $j_\epsilon(\tau) = \tau^2$ if $\epsilon = -1$ and $j_\epsilon(\tau) = |\tau|\overline{\tau}$ if $\epsilon = +1$.

If $c_\phi(0, r) = 0$ for all $r^2 \equiv 0 \mod 4m$ then ϕ is called a cusp form. The set of all such cusp forms is denoted by $P^-_{2,m}$ if $\epsilon = -1$, whereas we reserve the symbol $P^+_{2,m}$ for a slightly smaller space which we will explain below. Likewise, we call a Jacobi form of sign -1 holomorphic, and skew-holomorphic if it has sign $+1$. Holomorphic Jacobi forms have been studied in [E-Z], skew-holomorphic were first introduced in [S1],[S3] (a more thorough study of these is in preparation and will appear elsewhere).

By $S_2(m)$ we denote the space of modular cusp forms of weight 2 on $\Gamma_0(m)$. By definition, this is the space of all holomorphic functions $f(\tau)$ defined on the upper half plane $\mathbb{H}$ such that for all $A \in \Gamma$ one has $f|A = f$, and for all $A \in \Gamma_1$ the function $(f|A)(u+iv)$ tends uniformly to 0 for v tending to infinity. Here $f|A$ refers to the (right) action of $\mathrm{GL}^+(2,\mathbb{R})$ on functions defined on the upper half plane given by

$$(f|A)(\tau) = \frac{\det(A)}{(c\tau+d)^2} f\left(\frac{a\tau+b}{c\tau+d}\right) \qquad (A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}).$$

For any modular form f in $S_2(m)$ and any $A \in \mathrm{GL}^+(2,\mathbb{Q})$, the function $f|A$ is periodic and has a Fourier expansion of the form

$$f|A = \sum_{n=1}^{\infty} a_{f|A}(n) q_N^n \qquad (q_N = \exp\left(2\pi i \frac{\tau}{N}\right))$$

for a suitable positive integer N. The Dirichlet series

$$L(f|A,s) := N^s \sum_{n=1}^{\infty} \frac{a_{f|A}(n)}{n^s}$$

converges for $\Re(s) > \frac{3}{2}$ and can be continued to a holomorphic function on $\mathbb{C}$. The space $S_2(m)$ decomposes into the direct sum of the two subspaces $S_2^-(m)$ and $S_2^+(m)$ consisting of those f such that $L(f,s)$ satisfies the functional equation

$$L^*(f,s) := (2\pi)^{-s} m^{s/2} \Gamma(s) L(f,s) = \epsilon L^*(f, 2-s)$$

with $\epsilon = -1$ and $\epsilon = +1$, respectively. For details cf. [H, in particular p.644 ff.] (or, for more modern terminology, cf. [Sh])

The third main space participating in this section is a subspace of the space $\mathrm{Co}(m)$ of Γ-co-invariants of $\mathbb{C}[\mathbb{P}^1(\mathbb{Q})]^0$. We set

$$\mathrm{CCo}(m) := \ker\left(\mathrm{H}_0(\Gamma, \mathbb{C}[\mathbb{P}^1(\mathbb{Q})]^0) \xrightarrow{i_*} \mathrm{H}_0(\Gamma, \mathbb{C}[\Gamma\backslash\mathbb{P}^1(\mathbb{Q})])\right),$$

where i_* is the map induced by the inclusion $i : \mathbb{C}[\mathbb{P}^1(\mathbb{Q})]^0 \to \mathbb{C}[\mathbb{P}^1(\mathbb{Q})]$. Let

$$g = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}.$$

The matrix g acts on $\mathbb{C}[\mathbb{P}^1(\mathbb{Q})]$ by linear extension of its natural action on $\mathbb{P}^1(\mathbb{Q})$. Since it normalizes Γ, this action induces an involution g_* on $\mathrm{Co}(m)$ and, as it is easily checked, on $\mathrm{CCo}(m)$ too. Thus the space $\mathrm{CCo}(m)$ decomposes into the direct sum of the (-1) and $(+1)$ eigenspaces $\mathrm{CCo}^+(m)$ and $\mathrm{CCo}^-(m)$ of g_*. More explicitly, $\mathrm{CCo}^\epsilon(m)$ consists of those classes $[\sum n_p(p)]$ that satisfy

$$[\sum n_p(p)] = -\epsilon[\sum n_p(-p)].$$

On each of these three kinds of spaces introduced in this section one has Hecke operators, a sequence of natural operators $T(n)$ $(n = 1, 2, 3, \dots,$ $\gcd(n, m) = 1)$, whose action on a Jacobi form ϕ, a modular form f and a co-invariant $\sigma = [Z]$, respectively, is given by

$$c_{T(n)\phi}(\Delta, \rho) = \sum_{d|n^2} \psi_\Delta(d) c_\phi\left(\frac{n^2}{d^2}\Delta, \rho'\right),$$

