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The k-dimensional Duffin and Schaeffer conjecture.

par A.D. POLLINGTON AND R.C. VAUGHAN

Résumé — **Nous montrons que la conjecture de Duffin et Schaeffer est vraie en toute dimension supérieure à 1.**

Abstract — *We show that the Duffin and Schaeffer conjecture holds in all dimensions greater than one.*

In 1941 Duffin and Schaeffer [1] made the following conjecture:

CONJECTURE. *Let $\{\alpha_n\}$ denote a sequence of real numbers with*

$$0 \leq \alpha_n < \frac{1}{2}$$

then the inequalities

$$|nx - a| < \alpha_n, \quad (a, n) = 1, \tag{1}$$

have infinitely many solutions for almost all x if and only if

$$\sum_{n=1}^{\infty} \frac{\alpha_n \varphi(n)}{n} \tag{2}$$

diverges.

If (2) converges then it easily follows from the Borel-Cantelli lemma that the set of x's satisfying infinitely many of the inequalities (1) has Lebesgues measure zero. Duffin and Schaeffer gave conditions on the α_n for which the conjecture is true and showed that the condition $(a, n) = 1$ is necessary. In 1970 Erdős [2] showed that the conjecture holds if $\alpha_n = \frac{\epsilon}{n}$ or 0. This was later extended by Vaaler [6] who showed that $\alpha_n = O(\frac{1}{n})$ is sufficient. In his book on metric number theory [5] Sprindzuk considers a k-dimensional analogue of the conjecture in which (1) is replaced by

$$\max(|x_1 n - a_1|, ..., |x_k n - a_k|) < \alpha_n \quad (a_i, n) = 1, \quad i = 1, ..., k \tag{3}$$

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and (2) by

(4) $$\sum_{n=1}^{\infty}(\frac{\alpha_n\varphi(n)}{n})^k$$

where the measure is now k-dimensional Lebesgues measure. He states that the study of such approximations subject to the conditions $(a_1, n) = \cdots = (a_k, n) = 1$ is probably a problem of the same degree of complexity as the case $n = 1$. This appears not to be the case. For we can now prove the k-dimensional analogue of the Duffin and Schaeffer conjecture.

We prove the following result:

THEOREM. *Let $k > 1$ and let $\{\alpha_n\}$ denote a sequence of real numbers with*

$$0 \leq \alpha_n < \frac{1}{2}$$

and suppose that

$$\sum_{n=1}^{\infty}(\frac{\alpha_n\varphi(n)}{n})^k$$

diverges. Then the inequalities

$$\max(|x_1 n - a_1|, ..., |x_k n - a_k|) < \alpha_n \quad (a_i, n) = 1, \quad i = 1, ..., k$$

have infinitely many solutions for almost all $x \in R^k$.

Unfortunately our method does not readily extend to the case $k = 1$. We are able find some more sequences $\{\alpha_n\}$ for which the conjecture holds, for example if $\alpha_n = 0$ or $1 \ll \alpha_n$, but not to settle the dimension one case. In the k-dimensional case, $k \geq 2$, Vilchinski has previously shown that we may take $\alpha_n = O(n^{-\gamma})$ for any $\gamma > 0$.

Put

$$E_n = E_n^{(1)} \times \cdots \times E_n^{(k)}$$

where

(5) $$E_n^{(i)} = \bigcup_{\substack{1 \leq a_i \leq n \\ (a_i, n) = 1}} \left(\frac{a_i - \alpha_n}{n}, \frac{a_i + \alpha_n}{n}\right).$$

Then E_n is the set counted in (3) and

$$(6) \qquad \lambda_k(E_n) = (\frac{2\alpha_n\varphi(n)}{n})^k.$$

Thus (4) becomes

$$(7) \qquad \sum_{k=1}^{\infty} \lambda_k(E_n)$$

diverges.

We are interested in

$$(8) \qquad \lambda_k\left(\bigcap_{m=1}^{\infty}\bigcup_{n=m}^{\infty} E_n\right).$$

By an ergodic theorem of Gallagher [3], see Sprindzuk [5], this is either zero or one. Gallagher proves his result for dimension one. The corresponding k-dimensional result is proved by Vilchinski [7]. In order to prove the theorem it therefore suffices to show that (8) is not zero.

