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Solution of a Quartic Diophantine Equation

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Solution of a Quartic Diophantine Equation;

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Recently W. V. Parker and the author obtained solutions of the Diophantine equation

$$(ax+by)(mx^2+nxy+py^2)=(cu+dv)(mu^2+nuv+pv^2) \quad [1].$$

The question arose as to whether or not the same method used could be applied when each side of the equation is composed of two linear and one quadratic factors. The answer to this question is in the affirmative but certain restrictions on the coefficients are necessary [2].

Consider the equation

$$(1) \qquad (ax+by)(x+py)[x^2+nxy+p(n-p)y^2] = (cu+dv)(u+pv)[u^2+nuv+p(n-p)v^2],$$

where all the letters represent integers and x, y, u, v are unknow. We may write (1) in the form

$$(2) \qquad \frac{ax+by}{cu+dv}=\frac{(u+pv)[u^2+nuv+p(n-p)v^2]}{(x+py)[x^2+nxy+p(n-p)y^2]}.$$

If

$$(3) \qquad \gamma^2 u = \alpha x - p(n-p)\beta y, \qquad \gamma^2 v = \beta x + (\alpha+n\beta)y,$$

(1) « Solution of a Cubic Diophantine Equation ». This paper has been accepted for publication by the Tôhoku Mathematical Journal.

(2) These restrictions make the quadratic factorable.

then

$$(\gamma^2 u)^2 + n(\gamma^2 u)v + p(n-p)v^2$$
$$= [x^2 + nxy + p(n-p)y^2][\alpha^2 + n\alpha\beta + p(n-p)\beta^2]$$

and

$$\gamma^2 u + \gamma^2 pv = (x + py)(\alpha + p\beta).$$

Hence we have

$$(4) \qquad \frac{ax+by}{cu+dv} = \frac{(u+pv)[u^2 + nuv + p(n-p)v^2]}{(x+py)[x^2+nxy+p(n-p)y^2]}$$
$$= \frac{(\alpha+p\beta)[\alpha^2+n\alpha\beta+p(n-p)\beta^2]}{\gamma^6}.$$

If we put $cu + dv = k$ we hawe from (3) and (4)

$$(5) \qquad \begin{cases} a\gamma^6 x + b\gamma^6 y = k(\alpha+p\beta)[\alpha^2+n\alpha\beta+p(n-p)\beta^2], \\ (c\alpha+d\beta)x + [d\alpha + (dn - cpn + cp^2)\beta]y = \gamma^2 k \end{cases}$$

If we let

$$k = \gamma^2 \begin{vmatrix} a\gamma^6 & b\gamma^6 \\ c\alpha + d\beta & d\alpha + (dn - cpn + cp^2)\beta \end{vmatrix},$$

then from (5) and (3) we have as a solution of (1)

$$(6) \qquad \begin{cases} x = \gamma^2(\alpha+p\beta)[\alpha^2+n\alpha\beta+p(n-p)\beta^2][d\alpha+(dn-cpn+cp^2)\beta] - b\gamma^{10}, \\ y = a\gamma^{10} - \gamma^2(\alpha+p\beta)[\alpha^2+n\alpha\beta+p(n-p)\beta^2](c\alpha+d\beta), \\ u = d(\alpha+p\beta)[\alpha^2+n\alpha\beta+p(n-p)\beta^2]^2 - \gamma^8[b\alpha+ap(n-p)\beta], \\ v = -c(\alpha+p\beta)[a^2+n\alpha\beta+p(n-p)\beta^2]^2 + \gamma^8[a\alpha+(an-b)\beta], \end{cases}$$

where α, β, γ are arbitrary integers.

If x_1, y_1 are integers such that $ax_1 + by_1 = 0$, $x_1 + py_1 = 0$, or $x_1^2 + nx_1y_1 + p(n-p)y_1^2 = 0$ and u_1, v_1 are integers such that $cu_1 + dv_1 = 0$, $u_1 + pv_1 = 0$, or $u_1^2 + nu_1v_1 + p(n-p)v_1^2 = 0$, then evidently (x_1, y_1, u_1, v_1) is a solution of (1). Such solutions are trivial and will not be considered here. If (x_1, y_1, u_1, v_1) is a solution of (1) then (x_1t, y_1t, u_1t, v_1t), where $t \neq 0$, is also a solution and is proportional to the given solution. We wish to show that given any solution proportional to given solution proportional to the given solution.

Suppose

$$x = \lambda, \qquad y = \mu, \qquad u = \rho, \qquad v = \sigma$$

is any solution of (1). If we choose

$$\alpha = [\rho\lambda + n\mu\rho + p(n-p)\mu\sigma][\lambda^2 + n\lambda\mu + p(n-p)\lambda^2],$$
$$\beta = (\lambda\sigma - \mu\rho)[\lambda^2 + n\lambda\mu + p(n-p)\mu^2], \qquad \gamma = \lambda^2 + n\lambda\mu + p(n-p)\mu^2,$$

we have

$$\alpha^2 + n\alpha\beta + p(n-p)\beta^2 = [\lambda^2 + n\lambda\mu + p(n-p)\mu^2]^3[\rho^2 + n\rho\sigma + p(n-p)\sigma^2]$$
$$= \frac{[\lambda^2 + n\lambda\mu + p(n-p)\mu^2]^4(\alpha\lambda + b\mu)(\lambda + p\mu)}{(c\rho + d\sigma)(\rho + p\sigma)},$$

so that when we substitute these values in (6) we have

$$(7) \qquad x = K\lambda, \qquad y = K\mu, \qquad u = K\rho, \qquad v = K\sigma,$$

where

$$(8) \quad K = \frac{[\lambda^2 + n\lambda\mu + p(n-p)\mu^2]^9}{c\rho + d\sigma}$$
$$\Big[\{(ad - bc)\lambda + [b(d - cn) + acp(n-p)]\mu\}\rho + \{[d(an - b) + acp(p - n)]\lambda + p(n-p)(ad - bc)\mu\}\sigma \Big].$$

Hence the solution (7) is proportional to the given solution provided $K \neq 0$.

