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On ruled fields

by JACK OHM

Résumé — Nous discutons de quelques résultats et problèmes en relation avec les fondements de la théorie des extensions rationnelles de corps d'une ou plusieurs variables.

Abstract — *Some results and problems that arise in connection with the foundations of the theory of ruled and rational field extensions are discussed.*

I would like to discuss here some results and problems that deal with the foundations of the theory of ruled fields and that have as their centerpiece Theorem 1.1 below. The proofs are elementary, with a certain kind of specialization argument as a common underlying theme.

A *ruled field* may be defined to be a triple (L, K, t) such that L is a field, K is a subfield of L, t is an element of L which is transcendental over K, and $L = K(t)$. The pair (K, t) will be called a *ruling* of L, and L will be said to be (K,t)-*ruled*. Occasionally, when it is not likely to cause confusion, we shall abbreviate this terminology to "$L = K(t)$ is a ruled field".

Fix an algebraically closed field Ω and a subfield k of Ω such that Ω has infinite degree of transcendance (abbreviated *dt*) over k, and consider the category whose objects are the subfields of Ω which contain k and have finite *dt* over k and whose morphisms are the (necessarily injective) k-homomorphisms. We can form a subcategory by taking its objects to be the ruled fields and by defining a morphism from a ruled field (L, K, t) to a ruled field (L', K', t') to be a homomorphism h of L to L' such that $h(K) \subset K'$ and $h(t) = t'$. Since such an h is completely determined by its restriction to K, we could equivalently have defined the morphism h to be merely a homomorphism from K to K'.

Problem. Let (L, K, t) and (L', K', t') be ruled fields. If there exists a homomophism ϕ (resp. an isomorphism) from L to L', does there exist a morphism (resp. an isomorphism) from (L, K, t) to (L', K', t')? The answer to the homomorphism part of the question is "yes"; see 1.1-(i) below. As for the isomorphism part, this is just the famous Zariski problem

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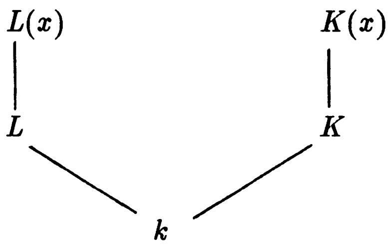
for fields and is now known to have a negative answer; cf. [BCSS, 1985]. However, we want to focus our attention here on the restricted isomorphism problem obtained by requiring $\phi(t) = t'$; we shall later refer to this as the Samuel Problem. Part (iii) of Theorem 1.1 asserts that this question has an affirmative answer if K/k is finitely generated and the base field k is infinite, but the question remains open without these hypotheses.

In addition to the notation *dt* for "degree of transcendence", we use $<$ for proper containment and *tr.* for "transcendental".

1. The central theorem.

1.1. THEOREM. *Let L and K be subfields of a field Q and extensions of a field k, let x be an element of Q which is tr. over K, and assume $dt(L/k) \leq dt(K/k) < \infty$:*



Then

i) (Roquette [R, 1964] for k infinite, Ohm [O, 1984] for k arbitrary) $L \subset K(x)$ implies L is k-isomorphic to a subfield of K ;

ii) (Nagata [N,1967]) $L(x) = K(x)$ implies either L is k-isomorphic to K or both L and K are ruled over k ;

iii) (Samuel [Sa,1953]) $L(x) = K(x)$, k infinite, and K/k finitely generated imply L is k-isomorphic to K.

The proof will be given in section 2, but first we want to mention some applications (in 1.2 and 1.3 below).

Note that the hypothesis $L(x) = K(x)$ of (ii) and (iii) implies x is tr. over L. One may also assume x is tr. over L in (i) ; for if x is algebraic over L, then by the hypothesis $dt(L/K) \leq dt(K/k) \leq dt(K(x)/k)$, there exists t in K which is tr. over L, and we can replace x by $x + t$. Note too that 1.1 may be regarded as a statement involving simple tr. base change ; for, since x is tr. over K, K and $k(x)$ are linearly disjoint over k, and similarly for L.

1.2. Two and one-half proofs that Lüroth's Theorem implies the Generalized Lüroth Theorem (the one-half since one of the proofs only works for k_0 infinite).

Each of the parts of 1.1 can be used as the induction step in deriving the Generalized Lüroth Theorem from the classical Lüroth Theorem.

LÜROTH'S THEOREM, LT. *(cf. [vdW] or [Sc]). Let* $k_0 \subset F \subset k_0(x)$ *be field extensions, with* x *tr. over* k_0*. Then there exists* t *in* $k_0(x)$ *such that* $F = k_0(t)$.

GENERALIZED LÜROTH THEOREM, GLT. *(Gordan, Netto, Igusa, cf. [Sc]). Let* $k_0 \subset F \subset k_0(x_1, \cdots, x_n)$ *be field extensions, with* $x_1, \cdots, x_n$ *algebraically independent over* k_0 *and* $dt(F/k_0) = 1$*. Then there exists* t *in* $k_0(x_1, \cdots, x_n)$ *such that* $F = k_0(t)$.

The proofs that LT implies GLT are by induction on n. Note first that we may assume $n > 1$ and that $x_2, \cdots, x_n$ are algebraically independant over F. Then we have :

(1.2.1)

$$\begin{array}{ccc} L(x_n) & & K(x_n) \\ | & & | \\ L = F(x_2, \cdots, x_{n-1}) & & K = k(x_1) \\ & \searrow \quad \swarrow & \\ & k = k_0(x_2, \cdots, x_{n-1}) & \end{array}$$

First proof, via 1.1-(i). By (i), L is k-isomorphic to a subfield of $K = k_0(x_1, \cdots, x_{n-1})$, so we are done by induction.

Second proof, via 1.1-(ii) ([N,1967]). By Lüroth's Theorem applied to $k_0(x_2, \cdots, x_n) \subset F(x_2, \cdots, x_n) \subset k_0(x_2, \cdots x_n)(x_1)$, there exists x_1' in $k_0(x_1, \cdots, x_n)$ such that

$$F(x_2, \cdots, x_n) = k_0(x_1', x_2, \cdots, x_n).$$

Thus, we may assume $L(x_n) = K(x_n)$ in diagram (1.2.1).

By 1.1-(ii), either $F \subset L \cong_k k_0(x_1, \cdots, x_{n-1})$, in which case we are done by induction, or L is ruled over k. In the latter case, since $dt(L/k) = 1$, L is simple tr. over the algebraic closure of k in L. But k is algebraically closed in $K(x_n)$ and a fortiori in L. Thus, L is then simple tr. over $k : L = k(t)$. Therefore $F \subset L = k_0(t, x_2, \cdots, x_{n-1})$, and we are again done by induction.