$$a_{T(n)f}(n') = \sum_{d|\gcd(n,n')} d a_f\left(\frac{nn'}{d^2}\right),$$

$$T(n)\sigma = \sum_A [A \cdot Z].$$

Here the first sum is over those divisors d of n^2 such that $\frac{n^2}{d^2}\Delta$ is an integer and such that there exists a ρ' with $n\rho \equiv d\rho' \pmod{2m \gcd(d, n)}$ and $\frac{n^2}{d^2}\Delta \equiv \rho'^2 \pmod{4m}$ (these two congruences uniquely determine ρ' modulo $2m$), and $\psi_\Delta(d) = f\left(\frac{\Delta/f^2}{d/f^2}\right)$ if $\gcd(d, \Delta) = f^2$ with $\Delta/f^2 \equiv 0, 1 \mod 4$ and $\psi_\Delta(d) = 0$ otherwise. In the third sum, the A run through a set of representatives for the Γ left cosets of the set of all 2×2-matrices with integral entries, determinant n and left lower entry divisible by m, and $A \cdot Z$ refers to the action of $\mathrm{GL}^+(2, \mathbb{Q})$ on $\mathbb{C}[\mathbb{P}^1(\mathbb{Q})]^0$. That the operator on co-invariants is well defined, i.e. that the right side of the defining equation does not depend on the particular choice of the representatives A, can be easily checked. For the Hecke operators on Jacobi and modular forms we refer to the above references for modular and Jacobi for

An element of any of these spaces is called a Hecke eigenform if it is a simultaneous eigenform of all Hecke operators $T(n)$ $(n = 1, 2, 3, \dots,$ $\gcd(n, m) = 1)$. There exists a basis consisting of Hecke eigenforms in the case of Jacobi and modular forms ([E-Z], [Sh]). In the case of skew-holomorphic Jacobi forms there exist certain *trivial cusp forms.* These are those Jacobi cusp forms of index m such that $c(\Delta,\rho)$ is different from 0 at most if Δ is a perfect square. By $P^{+}_{2,m}$ we denote the subspace spanned by all Jacobi cusp forms of index m and weight 2 which are Hecke eigenforms and are not trivial.

The three types of objects introduced in this section are connected by the following two theorems.

THEOREM 1. *Let $\epsilon \in \{\pm 1\}$. For any fixed fundamental discriminant D and any fixed integer r such that $D \equiv r^2 \mod 4m$ and $\operatorname{sign} D = \epsilon$ there is a Hecke equivariant map*

$$\mathcal{S}_{D,r}\colon P^{\epsilon}_{2,m} \to S^{\epsilon}_2(m)$$

given by

$$\phi \mapsto \sum_{n=1}^{\infty} \left\{ \sum_{d|n} \left(\frac{D}{d}\right) c_{\phi}\left(\frac{n^2}{d^2}D, \frac{n}{d}r\right) \right\} q^n.$$

There is a linear combination of these maps which is injective. For each Hecke eigenform f in $S^{\epsilon}_2(m)$ there exists a Hecke eigenform $\phi \in J^{\epsilon}_{2,m}$ which has the same eigenvalues as f under all Hecke operators.

Remark. It is possible to give a simple description of the sum of all the images of the $\mathcal{S}_{D,r}$, i.e. of the the image of any injective linear combination of the $\mathcal{S}_{D,r}$ ([S-Z]).

Proof. For the case $\epsilon = -1$ the theorem was proved in [S-Z]. For the case $\epsilon = +1$ it was proved in [S2] that the $\mathcal{S}_{D,r}$ map $P^{+}_{2,m}$ into $S^{+}_2(m)$. That they are Hecke equivariant is immediate from the definition. The remaining assertions for the case $\epsilon = +1$ will be published elsewhere.

Let f be a cusp form in $S_2(m)$. For $Z = (q) - (p) \in \mathbb{C}[\mathbb{P}^1(\mathbb{Q})]^0$ set

$$\int^{Z} f(\tau)\,d\tau = \int_0^{\infty} f(Aiv)\,d(Aiv),$$

where A is any matrix in $\mathrm{GL}^+(2,\mathbb{Q})$ taking 0 and ∞ to p and q, respectively. It is easily checked that this does not depend on the choice of A, that the integral is absolutely convergent, and that one has

$$\left(\int^{(p)-(p')} + \int^{(p')-(p'')} - \int^{(p)-(p'')}\right) f(\tau)\,d\tau = 0.$$

¿From the last identity we see that the symbol $\int^{Z} f(\tau)\,d\tau$ can be linearly continued to $\mathbb{C}[\mathbb{P}^1(\mathbb{Q})]^0$. From the definition it is immediate that the map $Z \mapsto \int^{Z} f(\tau)\,d\tau$ is Γ-invariant. Hence the symbol $\int^{Z} f(\tau)d\tau$ depends only on the Γ-co-invariance class $\sigma = [Z]$ of Z. We denote it by $\langle f \mid \sigma \rangle$. Likewise, we can use the (easily proved) identity

$$\int_0^\infty f(Aiv)\,dA(iv) = \frac{i}{2\pi} L\,(f|A,1)$$

to rewrite this symbol as

$$\langle f \mid \sigma \rangle = \frac{1}{2\pi i} \sum_{p \in \mathbb{Q}} n_p\, L\left(f\middle| \begin{pmatrix} 1 & p \\ 0 & 1 \end{pmatrix}, 1\right),$$

where $\sigma = [\sum n_p(p)]$.

