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A theory of integral invariants

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A theory of integral invariants;

By A. E. TAYLOR.

Integral invariants were first studied by Poincaré, who used them in his development of celestial mechanics, and they have been studied further by a number of authors, notably Goursat, Cartan, and De Donder [1]. Cartan introduced « complete » integral invariants, which were subsequently shown to be a special kind of Poincaré invariant in a manifold of one more dimension.

The purpose of this memoir is twofold. In the first place we propose to develop the theory of an integral invariant which is of a type intermediate to those of Poincaré and Cartan; I have chosen to call these invariants *associated*, because of their connection with an associated differential system. Secondly, the associated integral invariants are used to extend certain known results in the application of the theory of integral invariants to the theory of differential equations.

I wish to make special mention of theorem 3.2 and 5.3, which were suggested to me by Prof. A. D. Michal. His criticism has aided greatly in the completion of this work.

(1) For works on integral invariants see bibliography at the end of this paper. More complete references may be found in the books of Cartan and De Donder [1], [2]. (Numbers in parenthesis refer to the bibliography.)

I.

1. Preliminary considerations. — We start with the fundamental system of first order differential equations

$$\frac{dx^i}{dt} = \xi^i(x^1, \ldots, x^n, t) \qquad (i = 1, \ldots, n), \tag{1.1}$$

where the ξ^i are continuous, with continuous partial derivatives of as many orders as we shall require (in general only the first two); in a certain region of $n+1$ dimensional space. We further require that they do not vanish simultaneously in this region. Then the system admits a unique solution

$$x^i = f^i(t; x_0^1, \ldots, x_0^n) \equiv f^i(t; x_0), \tag{1.2}$$

taking on the initial values $x_0^i = f^i(t_0; x_0)$ where the f^i are continuous, with continuous derivatives of (at least) the first two orders, in a suitable region V_{n+1}. In what follows we shall restrict ourselves to manifolds lying entirely within V_{n+1}.

Consider a sub-space (E_p) of p $(p \leqq n)$ dimensions, within V_{n+1}, but not generated by a p-parameter family of integral curves. Let it be given analytically by

$$\begin{cases} x^i = x^i(u^1, \ldots, u^p), \\ t = t(u^1, \ldots, u^p), \end{cases} \tag{1.3}$$

where the u's range over a domain (e_p) of u-space, and the matrix

$$\begin{vmatrix} \frac{\partial x^1}{\partial u^1} & \cdots & \frac{\partial x^n}{\partial u^1} & \frac{\partial t}{\partial u^1} \\ \cdots & \cdots & \cdots & \cdots \\ \frac{\partial x^1}{\partial u^p} & \cdots & \frac{\partial x^n}{\partial u^p} & \frac{\partial t}{\partial u^p} \end{vmatrix} \tag{1.4}$$

is of maximum rank.

Next we define a set of differentials

$$\begin{cases} \delta_\sigma x^i = \frac{\partial x^i}{\partial u^\sigma} du^\sigma & [i = 1, \ldots, n), \\ \delta_\sigma t = \frac{\partial t}{\partial u^\sigma} du^\sigma & (\sigma = 1, \ldots, p \quad (\text{a free index})] \end{cases} \tag{1.5}$$

the $du^1, \ldots, du^p$ being arbitrary, and construct the differential form

$$(1.6)\quad \begin{cases} A_{\alpha_1\ldots\alpha_p}\,\delta_1 x^{\alpha_1}\ldots\delta_p x^{\alpha_p} \\ \displaystyle +\sum_{\sigma=1}^{p} A_{\alpha_1\ldots\alpha_{\sigma-1}0\alpha_{\sigma+1}\ldots\alpha_p}\,\delta_1 x^{\alpha_1}\ldots\delta_{\sigma-1}x^{\alpha_{\sigma-1}}\,\delta_\sigma t\,\delta_{\sigma+1}x^{\alpha_{\sigma+1}}\ldots\delta_p x_{\alpha_p}. \end{cases}$$

We are using the summation convention of tensor analysis; the α's are summed from 1 to n. The functions $A_{\alpha_1\ldots\alpha_p}$, $A_{\alpha_1\ldots 0\ldots\alpha_p}$ are assumed to be continuous functions of $x^1, \ldots, x^n, t$, with continuous partial derivatives of (at least) the first two orders; we shall also suppose that they are skew-symmetric, that is, two functions with different indices will be distinct, but if the indices are the same except as to order, then the functions are equal, or the negatives of one another, according as the two permutations of the indices differ by an even or odd number of inversions. It follows from this that any function with two equal subscripts is identically zero.