Third (half) proof, via 1.1-(iii), valid for k_0 infinite (Samuel [Sa,1953]). Same as the first part of the preceding proof.

REMARK. There is a polynomial version of LT, which asserts that if the field F contains a polynomial of $deg > 0$, then $F = k_0(t)$, with t a polynomial ; cf. [Sc, p. 10, Theorem 4] (where the result is attributed to E. Noether in char 0 and to Schinzel in char > 0). An analogous polynomial version of GLT can be derived from this result by using a sharpened version of 1.1-(i) for the induction step.

1.3. Subrational = unirational.

Part (i) of 1.1 can be used to prove this equivalence. Let us first review the terminology. A field extension $k \subset K$ is called *pure transcendental* if $dt(K/k) > 0$ and there exists a transcendence basis T of K/k such that $K = k(T)$, and *rational* if K/k is finitely generated and pure transcendental. An extension $k \subset K$ is called *subrational* (resp.*unirational*) if there exists an extension (resp. an algebraic extension) of K which is rational over k.

One thing to note immediately is that if $dt(K/k)$ is finite and there exists an extension L of K which is pure transcendental over k, then there exists such an L with L/k finitely generated. For, if $L = k(T)$, with T an algebraically independent set over k, we can choose a finite subset T_0 of T such that $k(T_0)$ contains a transcendence basis of K/k ; and since $k(T_0)$ is algebraically closed in $k(T)$, it follows that $K \subset k(T_0)$.

COROLLARY to 1.1.-(i) (Chevalley-Shimura [C,1954 ; p. 319] for k_0 infinite, Ohm [O, 1984] for k_0 arbitrary). *Suppose*

$$k < L \subset k(x_1, \cdots, x_n)$$

is an extension of fields, with $x_1, \cdots, x_n$ algebraically independent over k. If $dt(L/k) = m$, then L is k-isomorphic to a subfield of $k(x_1, \cdots, x_m)$.

2. Proof of Theorem 1.1.

The proof of 1.1 requires the following elementary lemmas.

2.1. LEMMA. *(cf. [ZS 1, p. 101, Theorem 29]). Let $k \subset L$ be a field extension of finite dt, let v be a valuation of L/k, and let L^* denote the residue field of v. Then either $dt(L^*/k) < dt(L/K)$, or v is trivial (i.e. is the 0-valuation) and the residue map $L \to L^*$ is an isomorphism.*

2.2. LEMMA. *Let K be a field, let x be an element tr. over K, and let $a_1, \cdots, a_n$ be nonzero elements of $K(x)$. Then for all but finitely many* *elements c in K, the $(x-c)$-adic valuation of $K(x)/K$ (i.e. the valuation whose ring is $K[x]_{(x-c)}$) has value 0 at $a_1, \cdots, a_n$ (or, equivalently, the $(x-c)$-adic residues of $a_1, \cdots, a_n$ are finite and nonzero).*

PROOF. Write $a_1, \cdots a_n$ as quotients of polynomials in $K[x]$, and choose c to avoid the zeros of these polynomials.

2.3. LEMMA - THE RULED RESIDUE THEOREM, RRT. *(Nagata [N,1967] for the discrete case, Ohm [O, 1983] for the general case). Let $(K_0, v_0) \subset (K_0(x), v)$ be a valued field extension with x tr. over K_0, and let $k_0 \subset k$ be the corresponding residue field extension. Then either k/k_0 is algebraic or k/k_0 is ruled.*

PROOF OF 1.1.-(ii)([N,1967]). Let v be the x-adic valuation of $K(x)/K$, and note that the residue field of v is K. Let L^* denote the residue field of $v|L$. By 2.1. either

$$dt(L^*/k) < dt(L/k) \ (= dt(K/k)),$$

or v is trivial on L and the residue map $L \to L^*$ is a k-isomorphism. In the latter case v must be the x-adic valuation of $L(x)/L$, and then the residue field of $L(x)$ is L^* and $K = L^*$. In the former case, K/L^* is not algebraic, hence is ruled by the RRT 2.3 ; and then K/k is a fortiori ruled.

PROOF OF 1.1.-(iii). Since K/k is finitely generated, we can choose $b_1, \cdots, b_n$ to be a set of nonzero generators of K/k. Write

$$b_i = f_i(x)/g_i(x), \ f_i(x), g_i(x) \text{ in } L[x].$$

Since k is infinite, by 2.2. there exists c in k such that under the $(x-c)$-adic valuation v of $K(x)/K$, f_i, g_i, and all the nonzero L-coefficients of $f_i(x), g_i(x)$ $(i = 1, \cdots, n)$ have finite nonzero v-residues. Thus, if $*$ denotes image under the v-residue map, then $b_i = b_i^* = f_i^*(c)/g_i^*(c)$ is in $L^*(c)$; and $L^*(c) = L^*$ since c is in $k \subset L^*$. But then $K \subset L^*$; and since K is the residue field of v, $L^* = K$. Then $dt(L^*/k) = dt(K/k) = dt(L/k)$, so by 2.1 the residue map $L \to L^* = K$ is a k-isomorphism.

PROOFof 1.1.-(i) for k infinite. We may assume $dt(K/k) \geq 1$, for otherwise $L \subset K$ = algebraic closure of k in $K(x)$. By adjoining to L some elements of a transcendence basis of K/k, we may further assume that $K(x)$ is algebraic over $L(x)$. Then, if $t_1, \cdots, t_n$ is a transcendence basis of K/k, we can write

$$\begin{aligned}(2.4)\quad & f_{i0}t_i^{m_i} + f_{i1}t_i^{m_i-1} + \cdots + f_{im_i} = 0 \ (i = 1, \cdots, n),\\ & f_{ij} \in L[x],\\ & f_{i0} \neq 0.\end{aligned}$$

As in the proof of 1.1-(iii), using the assumption that k is infinite, we see by 2.2 that there exists c in k such that the relations (2.4) are preserved under the residue map $*$ for the $(x-c)$-adic valuation of $K(x)/K$. It follows that K is algebraic over L^*, and hence as before that $L \to L^* \subset K$ is an isomorphism.