THEOREM 2. *Let $\epsilon \in \{\pm 1\}$. The association*

$$(f, \sigma) \to \langle f \mid \sigma \rangle$$

defines a perfect pairing between $S_2(m)$ and $\mathrm{CCo}^\epsilon(m)$. One has

$$\langle T(n)f \mid \sigma \rangle = \langle f \mid T(n)\sigma \rangle$$

for all $n \geq 1, \gcd(m,n) = 1$.

Proof. We have to show that the correspondence $f \mapsto \langle f \mid \cdot \rangle$ defines an isomorphism between $S_2(m)$ and $\mathrm{Hom}_{\mathbb{C}}(\mathrm{CCo}^\epsilon(m), \mathbb{C})$.

To prove injectivity, assume that f is orthogonal to $\mathrm{CCo}^\epsilon(m)$. Then we have in particular $\langle f \mid [Z_A] \rangle = 0$ for any $A \in \Gamma$, where

$$Z_A = (A0) - (0) - \epsilon((gAg0) - (0))$$

(with g as in the definition of $\mathrm{CCo}^\epsilon(m)$). But, for any f, by a simple calculation,

$$\int^{Z_A} f(\tau)d\tau = \int_0^\infty (f(\tau)\,d\tau + \epsilon f(-\overline{\tau})\,d\overline{\tau}) \circ M,$$

where M denotes any matrix in $\mathrm{GL}^+(2,\mathbb{Q})$ taking ∞ and 0 to $A0$ and 0, respectively. It is well-known that these integrals vanish for all $A \in \Gamma$ only if $f = 0$ (cf. [Sh], or [S1, proof of the Lemma]; the idea of the proof is that for any antiderivative F of the harmonic differential $f(\tau)\,d\tau + \epsilon f(-\overline{\tau})\,d\overline{\tau}$, one has $F(A\tau) - F(\tau) = \langle f \mid [Z_A]\rangle$, so F would be a Γ-invariant antiderivative if the right hand side of the last identity would vanish for all A, and so would induce a harmonic function on the natural compactification $X(\Gamma)$ of the Riemann surface $\Gamma\backslash\mathbb{H}$ which thus has to be a constant).

¿From the injectivity we deduce that the dimension of $\mathrm{CCo}^{\epsilon}(m)$ is greater or equal to the dimension of $S_2(m)$. The latter is known to be

$$\dim S_2(m) = 1 + \frac{1}{2}\big([\Gamma_1 : \Gamma] \\ - \#\Gamma\backslash\Gamma_1/\langle S\rangle - \#\Gamma\backslash\Gamma_1/\langle ST\rangle - \#\Gamma\backslash\mathbb{P}^1(\mathbb{Q})\big)$$

where we use the notation of Proposition 3 (cf. e.g.[Sh], or apply the Riemann Hurwitz formula to the natural map $X(\Gamma) \to X(\Gamma_1)$ and use the facts that cusp forms in $S_2(m)$ correspond to holomorphic differentials on $X(\Gamma)$ and that $X(\Gamma_1)$ has genus 0). But this formula implies a lower bound for

$$\dim \mathrm{Co}(m) = \dim \mathrm{CCo}^{+}(m) + \dim \mathrm{CCo}^{-}(m) + \#\Gamma\backslash\mathbb{P}^1(\mathbb{Q}) - 1,$$

and a short computation shows that this lower bound is equal to the upper bound that we gave in the proof of Proposition 3. Thus we have equality everywhere. Note that this also completes the proof of Proposition 3.

The compatibility with Hecke operators is easily checked noticing that a set of representatives for the Γ-left cosets of a set of all 2×2-matrices with integral entries, determinant n and left lower entry divisible by m is given by

$$\begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \qquad (ad = n,\ 0 \le b < d),$$

that for any such matrix

$$L\left(f\Big|\begin{pmatrix} 1 & \frac{ap+b}{d} \\ 0 & 1 \end{pmatrix}, 1\right) = L\left(f\Big|\begin{pmatrix} a & b \\ 0 & d \end{pmatrix}\begin{pmatrix} 1 & p \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1/a & 0 \\ 0 & 1/d \end{pmatrix}, 1\right) \\ = L\left(f\Big|\begin{pmatrix} a & b \\ 0 & d \end{pmatrix}\begin{pmatrix} 1 & p \\ 0 & 1 \end{pmatrix}, 1\right),$$

and that

$$T(n)f = \sum f\Big|\begin{pmatrix} a & b \\ 0 & d \end{pmatrix}$$

$(a, b, d$ varying in the above range). This completes the proof of Theorem 2.