It will be convenient to adopt a symbolism for the differential forms which we shall consider (¹). In doing so we follow Goursat where possible. We write

$$(1.7)\quad \begin{cases} \omega_p = A_{\alpha_1\ldots\alpha_p}\,\delta_1 x^{\alpha_1}\ldots\delta_p x^{\alpha_p}, \\ \omega_{p-1}^{(\sigma)} = A_{\alpha_1\ldots\alpha_{\sigma-1}0\alpha_{\sigma+1}\ldots\alpha_p}\,\delta_1 x^{\alpha_1}\ldots\delta_{\sigma-1}x^{\alpha_{\sigma-1}}\delta_{\sigma+1}x^{\alpha_{\sigma+1}}\ldots\delta_p x^{\alpha_p}, \\ \displaystyle \Omega_p = \omega_p + \sum_{\sigma=1}^{p}\omega_{p-1}^{(\sigma)}\,\delta_\sigma t, \end{cases}$$

thus giving a concise expression for (1.6). We see that Ω_p may be regarded as a differential form analogous to ω_p in the variables $x^1, \ldots, x^n, t$). We shall also have occasion to consider the « contracted » form

$$(1.8)\quad (\omega_p, \xi)_\sigma = A_{\alpha_1\ldots\alpha_{\sigma-1}k\alpha_{\sigma+1}\ldots\alpha_p}\xi^k\,\delta_1 x^{\alpha_1}\ldots\delta_{\sigma-1}x^{\alpha_{\sigma-1}}\,\delta_{\sigma+1}x^{\alpha_{\sigma+1}}\ldots\delta_p x^{\alpha_p}.$$

By the derived form of a form ω we mean

$$(1.9)\quad \begin{cases} \omega'_1 = a_{\alpha_1\ldots\alpha_{p+1}}\,\delta_1 x^{\alpha_1}\ldots\delta_{p+1}x^{\alpha_{p+1}}, \\ a_{\alpha_1\ldots\alpha_{p+1}} = \dfrac{\partial A_{\alpha_1\ldots\alpha_p}}{\partial x^{\alpha_{p+1}}} - \dfrac{\partial A_{\alpha_{p+1}\alpha_2\ldots\alpha_p}}{\partial x^{\alpha_1}} - \ldots - \dfrac{\partial A_{\alpha_1\ldots\alpha_{p-1}\alpha_{p+1}}}{\partial x^{\alpha_p}}, \end{cases}$$

(¹) Our notations for ω and ω' coincide essentially with those of Goursat, [3], Ch. III.

Ω'_p is defined as the derived form of Ω_p, regarded as a form in the variables $x^1, \ldots, x^n, t$. This involves the further expression

$$
(1.10)\quad \begin{cases} \overline{\omega}^{(\sigma)}_{p-1} = a_{\alpha_1 \ldots \alpha_{\sigma-1} 0 \alpha_{\sigma+1} \ldots \alpha_{p+1}} \delta_1 x^{\alpha_1} \ldots \delta_{\sigma-1} x^{\alpha_{\sigma-1}} \delta_{\sigma+1} x^{\alpha_{\sigma+1}} \ldots \delta_{p+1} x^{\alpha_{p+1}}, \\ a_{\alpha_1 \ldots 0 \ldots \alpha_{p+1}} = \dfrac{\partial A_{\alpha_1 \ldots 0 \ldots \alpha_p}}{\partial x^{\alpha_{p+1}}} - \ldots - \dfrac{\partial A_{\alpha_1 \ldots \alpha_{p+1} \ldots \alpha_p}}{\partial t} - \ldots - \dfrac{\partial A_{\alpha_1 \ldots 0 \ldots \alpha_{p-1} \alpha_{p+1}}}{\partial x^{\alpha_p}}. \end{cases}
$$

Then

$$
(1.11)\quad \Omega'_p = \omega'_p + \sum_{\sigma=1}^{p+1} \overline{\omega}^{(\sigma)}_{p-1} \delta_\sigma t.
$$