PROOF OF 1.1-(i) for arbitrary k. The preceding proof should be modified as follows. Instead of choosing the element c from k, one should use 2.2 to choose c from the infinite set $\{t_1^j, j \geq 1\} \subset K$. By applying the residue map $*$ to the relation 2.4, one sees that $t_1, \cdots, t_n$ are algebraic over $L^*[c]$. Moreover, if $c = t_1^j$ for j sufficiently large, then the first equality of 2.4 shows that t_1, and hence also c, is algebraic over L^*. Thus, $t_1, \cdots, t_n$ are algebraic over L^*, and therefore K is algebraic over L^*. By 2.1 this again implies $L \to L^*$ is an isomorphism.

$$(2.5)\qquad \begin{array}{ccccccccc} k & — & L & — & & — & & — & K(x)\\ \downarrow & & \downarrow & & & & & & \downarrow\\ k & — & L^* & — & L^*[c] & — & L^*[t_1] & — & K.\end{array}$$

3. Generalizations.

3.1. COROLLARY to 1.1.-(iii) ([Sa, 1953]). *Let L and K be subfields of a field Q and extensions of a field k, let $x_1, \cdots, x_n$ be elements of Q which are algebraically independent over K, and assume $dt(L/k) \leq dt(K/k)$. Then K/k finitely generated, k infinite, and $L(x_1, \cdots, \cdots, x_n) = K(x_1, \cdots, x_n)$ imply L is k-isomorphic to K.*

PROOF. Reduce to the 1-variable case of 1.1.-(iii) by adjoining $x_1, \cdots, x_{n-1}$ to k, L and K ; and then apply induction on n.

3.2. The following result includes 1.1. for the case that K/k is finitely generated and k is infinite (as in 1.1-(iii)).

THEOREM. *Let L and K be subfields of a field Q and extensions of a field k, let x be an element of Q, tr. over K, and assume $dt(L/k \leq dt(K/k)$. Then*

K/k finitely generated, k infinite, and $L \subset K(x)$ imply L is k-isomorphic to a subfield L^* of K such that $[K : L^*] \leq [K(x) : L(x)]$.

(The inequality is meant to be vacuous if $[K(x) : L(x)] = \infty$. Note that the inequality would be immediate if $L^\star$ were the residue field of $L(x)$, rather then that of L, under the $(x - c)$-adic valuation v that appears in the proof below.)

PROOF. As in 1.1.-(i), we may assume $[K(x) : L(x)] < \infty$. Then, if $b_1, \cdots, b_n$ is a set of generators for K/k, we can write out the irreducible polynomial relation for b_1 over $L(x)$, for b_2 over $L(x)[b_1]$, etc. By 2.2. we can choose c in k such that all the nonzero elements of $K(x)$ appearing in these expressions have value 0 under the $(x-c)$-adic valuation v of $K(x)/K$. By applying the v-residue map * to these relations, we conclude that K is algebraic over $L^\star$ and $[K : L^\star] \leq [K(x) : L(x)]$. Moreover, since K is algebraic over $L^\star$, by 2.1 $L \to L^\star$ is an isomorphism.

3.3. Roquette's original formulation of 1.1-(i) is

THEOREM. *(Generalization of 1.1.-(i)). Let L and K be subfields of a field Q and extensions of a field k, let X be a set of elements of Q which are algebraically independent over K, and assume $dt(L/k) \leq dt(K/k) < \infty$. Then $L \subset K(X)$ implies L is k-isomorphic to a subfield $L^\star$ of K.*

PROOF. (By reduction to 1.1): Since $dt(L/k) < \infty$, the elements of a tr. basis of L/k are in $K(x_1, \cdots, \cdots, x_n)$, for some $\{x_1, \cdots, x_n\} \subset X$. But $K(x_1, \cdots, x_n)$ is algebraically closed in $K(X)$, so then $L \subset K(x_1, \cdots, x_n)$. By induction, we are reduced to the case that X consists of a single element x, i.e. to 1.1-(i).

3.4. A different generalization of 1.1-(i) is the

THEOREM. *Let L and K be subfields of a field Q and extensions of a field k, let X be a transcendence basis of Q/K, and assume $dt(L/K) \leq dt(K/k) < \infty$. Then there exists an algebraic extension $M^\star$ of K such that L is k-isomorphic to a subfield of $M^\star$ and $[M^\star : K] \leq [Q : K(X)]$.*

(Note that the inequality is the significant part of the conclusion. If $[Q : K(x)] = \infty$, then the inequality is intended to be vacuous.)

PROOF. The theorem is trivial, with $M^\star = Q$, if Q is algebraic over K so we may assume X is nonempty. Since $dt(L/k)$ is finite, the elements of

L are algebraic over $K(x_1, \cdots, x_n)$ for some $\{x_1, \cdots, x_n\} \subset X$. Replace Q by M, the algebraic closure of $K(x_1, \cdots, x_n)$ in Q, and note that $[M : K(x_1, \cdots, x_n)] \leq [Q : K(X)]$ since M and $K(X)$ are linearly disjoint over $K(x_1, \cdots, x_n)$ (cf. [ZS 1, p. 111, Cor. 1]). Now reduce further to the case that X consists of a single element x by induction on n. As in the proof of 1.1-(i), we may also assume $dt(L/k) = dt(K/k)$ and x is tr. over L. Now choose an appropriate c in K (to be prescribed shortly) and extend the $(x - c)$-adic valuation of $K(x)/K$ arbitrarily to a valuation v of M. Since M is algebraic over $K(x)$, the residue field $M^\star$ of v is algebraic over the residue field K of $K(x)$, and $[M^\star : K] \leq [M : K(x)]$ (cf. [ZS 2, p. 26, Cor 2]). Moreover, the residue field $L^\star$ of $v|L$ is contained in $M^\star$; so if we can choose c in K such that $M^\star$ is algebraic over $L^\star$, then by 2.1 the residue map $L \to L^\star$ will be the required isomorphism.

The choice of c in K is made as in the proof of 1.1.-(i), except that now the following generalization of Lemma 2.2 is neededed.

LEMMA. *Let $K \subset K(x)$ be a field extension, with x tr. over K, let M be an algebraic extension of $K(x)$, and let $a_1, \cdots, a_n$ be nonzero elements of M. Then for all but finitely many c in K, any extension of the $(x - c)$-adic valuation of $K(x)/K$ to M will have value 0 at $a_1, \cdots, a_n$.*

PROOF. Choose c to avoid the zeros of the numerators and denominators of the nonzero coefficients appearing in the monic irreducible polynomial for a_i over $K(x)$ $(i = 1, \cdots, n)$. Then a_i and $1/a_i$ will be integral over the valuation ring for the $(x - c)$-adic valuation of $K(x)/K$, hence will both be in the valuation ring of any extension of this valuation to M.

3.5. The theorem of 3.4 yields a corresponding generalization of the corollary of 1.3. First we need to extend to arbitrary field extensions the notion of degree of an algebraic extension.

DEFINITION. Let $k \subset K$ be a field extension. As usual, if K/k is algebraic, we define the *deg of K/k*, denoted $[K : k]$, to be ∞ if K is an infinite dimensional k-vector space, and otherwise the vector space dimension of K/k. If K/k is not algebraic, we define

$$deg \text{ of } K/k = min\{[K : k(X)] \mid X \text{ is a transcendence basis of } K/k\}.$$

Moh-Heinzer [MH, 1982 ; p. 64] call $[K : k]$ the "deg of irrationality of K/k". Note that K/k is pure tr. iff $K \neq k$ and $[K : k] = 1$.