Since (7) diverges, and $\lambda_k(E_n) \to 0$ as $n \to \infty$, given any $1 > \eta > 0$, for every N we can find a finite set Z so that if $z \in Z$ then $z > N$, and

$$(9) \qquad \eta^2 < \Lambda(Z) = \sum_{n\in Z} \lambda_k(E_n) < \eta.$$

By the Cauchy-Schwarz inequality

$$(10) \qquad \lambda_k\left(\bigcup_{n\in Z} E_n\right) \geq \frac{(\sum_n \lambda_k(E_n))^2}{(\sum_n\sum_m \lambda_k(E_n \cap E_m))},$$

this is Lemma 5 of Sprindzuk [5]. So provided there is some absolute constant c for which

$$(11) \qquad \sum_{\substack{n\neq m\\ m,n\in Z}} \lambda_k(E_n\cap E_m) \leq c\sum_{n\in Z}\lambda_k(E_n)$$

the theorem is proved.

Thus we need to bound

$$\sum_{\substack{n\neq m\\ m,n\in Z}} \lambda_k(E_n\cap E_m).$$

In the subsequent discussion constants in the Vinogradov $\ll$ symbol depend at most on the dimension k.

Note that

$$\lambda_k(E_n)=\prod_{i=1}^{k}\lambda(E_n^{(i)}) \tag{12}$$

and

$$\lambda_k(E_n\cap E_m)=\prod_{i=1}^{k}\lambda(E_n^{(i)}\cap E_m^{(i)}). \tag{13}$$

We now concentrate on $k=1$ and use (12) and (13) to obtain a bound for $k>1$.

We wish to obtain an estimate for

$$\lambda(E_n\cap E_m)\quad \text{with } n>m.$$

Put

$$\delta=\min(\frac{\alpha_n}{n},\frac{\alpha_m}{m})\quad\text{and}\quad \Delta=\max(\frac{\alpha_n}{n},\frac{\alpha_m}{m}), \tag{14}$$

and

$$d=(m,n)\quad m'=\frac{m}{d}\quad n'=\frac{n}{d}.$$

Then

$$\lambda(E_n\cap E_m)\le 2\delta \sum_{\substack{|\frac{a}{n}-\frac{b}{m}|<2\Delta\\ (a,n)=1\\ (b,m)=1}} 1. \tag{15}$$

By estimating

$$\sum_{\substack{|\frac{a}{n}-\frac{b}{m}|<2\Delta\\ (a,n)=1\\ (b,m)=1}} 1 \tag{16}$$

we obtain

LEMMA 1. *With E_n as above and $d = (m, n)$ there is an absolute constant c so that*

$$\lambda(E_n \cap E_m) \le c\lambda(E_n)\lambda(E_m) \prod_{\substack{p|m'n' \\ p > \frac{2mn\Delta}{d}}} (1 - \frac{1}{p})^{-1}. \tag{17}$$

Let $g(n)$ be defined as the least number so that

$$\sum_{\substack{p|n \\ p > g(n)}} \frac{1}{p} < 2$$

Put

$$t = \max(g(n'), g(m')).$$

Then by Lemma 1

$$\lambda(E_n \cap E_m) \ll \lambda(E_n)\lambda(E_m) \prod_{\substack{p|m'n' \\ \frac{2mn\Delta}{d} < p < t}} (1 - \frac{1}{p})^{-1}. \tag{18}$$

In particular if $\frac{2mn\Delta}{d} \ge t$ then

$$\lambda(E_n \cap E_m) \ll \lambda(E_n)\lambda(E_m). \tag{19}$$

Now

$$\prod_{\substack{p|\frac{nm}{d^2} \\ p<t}} (1 - \frac{1}{p})^{-1} \le \prod_{p<t} (1 - \frac{1}{p})^{-1} \ll \log t. \tag{20}$$