Evidently $\overline{\omega}^{(\sigma)}_{p-1}$ differs from $(\omega^{(\sigma)}_{p-1})'$ only by terms involving the partial derivatives of the $A's$ with respect to t. In fact

$$
(1.12)\quad (\omega^{(\sigma)}_{p-1})' = \overline{\omega}^{(\sigma)}_{p-1} + \frac{\partial A_{\alpha_1 \ldots \alpha_{\sigma-1} \alpha_{p+1} \alpha_{\sigma+1} \ldots \alpha_p}}{\partial t} \delta_1 x^{\alpha_1} \ldots \delta_{\sigma-1} x^{\alpha_{\sigma-1}} \ldots \delta_{p+1} x^{\alpha_{p+1}}.
$$

Corresponding to the differential forms ω and Ω we have the integral forms $\int \omega$, $\int \Omega$ extended over the manifold (E_p). For example, if $p = 1, 2$, we have, respectively

$$
\int A_\alpha \delta x^\alpha + A_0 \delta t,
$$

$$
\int\int A_{\alpha_1 \alpha_2} \delta_1 x^{\alpha_1} \delta_2 x^{\alpha_2} + A_{0 \alpha_2} \delta_1 t \delta_2 x^{\alpha_2} + A_{\alpha_1 0} \delta_1 x^{\alpha_1} \delta_2 t.
$$

2. THE DEFINITIONS. — The manifold (E_p) is cut but once by any given integral curve. Thus it determines a p-parameter family of integral curves of the set (**1**.2). The equations of this family may be obtained by substituting (**1**.3) in (**1**.2) and solving for the x^i_0 in terms of $u^1, \ldots, u^p : x^i_0 = x^i_0(u^1, \ldots, u^p) \equiv x^i_0(u)$. These are the equations of the manifold (E^0_p) which is the « projection » of (E_p) on the space $t = t_0$ by integral curves. If these values are then substituted back in (**1**.2) we have

$$
(2.1)\quad x^i = f^i[t; x_0(u)],
$$

as the equation of the family which we are seeking.

Let us now consider a p-dimensional manifold $(E_p)_1$ derived from (E_p) in the following manner : let each point ot (E_p) be carried over into the point on its integral curve for which $t = t(u) + T$ where T is a fixed number. The equations of this new manifold are seen to be

$$(2.2) \qquad \begin{cases} x^i = f^i[T + t(u), x_0(u)], \\ t = t(u) + T. \end{cases}$$

If $\int \Omega$ is extended over this manifold, its value will not in general be the same as when extended over (E_p). It may be, however, that these two integrals are equal, no matter what the value of T, for an arbitrary original (E_p). In this case we say that the integral is an associated invariant of p^{th} order of the system (1.1). We designate it by I_p^* :

$$(2.3) \qquad I_p^* = \int \Omega.$$

A particular case arises if we restrict (E_p) to be a manifold throughout which t is constant. Then our original (E_p) may without loss of generality be taken as (E_p^0) and $T = t - t_0$. Under these circumstances we call the integral a *Poincaré* integral invariant, and designate it by I_p,

$$(2.4) \qquad I_p = \int \omega.$$

Even more generally, however, we may consider manifolds defined by (2.2), where T is not a constant, but an arbitrary uniform function of $u^1, \ldots, u^p$. This amounts to making a deformation of (E_p) along the integral curves. If the integral $\int \Omega$, extended over an arbitrary manifold of this kind, has the same value as when extended over (E_p) [and also over (E_p^0)], then it is called a complete integral invariant of the system (1.1). For this invariant we use the notation I_p^c.

The above definitions have been laid down on the supposition that (E_p) need not be closed. If the conditions of the definition are fulfilled when and only when (E_p) is closed, the invariants are said to be *relative*, and we use the notations J_p, J_p^*, J_p^c to distinguish them from those defined above, which will be called *absolute*.

3. THE INVARIANCY CONDITIONS FOR I_p AND I_p^*. — Consider the diffe-

rential system

$$(3.1)\qquad \frac{dx^1}{\xi^1}=\ldots=\frac{dx^n}{\xi^n}=\frac{dt}{1}=d\tau.$$

This will be called the system *associated* with (1.1). The necessary and sufficient conditions for an associated integral invariant (absolute or relative) of (1.1) is that

$$\frac{d}{dT}\int\Omega=0$$

the integral being extended over the domain defined by (2.2).