COROLLARY TO 3.4. (GENERALIZATION OF 1.3.). *Let $k \subset L \subset Q$ be field extensions such that $dt(L/k) < \infty$. Then there exists an algebraic extension M^* of L such that $[M^* : k] \leq [Q : k]$.*

PROOF. Let X be a transcendence basis of Q/k such that $[Q : k(X)] = [Q : k]$. If $dt(L/k) = n$, let $K = k(x_1, \cdots, x_n)$ for some $\{x_1, \cdots, x_n\} \subset X$; and apply 3.4. Q.E.D.

The statement of this corollary for the special case that $dt(L/k) = 1$ and k is infinite appears implicitly in the proof of [MH, 1982; Theorem 2]. (This part of their proof contains a slip, in line 17 of p. 64, which is now corrected by the above corollary.)

4. The Samuel problem and the Zariski problem.

We shall call the following the

4.1. n-dim Samuel problem. Let L and K be subfields of a field Q and extensions of a field k, let x be an element of Q tr. over L and K, and assume K/k is finitely generated of *dt n*. Does $L(x) = K(x)$ imply L is k-isomorphic to K ?

We have seen in 1.1. that the answer is "yes" if k is infinite, but I do not know if this remains true for k finite. This problem is related to the well-known (and difficult)

4.2. n-dim Zariski problem. Let L and K be subfields of a field Q, let x and y be elements of Q such that x is tr. over L and y is tr. over K, and assume K/k is finitely generated of *dt n*. Does $L(y) = K(x)$ imply L is k-isomorphic to K ?

These questions were first discussed in the paper [Se, 1949] of B. Segre. The Zariski problem is now known to be false in general [BCSS, 1985], and the counterexample is very complicated.

Note that the Zariski problem can be rephrased in terms of a single variable as follows : If x in Q is tr. over L and K, does $L(x) \simeq_k K(x)$ imply $L \simeq_k K$? The corresponding rephrasing of the Samuel problem reads : if x in Q is tr. over L and K, does $L(x) \simeq_k K(x)$ *under an isomorphism that takes x to x* imply $L \simeq_k K$? Thus the Zariski problem is suited to cancelling a sequence of variables, while the Samuel problem is not.

4.3. THEOREM. *For any $n \geq 0$, an affirmative answer to the n-dim Zariski problem implies an affirmative answer to the $(n+1)$-dim Samuel problem.*

PROOF. Assume the situation of 4.1., with $dt(K/k) = n+1$, and suppose $L \not\cong_k K$. By 1.1.-(ii), L/k and K/k are ruled ; thus, $L = L_0(y)$ and $K = K_0(z)$, where K_0 and L_0 are extensions of k, and y is tr. over L_0 and z tr. over K_0. Moreover, $dt(K_0(x)/k(x)) = dt(K_0/k) = n$.

$$\begin{array}{ccccc} L_0(x)(y) & & = & & K_0(x)(z) \\ | & & & & | \\ L_0(x) & & & & K_0(x) \\ & \diagdown & & \diagup & \\ & & k(x) & & \end{array}$$

Therefore by the $n-dim$ Zariski problem, $L = L_0(x) \cong_k K_0(x) = K$.

4.4. COROLLARY. *The n-dim Samuel problem has an affirmative answer for $n = 0, 1, 2$.*

Proof : $n = 0$: Then $K = L =$ algebraic closure of k in $K(x)$.

$n = 1$: Apply 1.1.-(ii) to conclude either $K \cong_k L$ or K and L are both simple transcendental over the algebraic closure of k in $K(x)$; in either case $K \cong_k L$.

$n = 2$: The 1-dim Zariski problem is known to have an affimative answer (cf. §7.6), so 4.3 applies.

REMARK. We have seen in 3.1 that the Samuel problem for m variables easily reduces to the Samuel problem for 1 variable. An analogous result is true for the 1-dim Zariski problem.

Proof. We may assume $L \not\subset K$. Then $K < KL \subset K(x_1, \cdots, x_m)$ and $dt(KL/K) = 1$ imply $KL = K(x)$ for some x not in K, by GLT. Similarly, $KL = L(y)$.

5. Subruled=uniruled.

This is the ruled analogue of 1.3. Roughly speaking, a field will be called subruled if it is non-trivially a subfield of a ruled extension and uniruled if it is a subfield of a ruled algebraic extension. To be precise,

5.1 DEFINITION A field extension $k \subset L$ will be called *subruled* if there exists an extension K of L and a subfield K_0 of K such that $k \subset K_0 \subset K = K_0(x)$, x tr. over K_0, and $L \not\subset K_0$; and an extension $k \subset L$ will be called *uniruled* if there exists an algebraic extension K of L which is ruled over k.

Note that if the extension L/k is finitely generated and subruled, then the extension K/k of 5.1 can also be chosen finitely generated. (Question : Does an analogous statement hold for finite transcendence deg?) A uniruled extension is clearly subruled, and for finitely generated extensions the converse is true:

5.2. THEOREM. *A finitely generated subruled extension is uniruled.*

Before proceeding to the proof, we need some preliminary remarks. Let v be a valuation of a field K, let V be the valuation ring of v, and let $a_1, \cdots, a_n$ be nonzero elements of K. We shall write $v(a_1) \gg v(a_2) \gg \cdots \gg v(a_n)$ if there exists a chain of prime ideals $\mathfrak{p}_n > \cdots > \mathfrak{p}_1 > \mathfrak{p}_0 = 0$ of v such that $a_i \in \mathfrak{p}_i \setminus \mathfrak{p}_{i-1} (i = 1, \cdots, n)$. Note that when this is the case, then $rk\ v \geq n$. It is easily seen that if $L \subset L(t_1, \cdots, t_n)$ is a field extension with $t_1, \cdots, t_n$ algebraically independent over L, then there exists a $rk\ n$ valuation v_0 of $L(t_1, \cdots, t_n)/L$ having residue field L and such that $0 < v_0(t_1) \ll \cdots \ll v_0(t_n)$.

PROOF OF 5.2. (Nagata [N,1967 ; p. 88]). Let $k \subset L$ be the given extension, and suppose we are in the subruled situation of 5.1. Since $L \not\subset K_0$, $K_0 < L(K_0) \subset K_0(x)$, and $K_0(x)$ is algebraic over $L(K_0)$. Therefore there exists a transcendence basis $t_1, \cdots, t_n$ of $K_0(x)/L$ consisting of t_i in K_0. By our preliminary remarks, there exists a $rk\ n$ valuation v_0 of $L(t_1, \cdots, t_n)/L$ such that $0 < v_0(t_1) \ll \cdots \ll v_0(t_n)$ and the residue field of v_0 is L ; extend v_0 to a valuation v of $K_0(x)$ and let * denote image under the v-residue map.