Remark. The perfect pairings of the theorem can also be viewed as injections from $S_2(m)$ into $\mathrm{H}^0\left(\Gamma, \mathrm{Hom}_{\mathbb{C}}(\mathbb{C}[\mathbb{P}^1(\mathbb{Q})]^0, \mathbb{C})\right)$, the dual space of $\mathrm{Co}(m)$. They are, in essence, variations of the classical Eichler-Shimura isomorphism which establishes a connection between $S_2(m)$ and the first cohomology group $\mathrm{H}^1(\Gamma, \mathbb{C})$ ([Sh1]). That there has to be a natural map between $\mathrm{H}^1(\Gamma, \mathbb{C})$ and the dual of $\mathrm{Co}(m)$ can immediately be recognized by looking at the long exact sequence of cohomology groups derived from the dual of the short sequence of Γ-modules

$$0 \to \mathbb{C}[\mathbb{P}^1(\mathbb{Q})]^0 \to \mathbb{C}[\mathbb{P}^1(\mathbb{Q})] \to \mathbb{C} \to 0$$

(with the obvious maps).

4. Closing the circle

So far we have described three types of correspondences, the $\mathcal{S}_{D,r}$, the pairings between $S_2(m)$ and the $\mathrm{CCo}^\epsilon(m)$ and the correspondences $\sigma \mapsto I_{\sigma,D,r}(\Delta, \rho)$. The are linked by the fact that the latter give, roughly stated, the adjoints of the former maps with respect to the pairings.

To make this precise we have to introduce two more notions. Denote by $J^\epsilon_{2,m}$ the space of all Jacobi forms of weight 2, index m and sign ϵ, and by $S^\epsilon_{2,m}$ the subspace of cusp forms (which, of course, equals $P^\epsilon_{2,m}$ for $\epsilon = -1$). Then there is the Petersson scalar product

$$\langle \cdot \mid \cdot \rangle : S^\epsilon_{2,m} \times J^\epsilon_{2,m} \to \mathbb{C},$$

a map which is linear in the first and antilinear in the second argument, whose restriction to $S^\epsilon_{2,m} \times S^\epsilon_{2,m}$ defines a positive definite scalar product so that the Hecke operators become hermitian, i.e

$$\langle \phi \mid T(n)\psi \rangle = \langle T(n)\phi \mid \psi \rangle$$

for all $\phi, \psi, T(n)$ ([E-Z],[S2]). It is easily checked that

$$\phi(\tau, z) \mapsto \overline{\phi(-\overline{\tau}, \overline{z})}$$

defines an antilinear operator on $J^\epsilon_{2,m}$. We denote this operator by ι.

The precise connection between the three types of correspondences can now be formulated as follows.

THEOREM 3. *For a fixed $\epsilon = \pm 1$, $\sigma \in \mathrm{CCo}^{\epsilon}(m)$, and m-admissible D, r with fundamental D and* sign $D = \epsilon$, *there is one and only one Jacobi form $\mathcal{L}_{D,r}(\sigma)$ in $P^{\epsilon}_{2,m}$ whose (Δ, ρ)-th Fourier coefficient, for any Δ such that neither Δ nor $D\Delta$ is a perfect square, equals the number*

$$I_{C,D,r}(\Delta, \rho) = \sum_{Q \in \mathcal{Q}(D\Delta, r\rho)} \chi_D(Q) C_Q \cdot C$$

introduced in sec. 1, where C is any chain of split Heegner cycles with $[\partial C] = \sigma$. There is a constant c, depending only on ϵ and m such that

$$c \langle \phi \mid \iota \mathcal{L}_{D,r}(\sigma) \rangle = \langle \mathcal{S}_{D,r}(\phi) \mid \sigma \rangle$$

for all $\phi \in P^{\epsilon}_{2,m}$ and all $\sigma \in \mathrm{CCo}^{\epsilon}(m)$.

Proof. The uniqueness of $\mathcal{L}_{D,r}(\sigma)$ follows from the fact that any non-trivial cusp $\phi \neq 0$ has a non-vanishing Fourier coefficient $c_\phi(\Delta, \rho)$ with Δ and $D\Delta$ not being a square (this is an easy consequence of [S-Z,Lemma 3.2] for $\epsilon = -1$ and the analogous statement for skew-holomorphic Jacobi forms will be published elsewhere).