Let us consider the geometrical significance of the associated system. Its solution may be written

$$(3.2)\qquad \begin{cases} x^i=f^i(\tau-\tau_0+t_0;\,x_0^1,\,\ldots,\,x_0^n), \\ t=\quad\tau-\tau_0+t_0, \end{cases}$$

where the f^i are the same functions that occur in (1.2). If $(\mathcal{E}_p^0)$ is a p-dimensional manifold imbedded in the $(n+1)$-dimensional continuum of the variables $x_0^1,\ldots,x_0^n$, t_0, it determines a p-parameter family of integral curves of the set (3.2), emanating from $(\mathcal{E}_p^0)$. If $(\mathcal{E}_p^0)$ is defined by

$$x_0^i=x_0^i(\alpha^1,\,\ldots,\,\alpha^p),$$
$$t_0=t_0\,(\alpha^1,\,\ldots,\,\alpha^p),$$

then the manifold $(\mathcal{E}_p^0)$ is carried over into a manifold $(\mathcal{E}_p)$ defined by

$$(3.3)\qquad \begin{cases} x^i=f^i[\tau-\tau_0+t_0(\alpha);\,x_0(\alpha)], \\ t=\quad\tau-\tau_0+t_0(\alpha). \end{cases}$$

If now we regard $\int\Omega$ as an integral form in the variables $x^1,\ldots,x^n,t$, the necessary and sufficient condition that this be a Poincaré integral invariant (absolute or relative) of the associated system (3.1) is evidently that

$$\frac{d}{d\tau}\int\Omega=0,$$

where the integration is over $(\mathcal{E}_p)$. The differentiation of the integral is carried out by the usual rules for differentiating under the sign of integration. However, from the foregoing considerations, and a comparison of (**2**.2) and (**3**.3) we conclude that the conditions

$$\frac{d}{dT}\int_{(E_p)_1}\Omega = 0, \qquad \frac{d}{d\tau}\int_{(\mathcal{E}_p)}\Omega = 0,$$

are equivalent. Hence we have proved the fundamental theorem :

THEOREM **3**.1. — *An associated integral invariant of the system* (**1**.1) *may be regarded as a Poincaré integral invariant of the associated system* (**3**.1) *Conversely, a Poincaré integral invariant of* (**3**.1) *in which the differential form does not depend explicitly on* τ *may be regarded as an associated invariant* (¹) of (**1**.1).

Suppose that I_p is an absolute Poincaré invariant of (**1**.1). The necessary and sufficient conditions that this be so are well known (²). They are

$$(\mathbf{3}.4)\quad \begin{cases} \left(\dfrac{\partial A_{\alpha_1\ldots\alpha_p}}{\partial x^k} - \dfrac{\partial A_{k\alpha_2\ldots\alpha_p}}{\partial x^{\alpha_1}} - \ldots - \dfrac{\partial A_{\alpha_1\ldots\alpha_{p-1}k}}{\partial x^{\alpha_p}}\right)\xi^k + \dfrac{\partial A_{\alpha_1\ldots\alpha_p}}{\partial t} \\ \qquad + \dfrac{\partial(A_{k\alpha_2\ldots\alpha_p}\xi^k)}{\partial x^{\alpha_1}} + \ldots + \dfrac{\partial(A_{\alpha_1\ldots\alpha_{p-1}k}\xi^k}{\partial x^{\alpha_p}} = 0 \\ \qquad (\alpha, k = 1, \ldots, n). \end{cases}$$

For some purposes it is more convenient to write them in the form

$$(\mathbf{3}.5)\qquad X(A_{\alpha_1\ldots\alpha_p}) + A_{k\alpha_2\ldots\alpha_p}\frac{\partial\xi^k}{\partial x^{\alpha_1}} + \ldots + A_{\alpha_1\ldots\alpha_{p-1}k}\frac{\partial\xi^k}{\partial x^{\alpha_p}} = 0,$$

where

$$(\mathbf{3}.6)\qquad X(f) = \xi^k\frac{\partial f}{dx^k} + \frac{\partial f}{\partial t}.$$

From equations (**3**.5) and Theorem **3**.1 we can deduce the conditions for an absolute associated invariant I_p^*. They are

$$(\mathbf{3}.7)\quad \begin{cases} X(A_{\alpha_1\ldots\alpha_{\sigma-1}0\alpha_{\sigma+1}\ldots\alpha_p}) + A_{k\alpha_2\ldots0\ldots\alpha_p}\dfrac{\partial\xi^k}{\partial\alpha_1} + \ldots \\ + A_{\alpha_1\ldots k\ldots\alpha_p}\dfrac{\partial\xi^k}{\partial t} + \ldots + A_{\alpha_1\ldots0\ldots\alpha_{p-1}k}\dfrac{\partial\xi^k}{\partial x^{\alpha_p}} = 0 \\ \qquad (\alpha, k = 1, \ldots, n; \sigma = 1, \ldots, p) \end{cases}$$

together with (**3**.5).