We now return to the k-dimensional case. By (12) and (13)

$$\sum_{\substack{n \ne m \\ \log t \le \Lambda(Z)^{-1/k}}} \lambda_k(E_n \cap E_m) \ll \Lambda(Z)^{-1} \sum_{n \ne m} \lambda_k(E_n)\lambda_k(E_m) \tag{21}$$

$$\ll \Lambda(Z).$$

From now on we shall assume that

$$\log t > \Lambda(Z)^{-1/k}.$$

We distinguish four cases:

(a) $$\Delta = \frac{\alpha_m}{m}, \quad \alpha_n < \alpha_m$$

(b) $$\Delta = \frac{\alpha_m}{m}, \quad \alpha_n \geq \alpha_m$$

(c) $$\Delta = \frac{\alpha_n}{n}, \quad \alpha_n < \alpha_m$$

(d) $$\Delta = \frac{\alpha_n}{n}, \quad \alpha_n \geq \alpha_m.$$

Recall that $m < n$ so (c) is impossible. We will consider the other three cases seperately. The first case corresponds to the situation considered by Erdős .

Case (a).

We have
$$\frac{mn\Delta}{d} = n'\alpha_m.$$
We need to consider pairs m, n with

(22) $$1 < n'\alpha_m < t$$

Let $A_{u,v}$ denote that part of the sum
$$\sum_{\substack{m \neq n \\ case(a)}} \lambda_k(E_n \cap E_m)$$
for which $g(m') = u$ and $g(n') = v$. Then, by Lemma 1 and (22)
$$A_{u,v} \ll \log^k t \sum_m \lambda_k(E_m) \sum\nolimits_{\substack{(u) \\ m'|m \\ 1 \leq m' < t\alpha_m^{-1}}} \sum\nolimits_{\substack{(v) \\ \alpha_m{}^{-1} \leq n' < t\alpha_m{}^{-1} \\ n = n'd}} \lambda_k(E_n).$$
Where $\sum_{(u)}$ means the sum of those m' with $g(m') = u$ and $m = m'd$. Since
$$\lambda_k(E_n) = \left(2\alpha_n \frac{\varphi(n)}{n}\right)^k \ll \alpha_m^k$$

then

$$A_{u,v} \ll \log^k t \sum_m \lambda_k(E_m)\alpha_m^k \sum\nolimits_{\substack{m'|m \\ 1\le m'<t\alpha_m^{-1}}}^{(u)} \sum\nolimits_{\substack{\alpha_m{}^{-1}\le n'<t\alpha_m{}^{-1} \\ n=n'd}}^{(v)} 1.$$

To estimate the inner sum we will use the following result due to Erdős which also appears in Vaaler [6]. Let $N(\xi, v, x)$ be the number of integers $n \le x$ which satisfy

$$\sum_{\substack{p|n \\ p\ge v}} \frac{1}{p} \ge \xi.$$

Then

Lemma 2. *For any $\epsilon > 0$ and any $\xi > 0$ there exists a positive integer $v_0 = v_0(\xi, \epsilon)$ such that for all $x > 0$ and all $v > v_0$,*

$$N(\xi, v, x) \le x \exp\{-v^{\beta(1-\epsilon)}\}$$

where $\log \beta = \xi$.

Applying this result with $\xi = 1$ and $\beta(1-\epsilon) = \frac{3}{2}$ we have

Corollary. *Given $x > 0$ and $v \ge 1$*

(23) $$N(1, v, x) \ll x \exp(-v^{\frac{3}{2}}).$$

Using (23) and partial summation

$$\sum\nolimits_{1\le n'\le t\alpha_n^{-1}}^{(v)} \ll t\alpha_m^{-1} \exp(-v^{\frac{3}{2}}).$$

Applying (23) again

$$A_{u,v} \ll t^2 \log^k t \exp(-u^{\frac{3}{2}}) \exp(-v^{\frac{3}{2}}) \sum_m \lambda_k(E_m),$$

since $k \ge 2$. Then summing over u and v we have

(24) $$\sum_{\substack{m\ne n \\ case(a)}} \lambda_k(E_n \cap E_m) \ll \Lambda(Z)$$

if η is sufficiently small.

The other cases are similar and we have

$$\sum_{n \neq m} \lambda_k(E_n \cap E_m) \leq c \sum_n \lambda_k(E_n) \tag{25}$$

where c is a constant depending only on k. This completes the proof of the Theorem .

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