It remains to show the existence of a $\mathcal{S}_{D,r}(\sigma)$ with the claimed properties. In [S1] we constructed for any A in Γ_1 a Jacobi form $\phi_{A,D,r}$ in $J^{\epsilon}_{2,m}$ whose (Δ, ρ)-th Fourier coefficient (ΔD not being a square) equals $I_{C_A,D,r}(\Delta, \rho)$, where C_A is a chain such that $\partial C_A = Z_A$ with

$$Z_A = \frac{1}{2}\left((A\infty) - (A0) - \epsilon((-A\infty) - (-A0))\right).$$

It was shown ([S1, end of sec. 2]), that

$$c \langle \phi \mid \phi_{A,D,r} \rangle = \langle \mathcal{S}_{D,r}(\phi) \mid [Z_A] \rangle$$

for all cusp forms ϕ and all A, with a constant c depending only on ϵ and m. According to Proposition 3 we can write any $\sigma \in \mathrm{CCo}^{\epsilon}(m)$ as

$$\sigma = \sum_A n_A [Z_A]$$

with suitable numbers n_A and A running through a set of representatives for $\Gamma \backslash \Gamma_1$. If we set

$$\psi := \sum_A n_A \iota \phi_{A,D,r}$$

then $pr(\psi)$ satisfies $c\langle\phi \mid \iota pr(\psi)\rangle = \langle \mathcal{S}_{D,r}(\phi) \mid \sigma\rangle$ for all ϕ, where pr denotes orthogonal projection from $J^{\epsilon}_{2,m}$ to $P^{\epsilon}_{2,m}$.

Clearly, the (Δ,ρ)-th coefficient of ψ equals $I_{C,D,r}(\Delta,\rho)$ for $D\Delta$ not a square. To investigate the result of applying pr to ψ we cite that the $0,r$-th coefficient of $\phi_{A,D,r}$ is of the form $\nu(A\infty)-\nu(A0)$ for a Γ-invariant map ν on $\mathbb{C}[\mathbb{P}^1(\mathbb{Q})]$. Thus, since σ is in $\mathrm{CCo}(m)$, i.e. in the kernel of the natural map $\mathrm{Co}(m) \to \mathbb{C}[\Gamma\backslash\mathbb{P}^1(\mathbb{Q})]$, we see that the $(0,r)$-th coefficients of ψ vanish, i.e. that ψ is a cusp form. Therefore the effect of applying pr to ψ is adding a trivial cusp form, i.e. a cusp form whose (Δ,ρ)-th coefficients are 0 except for Δ being a square. Thus, setting $\mathcal{L}_{D,r}(\sigma) = pr\psi$ proves Theorem 3.

There are two important corollaries to this theorem.

COROLLARY 1. *Let $\epsilon = \pm 1$. Then for any fixed m-admissible pair D,r, D being a fundamental discriminant and* $\operatorname{sign} D = \epsilon$, *the correspondence $\sigma \mapsto \mathcal{L}_{D,r}(\sigma)$ defines a Hecke equivariant map*

$$\mathcal{L}_{D,r}\colon \mathrm{CCo}^{\epsilon}(m) \to P^{\epsilon}_{2,m}.$$

There is a linear combination of these maps which is surjective.

Proof. By Theorem 1 there is a linear combination $\mathcal{S} = \sum_i n_i \mathcal{S}_{D_i,r_i}$ which is injective. Since, by Theorem 2, the pairing between $S_2(m)$ and $\mathrm{CCo}^{\epsilon}(m)$ is perfect, we deduce from the identity of Theorem 3 connecting $\mathcal{S}_{D,r}$ and $\mathcal{L}_{D,r}$ that the orthogonal complement of the image of $\mathcal{L} := \sum_i n_i \mathcal{L}_{D_i,r_i}$ equals the kernel of $\mathcal{S}$. Hence $\mathcal{L}$ is surjective. Via the identity of Theorem 3 we deduce also the Hecke equivariance of $\mathcal{L}_{D,r}$ from the Hecke equivariance of $\mathcal{S}_{D,r}$ and the compatibility of the Hecke operators with the pairing and scalar product ocuring in this identity.

The image of $\mathcal{L}_{D,r}$ is the orthogonal complement of the kernel of $\mathcal{S}_{D,r}$. We can give a more explicit statement if we restrict to newforms.

A Hecke eigenform in $f \in S_2(m)$ is called a newform of level m if the subspace of all modular forms in $S_2(m)$ having the same eigenvalues as f under all Hecke operators $T(n)$ ($\gcd(n,m)=1$) is one dimensional. It is a well-known fact that any Hecke eigenform in $S_2(m)$ is of the form $\sum n_d f(d\tau)$ with a newform f of a level m' dividing m, suitable numbers n_d, and d running through the divisors of m/m'. Thus, for studying modular forms, it suffices to focus on newforms. Let f be a newform, say in $S^{\epsilon}_2(m)$, i.e. let ϵ denote the sign in the functional equation of $L(f,s)$. Theorem 1 implies that there is one and only one Jacobi form ϕ (up to multiplication by a constant) having the same eigenvalues as f with respect to all $T(n)$. The Jacobi eigenforms in $P^{\epsilon}_{2,m}$ having the same eigenvalues as the newforms in $S^{\epsilon}_{2}(m)$ are called *Jacobi newforms*.