(¹) Michal has investigated invariants of the system (**3**.1) : see (9).

(²) *See* (3), p. 219.

The invariant I_p^* is characterized by the following theorem.

Theorem 3.2. — *If* $I_p^* = \int \Omega_p$ *is a known invariant, then* $I_p = \int \omega_p$ *and* $I_{p-1} = \int [\omega_{p-1}^{(\sigma)} + (\omega_p, \xi)_\sigma]$ (σ *arbitrary*) *are absolute Poincaré invariants of order p and p* — 1 *respectively. Conversely*, if $I_p = \int \omega_p$ *and* $I_{p-1} = \int \omega_{p-1}$ *are known invariants, we can construct an invariant* I_p^*.

Proof : It is sufficient to consider $\sigma = p$, since any other value gives coefficients which are the same except perhaps as to sign. Denote the coefficients of $[\omega_{p-1}^{(p)} + (\omega_p, \xi)_p]$ by $B_{\alpha_1 \dots \alpha_{p-1}}$. Then

$$B_{\alpha_1 \dots \alpha_{p-1}} = A_{\alpha_1 \dots \alpha_{p-1} 0} + A_{\alpha_1 \dots \alpha_{p-1} k} \xi^k.$$

We must apply conditions (3.5) to these coefficients. Now, by (3.5) and (3.7),

$$X(A_{\alpha_1 \dots \alpha_{p-1} k} \xi^k) + A_{j \alpha_2 \dots \alpha_{p-1} k} \xi^k \frac{\partial \xi^j}{\partial x^{\alpha_1}} + \dots + A_{\alpha_1 \dots \alpha_{p-2} j k} \xi^k \frac{\partial \xi^j}{\partial x^{\alpha_{p-1}}} = A_{\alpha_1 \dots \alpha_{p-1} k} \frac{\partial \xi^k}{\partial t},$$

$$X(A_{\alpha_1 \dots \alpha_{p-1} 0}) + A_{k \alpha_2 \dots \alpha_{p-1} 0} \frac{\partial \xi^k}{\partial x^{\alpha_1}} + \dots + A_{\alpha_1 \dots \alpha_{p-2} k 0} \frac{\partial \xi^k}{\partial x^{\alpha_{p-1}}} = - A_{\alpha_1 \dots \alpha_{p-1} k} \frac{\partial \xi^k}{\partial t};$$

so that the $B_{\alpha_1 \dots \alpha_{p-1}}$ are indeed coefficients of an absolute integral invariant of order $p - 1$. On the other hand, if $\int \omega_{p-1}$ is an invariant I_{p-1}, with coefficients $B_{\alpha_1 \dots \alpha_{p-1}}$, and I_p is known, then we may define

$$A_{\alpha_1 \dots \alpha_{\sigma-1} 0 \alpha_{\sigma+1} \dots \alpha_p} = B_{\alpha_1 \dots \alpha_{\sigma-1} \alpha_{\sigma+1} \dots \alpha_p} - A_{\alpha_1 \dots k \dots \alpha_p} \xi^k,$$

and these coefficients will satisfy equations (3.7), thus enabling us to construct an I_p^*.

From the equations written out in the foregoing proof it is easily seen that if

$$(3.8) \qquad A_{\alpha_1 \dots \alpha_{p-1} k} \frac{\partial \xi^k}{\partial t} = 0,$$

for arbitrary $\alpha_1, \dots, \alpha_{p-1}$ then the invariant I_{p-1} arising from I_p^* may be split into two separate invariants.

Theorem 3.3. — *If* $I_p^* = \int \Omega_p$ *is a known invariant, and if equations*

(3.8) *are satisfied, then* $\int \omega_{p-1}^{(p)}$ and $\int (\omega_p, \xi)_p$ *are both absolute Poincaré invariants.*

It is also worthy of note that from the invariant $I_p = \int \omega_p$ we obtain the invariant $I^*_{p+1} = \int \omega_p \delta_{p+1} t$ whose integrand is an Ω_{p+1}, most of the terms of which are zero. For the case $p = 1$ *see* Theorem **8**. 1.