Since $K_0(x)$ is algebraic over $L(t_1, \cdots, t_n)$, the residue field $K_0(x)^\star$ of v is algebraic over the residue field L of $L(t_1, \cdots, t_n)$; so it remains to prove $K_0(x)^\star$ is ruled over k. Consider $w = v|K_0$. $Rk\ w = n$ since $t_1, \cdots, t_n$ are in K_0 and $0 < w(t_1) \ll \cdots \ll w(t_n)$; and therefore (by induction on Lemma 2.1) $dt(K_0^\star/k) \leq dt(K_0/k) - n$. But $dt(K_0/k) - n < dt(K_0(x)/k) - n = dt(L/k) = dt(K_0(x)^\star/k)$; so $K_0(x)^\star/K_0^\star$ is not algebraic, and hence by the RRT 2.3, $K_0(x)^\star/K_0^\star$ is ruled, and a fortiori $K_0(x)^\star/k$ is ruled. Q.E.D.

REMARK. The following is not completely obvious from the definitions, so let us make it explicit : unirational implies uniruled. Proof : Suppose $k < L \subset K$ with $K = k(X)$, X an algebraically independent set over k. Choose an element b in $L \setminus k$. Some x in X occurs in the rational expression for b in $k(X)$, so b is not in $k(X \setminus \{x\}) = K_0$. Thus, $L \not\subset K_0$ and $L \subset K_0(x)$, as required.

6. Separability considerations.

6.1. THEOREM. *(Separable version of 3.3 and 1.1.-(i)). Let L and K be subfields of a field Q and extensions of a field k, let X be a set of elements of Q which are algebraically independent over K, and assume $dt(L/k) \leq dt(K/k)$. If $L \subset K(X)$, and if K/k is finitely generated and K(X)/L is separable, then L is k-isomorphic to a subfield L^* of K such that K/L* is separable.*

PROOF. First, some simplifications :

i) We may assume $dt(L/k) > 0$, since otherwise $L \subset K$ =algebraic closure of k in $K(X)$; and we may assume char $= p > 0$ since otherwise 3.3 applies. Further, as in the proof of 3.3, we can reduce to the case that X consists of a single element x.

ii) We may assume $dt(L/k) = dt(K/k)$. Since $K(x)/L$ is finitely generated and separable, the extension has a separating transcendence basis. By adjoining to L some elements of this basis and replacing L by the resulting field, we achieve the reduction.

iii) We may assume $K(x)$ is separably algebraic over $L(x)$. Let $e_1, \cdots, e_n$ be a generating set for K/k. Then $x - e_1, \cdots, x - e_n, x$ is a generating set for $K(x)/k$ and a fortiori for $K(x)/L$. Any such generating set contains a separating transcendence basis for $K(x)/L$ (cf. [ZS 1, pp. 112-113] ; so by replacing x by some $x - e_i$ if necessary, we may assume x is a separating transcendence basis for $K(x)/L$.

Now we are ready to proceed with the central part of the proof.

Let $e_1, \cdots, e_n$ be a generating set for K/k ; by (i) we may assume e_1 is tr. over k. Let $f_i(Y)$ in $L[x][Y]$ be the (separable) irreducible polynomial for e_i over $L[x]$. Since $f_i(Y)$ is separable in $L[x][Y]$, there exist $g_i(Y)$ and $h_i(Y)$ in $L(x)[Y]$ such that

$$1 = g_i(Y)f_i(Y) + h_i(Y)f_i'(Y),$$

where $f_i'(Y)$ is the derivative of $f_i(Y)$. A typical coefficient b of $g_i(Y)$, $f_i(Y), h_i(Y), f_i'(Y)$ can be written $b = A(x)/B(x)$, with $A(x), B(x)$ in $L[x]$.

$$(6.1.1.) \qquad \begin{cases} f_i(e_i) = 0 \\ \quad 1 = g_i(Y)f_i(Y) + h_i(Y)f_i'(Y) \\ \quad b = A(x)/B(x) \end{cases}$$

By 2.2 there exists c in the infinite set $\{e_1^p, e_1^{p^2}, \cdots\}$ such that the $(x-c)$-adic valuation v of $K(x)/K$ has value 0 at every nonzero element of $K(x)$ in sight, i.e. at the nonzero coefficients b of the $g_i(Y), f_i(Y), h_i(Y)$, $f_i'(Y)$, at all the nonzero $A(x), B(x)$, and at all the nonzero L-coefficients of the $A(x), B(x)$.

By applying the v-residue map $*$ to the expressions (6.1.1), we conclude that $e_1, \cdots e_n$ are separably algebraic over $L^*[c]$; hence the residue field K of v is separately algebraic over $L^*[c]$. Moreover, by choosing $c = e_1^{p^i}$ with i sufficiently large, we can force the equality $f_1^*(e_1) = 0$ to be a non-trivial algebraic relation for e_1 over L^*. Therefore, c is algebraic over L^* too, and $L^*[c] = L^*(c)$.

Thus, e_1 is both separable and purely inseparable over $L^*(c)$; so e_1 is in $L^*(c)$. Write

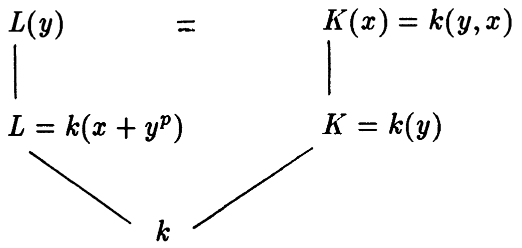
$$\begin{aligned} e_1 &= a_0{}^* + a_1{}^* c + \cdots + a_s{}^* c^s, \ a_i{}^* \text{ in } L^*, \\ &= a_0{}^* + a_1{}^* e_1^{p^i} + \cdots + a_s{}^*(e_1^{p^i})^s, \end{aligned}$$

and note that this is a *separable* algebraic relation for e_1 over L^*. Then $c = e_1^{p^i}$ is also separably algebraic over L^*. Thus, $K/L^*(c)$ is separably algebraic and $L^*(c)/L^*$ is separably algebraic ; so K/L^* is separably algebraic. Finally, by 2.1 this implies $L \to L^*$ is an isomorphism. Q.E.D.

Question. Can the hypothesis of 6.1 that K/k is finitely generated be replaced by the weaker hypothesis used in 3.3 that $dt(K/k)$ is finite ?