When does a Jacobi newform ϕ occur in the image of $\mathcal{L}_{D,r}$? It is a fact that the first coefficient $a_f(1)$ of a modular newform f is always different from zero. Thus $\mathcal{S}_{D,r}(\phi) = 0$ if $c_\phi(D,r)$, the first Fourier coefficient of $\mathcal{S}_{D,r}(\phi)$, vanishes. On the other hand $\mathcal{S}_{D,r}(\phi) = 0$ is equivalent to ϕ being orthogonal to the image of $\mathcal{L}_{D,r}$, which, for a newform, is easily checked to be equivalent to ϕ not being in the image of $\mathcal{L}_{D,r}$. Thus, we have proved

COROLLARY 2. *Let $\epsilon = \pm 1$ and D, r an m-admissible pair, D being a fundamental discriminant,* sign $D = \epsilon$. *Then a Jacobi newform ϕ in $P^{\epsilon}_{2,m}$ can be obtained as $\mathcal{L}_{D,r}(\sigma)$ with a sutable σ if and only if its Fourier coefficient $c_\phi(D,r)$ is different from 0.*

5. What is it good for ?

What can one do with Theorem 3 and its Corollaries? The main point in Theorem 3 is that the $\mathcal{L}_{D,r}(\sigma)$ are given by explicit, surprisingly simple formulas. There are three applications of this which come immediately to mind.

APPLICATION 1. *It is possible to compute closed formulas for all the Hecke eigenforms in $S_2(m)$ and $P^{+}_{2,m} \oplus P^{-}_{2,m}$ with fairly mild computational expenses.*

Indeed, it suffices to compute the Hecke eigenforms in CCo(m). The claimed closed formulas for Jacobi Hecke eigenforms are then obtained, according to Corollary 2, by applying sufficiently many maps $\mathcal{L}_{D,r}$ to these eigenforms. Applying the maps $\mathcal{S}_{D,r}$ to the Jacobi Hecke eigenforms so obtained yields (at least) the newforms in $S_2(m)$ (which, by the remarks in the foregoing section, suffice to produce all Hecke eigenforms if so desired).

In passing it should be noted that one can compute an explicit bound for the D's at most needed to be sure that every Jacobi or modular newform is hit at least once by the images of the $\mathcal{L}_{D,r}$ and $\mathcal{S}_{D,r}$. This bound is obtained by applying Corollary 2 (and the remarks preceding this Corollary) and the fact that for a Jacobi newform the nonvanishing of a (D,r)-th Fourier coefficient with $|D|$ below an effectively computable bound (depending only on the index m and the weight 2) is guaranteed.

To compute the Hecke eigenforms in $\mathrm{CCo}(m)$, or rather those eigenforms which correspond to newforms in $S_2(m)$, choose a basis for the space $\mathrm{CCo}(m)$. This is easily done using the map

$$\pi\colon \mathbb{C}[\Gamma\backslash\Gamma_1] \to \mathrm{Co}(m)$$

described in Proposition 3. Indeed, compute a basis $b_1, \ldots, b_s$ for

$$\ker(\rho(S)-1)+\ker(\rho(ST)-1),$$

where $\rho(A)$ denotes right-multiplication by the matrix A on $\mathbb{C}[\Gamma\backslash\Gamma_1]$. Then choose $r := [\Gamma_1 : \Gamma] - s$ cosets ΓA_i such that the $\Gamma A_i, b_j$ $(1 \leq i \leq r,\ 1 \leq j \leq s)$, considered as elements of $\mathbb{C}[\Gamma\backslash\Gamma_1]$, are linearly independent. From Proposition 3 we deduce that the $\pi(\Gamma A_i) = [(A_i\infty) - (A_i 0)]$ $(1 \leq i \leq r)$ form a basis for $\mathrm{Co}(m)$.

Next, as step 2, choose a number N and compute the matrices of all $T(p)$ with p prime, p not dividing m and $p < N$, with respect to the above basis. This is easily done using the formula

$$T(p)\circ\pi = \pi\circ\rho\left(\begin{pmatrix} \frac{1}{p} & 0 \\ 0 & 1\end{pmatrix}\right) - \sum_{\mu=0}^{p-1}\sum_{i=0}^{r_\mu} \pi\circ\rho\left(\begin{pmatrix} 1 & \frac{-\mu}{p} \\ 0 & \frac{1}{p}\end{pmatrix} C_{\mu,i}\right).$$

Here $\rho(M)$ for any M denotes that operator on $\mathbb{C}[\Gamma\backslash\Gamma_1]$ which maps a coset $\Gamma\begin{pmatrix} a & b \\ c & d\end{pmatrix}$ to that coset which contains a matrix with second row congruent to $(c,d)M$ modulo m, and the $C_{\mu,i}$ are defined by

$$C_{\mu,i} = \begin{pmatrix} a_0 & 1 \\ 1 & 0\end{pmatrix}\cdots\begin{pmatrix} a_i & 1 \\ 1 & 0\end{pmatrix}\begin{pmatrix} +1 & 0 \\ 0 & (-1)^{i-1}\end{pmatrix},$$

where $\frac{\mu}{p} = [a_0, \ldots, a_{r_\mu}]$ is the continued fraction decomposition of $\frac{\mu}{p}$ (with $a_{r_\mu} \geq 2$ if $\mu \neq 0$). The stated formula can be easily checked using the procedure described in the proof of Proposition 3 to express a co-invariant class in terms of the generators $[(A\infty) - (A0)]$.