Before proceeding further with our investigations it will be convenient to recall some points in the theory of integral invariants as developed by Goursat. In particular, we shall discuss invariants *attached* to the trajectories of (**1**. 1).

4. Attached integral invariants. — If an integral invariant of the system (**1**. 1) is also an integral invariant (of the same kind) of the system

$$\frac{dx^i}{dt} = \lambda(x^1, \ldots, x^n)\,\xi^i(x^1, \ldots, x^n, t) \tag{4.1}$$

where $\lambda(x^1, \ldots, x^n)$ is an arbitrary scalar function of such nature that $\lambda\xi^i$ fulfills the conditions originally imposed on ξ^i, then the integral invariant is said to be attached to the trajectories of (**1**. 1). Such invariants were considered by Goursat [1] only when the ξ^i do not depend on t. They may be interpreted geometrically as follows. For any fixed value of t equations (**1**. 1) and (**4**. 1) define the same direction field in the n-dimensional space of the x's. But (interpreting t as the time) the change undergone by a manifold during a given interval is not the same in the two cases. There is a re-allocation of points of simultaneity on the trajectories. Thus the integral curves in the space V_{n+1} are altered.

To obtain the conditions for attached absolute invariants we replace ξ^i by $\lambda\xi^i$ in (3.4). Because λ in arbitrary we easily infer that

$$A_{\alpha_1 \ldots \alpha_{p-1} k}\xi^k = 0, \qquad \frac{\partial A_{\alpha_1, \ldots, \alpha_p}}{\partial t} = 0. \tag{4.2}$$

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[1] *See* (3), p. 236.

Similarly, from (3.7) we obtain the conditions

$$(4.3) \qquad A_{\alpha_1\ldots 0\ldots\alpha_p,k}\xi^k = 0, \qquad \frac{\partial A_{\alpha_1\ldots 0\ldots\alpha_p}}{\partial t} = 0.$$

These are, of course, conditions which must hold in addition to (3.5) and (3.7). This enables us to state the theorem :

THEOREM 4.1. — *In order that absolute associated invariant* $I_p^* = \int \Omega_p$ *be attached to the trajectories of* (1.1), *it is both necessary and sufficient that the corresponding* $I_p = \int \omega_p$ *be attached, and that* $I_{p-1} = \int \omega_{p-1}^{(p)}$ *be an attached absolute invariant.*

From Theorem 3.3 and equations (4.2) we have the following theorem, due (in a slightly restricted form) to Poincaré and Goursat (1).

THEOREM 4.2. — *If* $I_p = \int \omega_p$ *is a known invariant, and if the equations*

$$(4.4) \qquad \xi^k \frac{\partial A_{\alpha_1\ldots\alpha_{p-1}k}}{\partial t} = 0,$$

are satisfied in addition to (3,8), *then* $I_{p-1} = \int (\omega_p, \xi)_p$ *is an attached absolute invariant.*

5. COMPLETE INTEGRAL INVARIANTS. — We might deduce the conditions for complete invariants by a direct analytical process (2).

But it is more elegant, as well as more instructive, to proceed in a different fashion. We shall give a new proof of the following theorem of Goursat (3).

(1) *See* (3), p. 242.

(2) *See* for instance the method of parametrization used in my paper on non-holonomic dynamical systems (12), p. 739-741. For this method of proof I am indebted to Prof. W. F. Osgood, to whose notes I had access.

(3) (4), p. 1089-1091. Goursat's proof, for the case of absolute invariants, may be summarized as follows. Given $I_p = \int \omega_p$, we deduce $I_{p+1}^* = \int \omega_p \, \delta_{p+1} t$.

THEOREM **5**.1. — *A complete integral invariant of* (**1**.1) *is a Poincaré invariant attached to the trajectories of the associated system* (**3**.1).