A solution of (1) such that $\lambda^2 + n\lambda\mu + p(n-p)\mu^2 = 0$ is one such that $K = 0$ but is a trivial solution and has been excluded. Hence if $K = 0$ we must have that the solution is also a solution of

$$\{(ad - bc)x + [b(d - cn) + acp(n-p)]y\}u + \{[d(an - b) + acp(p - n)]x + p(n-p)(ad - bc)y\}v = 0.$$

It follows then that

$$(9) \quad \begin{cases} u = S\{[d(b - an) + acp(n-p)]x + p(p-n)(ad - bc)y\} \\ v = S\{(ad - bc)x + [b(d - cn) + acp(n-p)]y\}. \end{cases}$$

We may show that

$$u + pv = SA(x + py),$$
$$cu + dv = SB(ax + by),$$
$$u^2 + nuv + p(n-p)v^2 = S^2C[x^2 + nxy + p(n-p)y^2],$$

where

$$A = (p-n)(ad-bc) + b(d-cn) + acp(p-n),$$
$$B = c^2p(n-p) + a(d-cn),$$
$$C = p(n-p)(ad-bc)^2 + [d(b-an) + acp(n-p)][b(d-cn) + acp(n-p)].$$

Since u, v, x, y, as given by (9), is a solution of (1), we have on substituting these values for u and v in (1)

$$(10) \qquad (ax+by)(x+py)[x^2+nxy+p(n-p)y^2] = MS^4(ax+by)(x+py)[x^2+nxy+p(n-p)y^2],$$

where $M = ABC$. Any value (x, y) will satisfy (10) is $MS^4 = 1$ and no value if $MS^4 \neq 1$. Hence in addition to the solutions given by (6), (1) is satisfied by all integral solution of

$$(11) \qquad \begin{cases} \sqrt[4]{M}\,u = [d(b-an) + acp(n-p)]x + p(p-d)(ad-bc)y, \\ \sqrt[4]{M}\,v = (ad-bc)x + [b(d-cn) + acp(n-p)]y, \end{cases}$$

if (11) has any integral solutions.

Consider the particular exemple

$$(12) \quad (x-2y)(x+2y)(x^2+5xy+6y^2) = (3u+v)(u+2v)(u^2+5uv+6v^2).$$

We note the trivial solutions

$$(13) \qquad \begin{cases} (x, y) = (2a, a); \quad (2a, -a); \ (3a, -a), \\ (u, v) = (b, -3b); \ (2b, -b); \ (3b, -b). \end{cases}$$

The solution of (12) is given by

$$(14) \qquad \begin{cases} x = \gamma^2(\alpha+2\beta)(\alpha^2+5\alpha\beta+6\beta^2)(\alpha-13\beta) + 2\gamma^{10}, \\ y = \gamma^{10} - \gamma^2(\alpha+2\beta)(\alpha^2+5\alpha\beta+6\beta^2)(3\alpha+\beta), \\ u = (\alpha+2\beta)(\alpha^2+5\alpha\beta+6\beta^2) + \gamma^8(2\alpha-6\beta), \\ v = \gamma^8(\alpha+7\beta) - 3(\alpha+2\beta)(\alpha^2+5\alpha\beta+6\beta^2). \end{cases}$$

Suppose $x=\lambda$, $y=\mu$, $u=\rho$, $v=\sigma$ is any non-trivial solution of (12). If in (13) we choose

$$\alpha = (\rho\lambda + 5\mu\rho + 6\mu\sigma)(\lambda^2+5\lambda\mu+6\mu^2), \qquad \beta = (\lambda\sigma - \mu\rho)(\lambda^2+5\lambda\mu+6\mu^2),$$
$$\gamma = \lambda^2 + 5\lambda\mu + 6\mu^2$$

we have $x = K\lambda$, $y = K\mu$, $u = K\rho$, $v = K\sigma$ where

$$K = \frac{(\lambda^2 + 5\lambda\mu + 6\mu^2)^q}{3\rho + \sigma}[1\lambda + 56\mu)\rho + (25\lambda + 42\mu)\sigma].$$

Hence (14) gives a solution of (12) proportional to any solution except those for which

$$(15) \qquad \begin{cases} \rho = S(25\lambda + 42\mu), \\ \sigma = -S(7\lambda + 56\mu). \end{cases}$$

Since

$$(\lambda - 2\mu)(\lambda + 2\mu)(\lambda^2 + 5\lambda\mu + 6\mu^2) = (3\rho + \sigma)(\rho + 2\sigma)(\rho^2 + 5\rho\sigma + 6\sigma^2)$$

we have from (15) that

$$(\lambda - 2\mu)(\lambda + 2\mu)(\lambda^2 + 5\lambda\mu + 6\mu^2) = S^4(\lambda - 2\mu)(\lambda + 2\mu)(\lambda^2 + 5\lambda\mu + 6\mu^2).$$

If $S = 1$, we have the non-trivial solution

$$(16) \qquad x = \lambda, \qquad y = \mu, \qquad u = 25\lambda + 42\mu, \qquad v = -7\lambda - 56\mu.$$

We may say then that given any solution of (12), a solution proportional to this solution is given by (13), (14), or (16).