In the third and final step, diagonalize the matrices of the $T(p)$ to find the canonical decomposition of $\mathrm{Co}(m)$ considered as a module over the algebra $\mathcal{H}(N)$ generated by the $T(p)$ with $p < N$.

Each one-dimensional piece in this canonical decomposition corresponds to a newform in $S_2(m)$. Thus, if the canonical decomposition contains as many one-dimensional components as there are newforms in $S_2(m)$ (whose number is given by a well-known explicit formula) then our task is fullfilled and we stop. Otherwise one increases N and restarts at step 2 with the new $T(p)$.

Each newform in $S_2(m)$ corresponds to a one-dimensional component of the canonical decomposition of $\mathrm{CCo}(m)$ with respect to $\mathcal{H}$, the algebra generated by all Hecke operators $T(n)$ ($\gcd(n,m) = 1$). Since $\mathrm{CCo}(m)$ and hence $\mathcal{H}$ is finite dimensional it is thus clear that the above algorithm eventually stops. (Here one also needs that each $T(n)$ can be written as a polynomial in the $T(p)$ where p runs through the prime divisors of n; this fact is also the reason why we compute only the $T(p)$ with p prime.)

The Γ-cosets in the above algrithm can be easily handeled by using the bijection

$$\Gamma\backslash\Gamma_1 \to \mathbb{P}(\mathbb{Z}/m\mathbb{Z}), \quad \Gamma\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto [c:d].$$

We would like to stress that the main point of the Application 1 is that it yields closed formulas for the Jacobi and modular forms by applying the $\mathcal{L}_{D,r}$ and $\mathcal{S}_{D,r}$ to the eigenforms in $\mathrm{CCo}(m)$. If one is solely interested in tabulating Fourier coefficients of modular forms, then, of course, one does not have to use the $\mathcal{L}_{D,r}$ and $\mathcal{S}_{D,r}$ at all, but one can simply apply the Hecke operator $T(n)$ to an eigenform in $\mathrm{CCo}(m)$ to find the n-th Forier coefficient of its corresponding modular form. The latter method (including the above algorithm to compute the eigenforms in $\mathrm{CCo}(m)$) can be found in a slightly different formulation in [M].

The described procedure to find explicit formulas for Jacobi and modular forms requires only computationally mild expenses in the sense that it starts from scratch using only linear algebra, and that it does not require any delicate tool to be launched, like ,e.g., the trace formula for the $T(n)$.

Before stating the next application we explain one more reason why it is interesting to compute Jacobi forms at all. The Jacobi form ϕ in $P^{\epsilon}_{2,m}$ having the same eigenvalues as a given newform f in $S^{\epsilon}_2(m)$ carries not only the information about the Hecke eigenvalues of f, but it carries also information about the arithmetically interesting values $L(f,D,1)$, where $L(f,D,s)$, for any discriminant D, denotes the L-series attached to the *twisted* modular form

$$f \otimes \left(\frac{D}{*}\right) = \sum_{n=1}^{\infty} \left(\frac{D}{n}\right) a_f(n) q^n.$$

More precisely, if D is a fundamental discriminant which is a square mod

$4m$, and relatively prime to m then one has

$$|c_\phi(D,r)|^2 = c_m \frac{\langle \phi \mid \phi \rangle}{\langle f \mid f \rangle} \sqrt{|D|}\, L(f,D,1)$$

with a constant c_m depending only on ϵ and m. Here r is any solution of $r^2 \equiv D \mod 4m$ and $\langle f \mid f \rangle$ denotes the *Petersson scalar product* of f on $S_2(m)$ (cf. [G-K-Z, Corollary 1 in sec.II.3] for the case $\epsilon = -1$, and [S2, Proposition 1] from which the above identity for the case $\epsilon = +1$ can be deduced analogously to the reasoning in [G-K-Z]; the above identity is the translation to the language of Jacobi forms of a well-known theorem of Waldspurger). Thus, given that one needs the values of the $L(f,D,1)$, one will be interested in describing as explicitly as possible the ϕ attached to the newform f. Such an explicit description can be calculated.

APPLICATION 2. *Given sufficiently many Fourier coefficients of a newform f in $S_2(m)$ it is possible to compute explicit formulas for the values $L(f,D,1)$ where D runs through the set of fundamental discriminants which are squares modulo $4m$ and relatively prime to m.*

. Indeed, having sufficiently many Fourier coefficient means that we have sufficiently many Hecke eigenvalues of f for identifying it by these finitely many eigenvalues. Hence, we simply have to compute a co-invariant σ in $\mathrm{CCo}(m)$ having the same eigenvalues with respect to the corresponding (finitely many) Hecke operators. Then the explicitly given Fourier coefficients of any (non-vanishing) $\mathcal{L}_{D',r'}(\sigma)$ provide the desired information.