Proof : The complete invariant $\int \Omega$ is characterized by the fact that its value is unchanged when the domain of integration (E_p) is deformed in an arbitrary, continuous manner along the integral curves defined by (**1**.2). Now the trajectories of (**3**.1) are the integral curves (**1**.2) of the system (**1**.1). If we replace (**3**.1) by the system

$$(\mathbf{5}.1)\qquad \frac{dx^1}{\rho\xi^1} = \ldots = \frac{dx^n}{\rho\xi^n} = \frac{dt}{\rho} = d\tau \qquad [\rho = \rho(x^1, \ldots, x^n, t)],$$

where ρ is arbitrary, save as to restrictions of continuity, the trajectories are unchanged, but the variation of τ along them is altered. Along a given trajectory, determined by $x_0^1, \ldots, x_0^n$, τ and t are related by

$$(\mathbf{5}.2)\qquad \begin{cases} \tau - \tau_0 = \displaystyle\int_{t_0}^{t} \frac{dt}{\varphi(t;\, x_0^1, \ldots, x_0^n)} = \psi(t,\, t_0,\, x_0^1,\, x_0^n) \\ \varphi(t;\, x_0^1, \ldots, x_0^n) = \rho[f^1(t;\, x_0), \ldots, f^n(t;\, x_0),\, t]. \end{cases}$$

Singling out a p-parameter family of trajectories, we see that $\tau =$ const. determines an (E_p), and that as τ changes, (E_p) is deformed continously along the trajectories. However, since $\int \Omega$ is a complete integral invariant, its value is unchanged. Therefore it may be regarded as Poincaré invariant attached to the trajectories of (**3**.1). This proof of the theorem is valid for both relative and absolute invariants.

To find the conditions for absolute complete invariants we write Ω as a differential form in the variables $x^1, \ldots, x^n$, t, and utilize (**4**.2). The result may be stated as follows.

THEOREM **5**.2 — *In order that* $\int \Omega_p$ *be an absolute complete invariant*

(*cf.* § **3**). Then by Theorem **3**.1 and **4**.2 we deduce an I_p^* which is a Poincaré invariant attached to the trajectories of (**3**.1). This turns out to be the I_p^c corresponding to I_p (*see* Theorem **5**.2 below).

of (1.1) *it is both necessary and sufficient that it be an* I_p^* *for which*

$$A_{\alpha_1\ldots\alpha_{\sigma-1}0\ldots\alpha_p} + A_{\alpha_1\ldots\alpha_{\sigma-1}k\ldots\alpha_p}\xi^k = 0 \qquad (\sigma = 1, \ldots, p). \tag{5.3}$$

The invariant can be written

$$I_p^c = \int A_{\alpha_1\ldots\alpha_p}[\delta_1 x^{\alpha_1} - \xi^{\alpha_1}\delta_1 t]\ldots[\delta_1 x^{\alpha_p} - \xi^{\alpha_p}\delta^{\alpha_p} t].$$

From this theorem, Theorem 5.2, and the remark following Theorem 5.3 we are able to infer the following interesting theorem.

THEOREM 5.3. — *If* $I_p^* = \int \Omega_p$ *is a known invariant, then we may write*

$$I_p^* = I_p^c + \int \sum_{\sigma=1}^{p} [\omega_{p-1}^{(\sigma)} + (\omega_p, \xi)_\sigma]\delta_\sigma t,$$

where I_p^c *corresponds to* $\int \omega_p$ *and the last integral is an associated invariant, the « Poincaré » portion of which is absent.*

Since the ξ's and A's are independent of τ, we can theorem 4.2 to deduce an I_{p-1}^c from I_p^*.

THEOREM 5.4. — *To each* $I_p^* = \int \Omega_p$ *corresponds an absolute invariant of order one less :*

$$I_{p-1}^c = \int \omega_{p-1}^{(p)} + (\omega_p, \xi)_p + \sum_{\mu=1}^{p-1} (\omega_{p-1}^{(\mu)}, \xi)_p \delta_\mu t.$$

From (5.3) we notice that if I_p^c is attached to the trajectories of (1.1), the coefficients $A_{\alpha_1\ldots0\ldots\alpha_p}$ all vanish, so that $\Omega_p \equiv \omega_p$. In other words, there is no distinction between an attached I_p^c and the corresponding I_p. This could have been foreseen, from the nature of attached invariants.

6. RELATIVE INVARIANTS. — In this paragraph we shall indicate briefly the extension of the theory to relative associated integral invariants. We make use of the generalized theorem of Stokes [1] this theorem

[1] *See* (5), p. 334.

asserts that the integral $\int \omega_p$, extended over a *closed* p-dimensional manifold (E_p) in (x) space, is equal to the integral $\int \omega'_p$, extended over the $(p+1)$ dimensional manifold bounded by (E_p). Similarly, the integral $\int \Omega_p$ extended over a closed manifold in (x, t) space is equal to the integral $\int \Omega'_p$, taken over the manifold bounded by the first one. This gives us a means of passing from relative (or absolute) invariants to absolute invariants of order one higher. It is almost at once evident that attached relative invariants go over into attached absolute invariants. The basic theorems are as follows.