6.2. In the proof of 1.1.-(i) for k infinite we chose c from k in forming the $(x-c)$-adic valuation whose residue map gave the desired isomorphism. However, in the proof of 6.1, even for k infinite, one is forced to take c from the larger field K, as one sees from the

EXAMPLE. Let k be a field of char $p > 0$ and x, y be indeterminates, and consider the extensions



Since $K(x)/L$ is simple tr., it is separable. However, any specialization over K of x to an element c in k will give K/L^* purely inseparable ; for, since any nonzero element of $k[x+y^p]$ remains nonzero when x is replaced by c, we see that $L^* = k(c+y^p) = k(y^p)$. On the other hand, we can readily achieve K/L^* separable by choosing c in K, *e.g.* $c = y^{p+1}$. Then $L^* = k(y^{p+1}+y^p)$ and $[K : L^*] = p+1$, so K/L^* is separable.

6.3. Separably subrational = separably unirational.

We shall call an extension $k \subset K$ *separably subrational* (*resp. separably unirational)* if there exists a separable extension (*resp.* a separable algebraic extension) of K which is rational over k.

COROLLARY TO 6.1. *(Separable version of 1.4.). Suppose $k \subset L \subset K(x_1, \cdots, x_n)$ is an extension of fields, with $x_1, \cdots, x_n$ algebraically independent over k. If $dt(L/k) = m$ and $k(x_1, \cdots, x_n)/L$ is separable, then L is k-isomorphic to a subfield L^* of $k(x_1, \cdots, x_m)$ such that $k(x_1, \cdots, x_m)/L^*$ is separable.*

REMARK. Zariski [Z,1958] gives examples of extensions K/k of *dt* 2 which are unirational but not separably unirational, and he proves that if k is algebraically closed, then a separably unirational K/k of *dt* 2 is rational. One should also note that some authors (e.g. [MB], [Sh]) use the term "unirational" for our separably unirational.

6.4. Separably subruled = separably uniruled (for k infinite) . There is also a separable analogue of 5.2, at least for k infinite (I do not know if this assumption is essential). The definitions of separably subruled and separably uniruled are the same as those of 5.1, except that we additionally require that the extension K/L of 5.1 be separable.

THEOREM. *(Separable version of 5.2). A separably subruled extension $k \subset L$, with L/k finitely generated and k infinite, is separably uniruled.*

PROOF. Suppose we are in the separably subruled situation of 5.1, i.e.

there exists a separable extension K of L such that $K = K_0(x)$, with x tr. over K_0 and $L \not\subset K_0$. Since $L \not\subset K_0$, $K_0 < L(K_0) \subset K_0(x)$; so by Lüroth's theorem we may assume $L(K_0) = K_0(x)$. Moreover, since L/k is finitely generated, we may also assume K_0/k is finitely generated. Then $L(K_0)/L$ is finitely generated and separable, hence there exists a separating transcendence basis $t_1, \cdots, t_n$ of $K_0(x)/L$ consisting of t_i in K_0 (cf. [ZS 1, pp. 112-113]). The remainder of the proof parallels that of 5.2, except that now to get the valuation v_0 of 5.2 we need the following lemma, whose proof I owe to a conversation with H.W. Lenstra.

LEMMA. *Let L be a field and k be an infinite subset of L, let $t_1, \cdots, t_n$ be algebraically independent elements over L, and let K be a finite separable algebraic extension of $L(t_1, \cdots, t_n)$. Then there exists a rk n valuation v_0 of $L(t_1, \cdots, t_n)/L$ and elements $a_1, \cdots, a_n$ in k such that*

i) the residue field of v_0 is L,

ii) $0 < v_0(t_1 - a_1) \ll v_0(t_2 - a_2) \ll \cdots \ll v_0(t_n - a_n)$, and

iii) every extension of v_0 to a valuation of K has a residue field which is separably algebraic over L.

PROOF. Let $D = L[t_1, \cdots, t_n]$ and $L(t) = L(t_1, \cdots, t_n)$. Since $K/L(t)$ is finite separable, there exists a primitive element : $K = L(t)(e)$ for some e in K. By multiplying e by a suitable element of D, we may assume the monic irreducible polynomial $f(Y)$ for e over $L(t)$ has coefficients in D. Moreover, since e is separable over $L(t), f(Y)$ and its derivative $f'(Y)$ are relatively prime in $L(t)[Y]$. Therefore there exist $d(t) \neq 0$ in D and $g(Y), h(Y)$ in $D[Y]$ such that

$$d(t) = g(Y)f(Y) + h(Y)f'(Y).$$

Since k is infinite, we can choose $a_1, \cdots, a_n$ in k such that $d(a_1, \cdots, a_n) \neq 0$. It is easily seen that there exists a rk n valuation v_0 of $L(t_1 - a_1, \cdots, t_n - a_n)/L$ such that (i) and (ii) are satisfied (take the value group to be the lexicographic sum of n copies of the integers and define $v_0(t_i - a_i) = (0, \cdots, 1_i, \cdots, 0)$), so it remains to verify (iii).

Let v be any extension of v_0 to K, and let * denote image under the v-residue map. Since $f(e) = 0$ and $f(Y)$ is monic with coefficients in D, which is contained in the valuation ring of v_0, e is integral over the valuation ring of v_0, and therefore $v(e) \geq 0$. Also,

$$0 \neq d(a_1, \cdots, a_n) = g^*(Y)f^*(Y) + h^*(Y)f'^*(Y)$$

in $L[Y]$ implies $f^*(Y)$ is a separable polynomial in $L[Y]$. Since $f^*(e^*) = 0$, we shall be done if we prove the
Claim : $L(e^*)$ is the residue field of v.

Let (K^h, v^h) be a henselization of (K, v), let $K_0 = L(t)$, let (K_0^h, v_0^h) be the unique henselization of (K_0, v_0) in (K^h, v^h), and let V_0^h be the valuation ring of v_0^h. Then $K^h = K_0^h(e)$ (cf. [E, p. 131]), the residue field of $v_0 = L =$ residue field of v_0^h, and the residue field of $v =$ residue field of v^h. Thus, it suffices to prove that the residue field of $v^h = L(e^*)$, or equivalently, that $[K_0^h(e) : K_0^h] = [L(e^*) : L]$. Factor $f(Y)$ over K_0^h :

$$f(Y) = f_1(Y)q(Y)$$

where $f_1(Y)$ and $q(Y)$ are in $K_0^h[Y]$, $f_1(Y)$ is irreducible in $K_0^h[Y]$, and $f_1(e) = 0$. Moreover, we can choose $f_1(Y)$ to be primitive in $V_0^h[Y]$, and then, since $f(Y)$ is in $D[Y] \subset V_0^h[Y]$, it follows from Gauss's Lemma that $q(Y)$ is in $V_0^h[Y]$. Thus,