Finally, applying the $\mathcal{S}_{D'',r''}$ to the $\mathcal{L}_{D',r'}(\sigma)$ so obtained we recover the newform with which we started. Thus, we may formulate this as

APPLICATION 3. *Given sufficiently many Fourier coefficients of a newform f in $S_2(m)$ it is possible to compute from these finitely many coefficients a completely explicit formula which gives all the Fourier coefficients of this form.*

It is not yet clear whether Theorem 3 and its Corollaries can also provide new theoretical insights. Yet the space of Γ-co-invariants $\mathrm{CCo}(m)$ of $\mathbb{C}[\mathbb{P}^1(\mathbb{Q})]^0$ looks rather handy, as is confirmed, for example, by Proposition 3, and any Hecke eigenform extracted from it can be immediately translated, via Theorem 3 and Corollaries, into a closed formula for the corresponding Jacobi and modular Hecke eigenforms. Thus, it seems to be worthwhile to investigate more closely the co-invariants and the action of the Hecke operators on it, and perhaps one may hope to find (non-trivial) families of Hecke eigenforms in passing. Another point is that the space $\mathrm{CCo}(m)$ has a natural $\mathbb{Q}$-structure (replace $\mathbb{C}$ by $\mathbb{Q}$ in the definition of $\mathrm{CCo}(m)$ and related spaces) and applying the $\mathcal{S}_{D,r} \circ \mathcal{L}_{D',r'}$ yields Hecke invariant subspaces of $S_2(m)$ which can be explicitly described. One may try to exploit this for obtaining information about the decomposition over $\mathbb{Q}$ of the algebra generated by the Hecke operat

6. An example: Tunnell's theorem

Let f denote a non-vanishing element of the one dimensional space $S_2(32)$. It is known that if a positive integer D is a congruent number, i.e. if it is the area of a right triangle with rational sides, then $L(f, D, 1) = 0$. A general conjecture in the theory of elliptic curves would imply that the converse statement holds true for squarefree D. These observations are the starting point of [T] where then modular forms of weight $\frac{3}{2}$ corresponding to f via Shimura's liftings were computed to obtain via Waldspurger's theorem explicit formulas for $L(f, D, 1)$. It may be amusing to mimic this procedure, but here with modular forms of weight $\frac{3}{2}$ replaced by Jacobi forms. The formulas so obtained are different from the one obtained by Tunnell. It may be interesting to investigate whether the formulas obtained here can be converted into those given in [T] by elementary means, i.e. directly without going through the theory of modular and Jacobi forms.

To find a formula for the Jacobi form ϕ and in particular for the $L(f, D, 1)$ attached to f, we note first of all that $S_2(32) = S_2^+(32)$, so that

$$\dim \mathrm{CCo}(m) = \dim \mathrm{CCo}^+(m) = \dim P_{2,m}^+ = 1.$$

Moreover, it is known that $L(f, 1)$ does not vanish, i.e. that the $(1,1)$-th coefficient of ϕ does not vanish. Hence, if σ is a non-zero element of $\mathrm{CCo}^+(m)$, then $\phi = \mathcal{L}_{1,1}(\sigma)$ and $f = \mathcal{S}_{D,r}\mathcal{L}_{1,1}(\sigma)$ for any D such that $c_\phi(D, r) \neq 0$ (up to multiplication by constants).

If we pick

$$\sigma = \frac{1}{2}([(-\frac{1}{3}) - (0)] - [(\frac{1}{3}) - (0)]),$$

then the (Δ, ρ)-th coefficient of $\mathcal{L}_{1,1}(\sigma)$ (Δ not a perfect square) equals

$$I(\Delta, \rho) := \frac{1}{2} \sum_{Q \in \mathcal{Q}(\Delta,\rho)} (\operatorname{sign} Q(1,3) - \operatorname{sign} Q(0,1)),$$

which, by a short calculation, can be rewritten as

$$I(\Delta, \rho) = \nu_+(\Delta, \rho) - \nu_-(\Delta, \rho),$$

where we put

$$\nu_{\pm}(\Delta,\rho) = \big\{(a,b,c) \in \mathbb{Z}^3 \mid b^2 - 4ac = \Delta,\ b^2 < D,\ \pm a > 0 \\ a \equiv 3\frac{b+\rho}{2} \mod 32,\ 3c \equiv \frac{b-\rho}{2} \mod 32\big\}.$$

We find $I(17,23) = 1$, hence $\sigma \neq 0$. So the above reasoning can be applied to this σ to obtain closed formulas for ϕ and f. In particular, we find

$$L(f,D,1) = \frac{\gamma}{\sqrt{D}}\left(\nu_+(D,r) - \nu_-(D,r)\right)^2$$

for any odd positive fundamental discriminant D which is a square modulo 8 (and hence mod 128), and where r is any solution of $r^2 \equiv D \mod 128$. Here γ is a constant, not depending on D, r. From this formula we deduce as in [T] the Theorem stated in the Introduction.

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