$$f^*(Y) = f_1^*(Y)q^*(Y)$$

(where * now denotes image under the v^h-residue map). We have already noted that $f^*(Y)$ is separable in $L[Y]$, so the same is true of $f_1^*(Y)$. Since $f_1(Y)$ is irreducible in $K_0^h[Y]$ and $f_1^*(Y)$ is separable in $L[Y]$, it then follows from Hensel's Lemma, (cf. [E, p. 118, Cor. 16.6]) that $f_1^*(Y)$ is irreducible in $L[Y]$. But $f(Y)$ is monic in $V_0^h[Y]$, so the leading coefficient of $f_1(Y)$ must be a unit of V_0^h, hence $deg\ f_1(Y) = deg\ f_1^*(Y)$. Thus, $[K_0^h(e) : K_0^h] = deg\ f_1(Y) = deg\ f_1^*(Y) = [L(e^*) : L]$. Q.E.D.

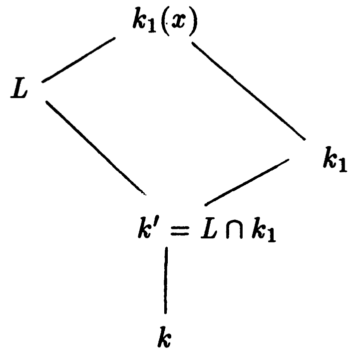
QUESTION. Does the above Theorem hold without the assumption that k is infinite ? Can the hypothesis that L/k is finitely generated be replaced by the weaker hypothesis that $dt(L/k)$ is finite ?

7. Separably uniruled extensions of transcendence degree 1.

7.1.THEOREM. *If $k \subset L$ is a separably uniruled extension of dt 1, then L/k' is finitely generated and separable, where k' is the algebraic closure of k in L.*

PROOF. Suppose L/k is separably uniruled via (k_1, x) :

(7.1.1.)



where k_1 is algebraic over k and x is tr. over k_1.

For any b in k_1, L and $k'(b)$ are linearly disjoint over k' (cf. [W, p. 6, Prop. 7]), so the irreducible polynomial for b over k' is also the irreducible polynomial for b over L. Since b is separable over L, it follows that b is separable over k' ; thus k_1/k' is separable. But then $k_1(x)/k'$ is separable, and hence also L/k'.

Since L/k' is regular, L and k_1 are linearly disjoint over k' (cf. [W, p. 18, Theorem 5]) ; hence if we choose t in L to be tr. over k', then L and $k_1(t)$ are linearly disjoint over $k'(t)$ (cf. [W, p. 5, Prop. 6]). But $k_1(x)/k_1(t)$, and therefore $k_1(L)/k_1(t)$, is finite algebraic. It follows that any vector space basis $b_1, \cdots, b_n$ of $k_1(L)/k_1(t)$, with b_i in L, is also a vector space basis of $L/K'(t)$. Q.E.D.

See Kang [K, 1987 ; p. 243, Cor.] for a "stable" version of 7.1.

7.2. There is a bit more that can be said about the diagram (7.1.1.), namely : $L(x) = k_2(x)$, where k_2 is the algebraic closure of k in $L(x)$. Since $k'(x) \subset L(x) \subset k_1(x)$, this follows from the

PROPOSITION. *Suppose $k(x) \subset K \subset k_1(x)$ are field extensions with $k \subset k_1$ separably algebraic and x tr. over k_1. Then $K = k_2(x)$, where k_2 is the algebraic closure of k in K.*

PROOF. We might as well assume $k = k_2$. Let b be an element of K. Then b is in $k(c_1, \cdots, c_n)(t)$, for some c_i in k_1 ; and since k_1/k is separable, $k(c_1, \cdots, c_n) = k(c)$ for some c in k_1. Thus, $k(x) \subset k(b, x) \subset k(c, x)$, and we want to prove b is in $k(x)$.

Since $k = k_2$ is algebraically closed in $k(b, x)$, $k(b, x)$ and $k(c)$ are linearly disjoint over k (cf. [W, p. 6, Prop. 7]). Therefore $[k(c) : k] = [k(b, x)(c) : k(b, x)]$. But $[k(x)(c) : k(x)] = [k(c) : k]$, so it follows that $k(x) = k(b, x)$.

7.3. Examples.

i) An example to show the Proposition of 7.2 is false without the separability assumption.

Let k_0 be a field of char $p > 0$, and let a, b, x be indeterminates. Let $k = k_0(a^p, b^p)$; $K = k(a + bx, x)$, and $k_1 = k_0(a, b)$:

$$k_1(x) = k_0(a,b)(x)$$

$$\Big|$$

$$K = k(a + bx, x)$$

$$\Big|$$

$$k(x) = k_0(a^p, b^p)(x).$$

Note that $K \neq k(x)$ because $a + bx \notin k(x)$, so it only remains to verify that k is algebraically closed in K. If we specialize x to 0 over k_1, the residue field K^* of K contains $k(a)$. On the other hand, $[K^* : k] \leq [K : k(x)] = p$, so we must have $K^* = k(a)$. Similarly, the residue field of K under the specialization of $1/x$ to 0 over k_1 is $k(b)$. Therefore the algebraic closure of k in K is contained in $k(a) \cap k(b) = k$. Q.E.D.

REMARK. Let $k \subset K$ denote a finitely generated extension of *dt* 1 with k algebraically closed in K. Lang-Tate [LT, 1952] have studied the case that K/k is inseparable of genus 0. They prove that K/k is inseparable of genus 0 iff char $k = 2$ and there exist x, y in K and a_0, b_0 in k such that $K = k(x, y)$, $y^2 - a_0x^2 = b_0$, and $[k(a_0^{1/2}, b_0^{1/2}) : k] = 4$. Moreover, they also prove that such extensions contain separable subfields of arbitrarily high genus, i.e. if K/k is inseparable of genus 0, then there exist fields L such that $k < L \subset K$ and L/k is separable of arbitrarily high genus.

Note that such an L/k of genus > 0 cannot be separably uniruled, for genus does not drop under separable base extension (cf. [C, p. 99, Theorem 5]). Thus, there exist finitely generated separable extension L/k of *dt* 1 such that L/k is uniruled (because K/k is) but not separably uniruled.

Finally, note that if one sets $y = a + bx$ in our Example (i), then $y^p = a^p + b^px^p$; so in char $p = 2$, we are exactly in the Lang-Tate situation.

ii) Example to show that the finitely generated assertion of 7.1 fails without the separability hypothesis.

The example is an elaboration of Example (i). Let k_0 be a field of char $p > 0$ and $a_1, b_1, a_2, b_2, \cdots ; x$ be a set of indeterminates. Let $k = k_0(a_1^p, b_1^p, a_2^p, b_2^p, \cdots)$, $K = k(a_1 + b_1 x, a_2 + b_2 x, \cdots ; x)$, and $k_1 = k_0(a_1, b_1, a_2, b_2, \cdots)$.

Then K/k is uniruled of *dt* 1, since $k_1(x)/K$ is algebraic. Moreover, we see as in (i) that k is algebraically closed in K. It remains to observe that K/k is not finitely generated. For otherwise we would have $K \subset k(a_1 + b_1 x, \cdots, a_n + b_n x, x)$ for some n, which is not the case since $a_{n+1} + b_{n+1} x$ is not in this field.

7.4. THEOREM. *Let $k \subset L$ be an extension of dt 1 with k algebraically closed in L. Then the following are equivalent :*

i) L/k is separably uniruled,

ii) there exist x, y in L and a, b in k such that $L = k(x, y)$ and

$$x^2 - ay^2 = b \text{ if char } k \neq 2$$
$$x^2 + xy - ay^2 = b \text{ if char } k = 2,$$

iii) there exists an element c which is separably algebraic of deg ≤ 2 over k and an element t tr. over $k(c)$ such that $L(c) = k(c, t)$.

Sketch of proof.

i) $\Rightarrow$ ii). By definition, there exist an algebraic extension k_1 of k and a tr. t over k_1 such that $k_1(t)$ contains L and is separably algebraic over L. Moreover, L/k is finitely generated and separable by 7.1, so k_1/k is also separable, and we may assume it is finitely generated. Therefore k_1/k has a primitive element : $k_1 = k(c)$. Then $k(c) < L(c) \subset k(c)(t)$; hence by Lüroth's theorem we may assume $L(c) = k(c, t)$.

Since k is algebraically closed in L, L and $k(c)$ are linearly disjoint over k. Therefore the passage from L to $L(c)$ is by base extension from k to $k(c)$. But genus is unchanged under separable base extension (cf. [C, p. 99, Theorem 5]), so genus of $L(c)/k(c) =$ genus of $k(c)(t)/k(c) = 0$ implies the genus of L/k is 0. It is well-known (cf. [A, p. 302]) that a genus 0 function field is the function field of a conic, and since L/k is separable, it is easily seen that the equation for this conic may be put in the form of (ii).

ii) $\Rightarrow$ iii). In the char $\neq 2$ case, let $c = \sqrt{a}$ and $t = b/(x - cy)$. In the char 2 case let c be a root of $z^2 + z - a$ and $t = b/(x - cy)$.

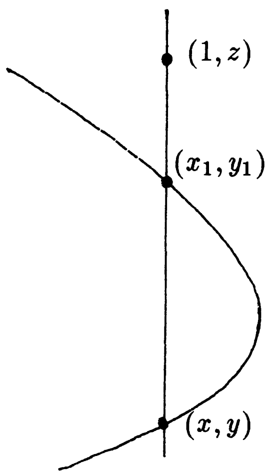
iii) $\Rightarrow$ i). Immediate. Q.E.D.

QUESTION. Is there a more elementary proof of (i) $\Rightarrow$ (ii), i.e. one that does not use the notion of genus ?

7.5. The next theorem asserts that any separably uniruled extension L/k of *dt* 1 can be non-trivially filled out to a Samuel problem diagram. (I thank A. Nobile for bringing the classical "method of sweeping lines" to my attention.)

THEOREM. *Let $k \subset K \subset K(z)$ be field extensions with $dt(K/k) = 1$, z tr. over K, and algebraically closed in K. Then K/k is separably uniruled implies there exists a field L such that $k < L$, $K \not\subseteq L$, z is tr. over L, and $L(z) = K(z)$.*

PROOF. By 7.4, $K = k(x,y)$, where $x^2 - ay^2 = b$ if char $k \neq 2$ and $x^2 + xy - ay^2 = b$ if char $k = 2$. Let us only consider the char $\neq 2$ case, since the char 2 case is similar. Choose a point in the plane having quite general coordinates from the field $k(x,y,z)$, say $(1,z)$, and find the point (x_1,y_1) of intersection of the conic $x^2 - ay^2 = b$ and the line joining $(1,z)$ to (x,y) :

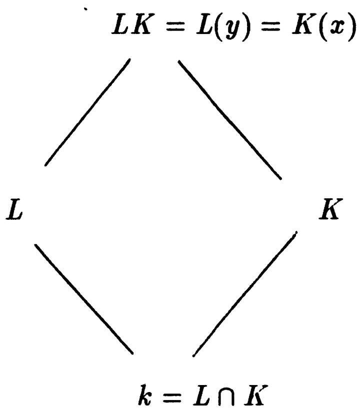


This involves solving $(x + t)^2 - a(y + zt)^2 = b$ for t, to find $t = -2(x - azy)/(1 - az^2)$. Then $x_1 = x + t$, $y_1 = y + zt$ is the sought-after point.

Now let $L = k(x_1,y_1)$. Then $x = x_1 - t$, $y = y_1 - zt$; and by symmetry, $-t = -2(x_1 - azy_1)/(1 - az^2)$. Thus, $L(z) = K(z)$. Moreover, $K \not\subset L$ since $x = x_1 - t \notin L$ because t is tr. over L.

7.6. THEOREM. *(1-dim Zariski problem ; cf. Deveney [D,1982]). Let L and K be finitely generated extensions of dt 1 of a field k, and let y be tr. over L and x be tr. over K. If $L(y) = K(x)$, then L is k-isomorphic to K.*

Sketch of proof : We may assume $k = L \cap K$ and, by Lüroth's theorem, that $LK = L(y) = K(x)$.



Since LK is separable over L and K, it is also separable over $L \cap K = k$ ([Wa, 1975 ; p. 39, Theorem 1.1.]). Then k is algebraically closed in K and K/k is separable, so L and K are linearly disjoint over k (cf. [W, p. 18, Theorem 5]). Thus, the extension from K to $LK = L(y)$ is a separable base change from k to L ; so $L(y)/L$ is of genus 0 implies K/k is of genus 0 (cf. [C, p. 99, Theorem 5]). Moreover, by 1.1.-(i) L is k- isomorphic to a subfield of K. Now the theorem follows from

THEOREM. *(Amitsur [A,1955 ; p. 42, Cor. 11.3.],* GENERALIZATION OF LÜROTH'S THEOREM). *Let $k \subset K$ be a finitely generated separable extension of dt 1 with k algebraically closed in K. If genus of K/k is 0 and L is a field such that $k < L \subset K$, then either L/k is simple tr. or L is k-isomorphic to K.*

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