THEOREM **6**.1. — *Let* $J^*_p = \int \Omega_p$ *be a known relative invariant. Then* $J_p = \int \omega_p$ *is a relative invariant, and*

$$I_p = \int \varpi^{(p+1)}_{p-1} + (\omega'_p, \xi)_{p+1},$$

$$I^*_{p+2} = \int \omega'_p \, \delta_{p+2} t,$$

are absolute invariants. Furthermore, if

$$a_{\alpha_1 \ldots \alpha_p k} \frac{\partial \xi^k}{\partial t} = 0, \tag{6.1}$$

the above invariant I_p *breaks up into two distinct invariants.*

This theorem is a consequence of theorems **3**.2, **3**.3, and the appended remarks. Analogous to theorem **4**.1 we have the result:

THEOREM **6**.2. — *In order that* J_p *be a relative associated invariant attached to the trajectories of* (**1**.1), *it is both necessary and sufficient that the corresponding invariants* $J_p = \int \omega_p$ *and* $J_{p-1} = \int \omega^{(p)}_{p-1}$ *be attached.*

Theorem **4**.2 enables us, under certain circumstances, to proceed from a relative Poincaré invariant, to an attached absolute invariant of the same order.

THEOREM **6**.3. — *If* $J_p = \int \omega_p$ *is a known invariant, and if the* *équations*

$$a_{\alpha_1\ldots\alpha_p k}\frac{\partial\xi^k}{\partial t}=0,\qquad \xi^k\frac{\partial a_{\alpha_1\ldots\alpha_p k}}{\partial t}=0, \tag{6.2}$$

are satisfied, then

$$I_p=\int(\omega'_p,\xi)_{p+1}$$

is an attached absolute invariant.

Theorem 5.2 gives us a means of characterizing complete relative invariants, and this result, together with Theorem 5.4, leads to the following result.

Theorem 6.4. — *To each invariant* $J^*_p=\int\Omega_p$ *corresponds an absolute complete invariant of the same order :*

$$I^c_p=\int\varpi^{(p+1)}_{p-1}+(\omega'_p,\xi)_{p+1}+\sum_{\sigma=1}^{p}(\varpi^{(\sigma)}_{p-1},\xi)_{p+1}\delta_\sigma t.$$

Relative integral invariants of the first order are know to be of especial interest when the différential system (1.1) is of Hamiltonian form, for such systems are characterized by the complete relative invariant (¹) $\int p_i\delta q^i-H\delta t$. It is a fact of considerable interest that with *any* system (1.1) which admits a complete relative invariant of the first order there is associated a Lagrangian function L and a canonical system admitting the same invariant. We shall prove this result as a consequence of our treatment of the relative associated invariant *without* the use of Stokes' theorem.

Theorem 6.5. — *In order that*

$$\int\Omega_1\equiv\int A_i\delta x^i+A_0\delta t,$$

be a relative associated invariant J^*_1 *of* (1.1) *it is both necessary and*

(¹) *See* [1], p. 7 and [12], p. 740.

sufficient that the differential equations

$$(6.3)\qquad \begin{cases} \dfrac{\partial L}{\partial x^i} = X(A_i) + A_k \dfrac{\partial \xi^k}{\partial x_i}, \\[2ex] \dfrac{\partial L}{\partial t} = X(A_0) + A_k \dfrac{\partial \xi^k}{\partial t}, \end{cases}$$

form a completely integrable system.

Proof : In the equations (1.2) let the initial values be made to depend on a parameter in such manner that

$$(C_0)\qquad x_o^i = x_o^i(u)$$

defines a regular, closed curve C_0 in the $(x_0^1, \ldots, x_0^n)$ space.

There is thus defined a tube of integral curves of the set (1.2). Consider two simple, closed circuits (1) C, C′ on this tube, where C is defined by

$$(6.4)\qquad x^i = f^i(t(u),\ \ x_0(u)) \qquad t = t(u),$$

and C′ is obtained by replacing $t(u)$ by $t(u)+T$, where T is a constant. The necessary and sufficient condition that $\int \Omega_1$ be an invariant is that

$$\int_{C'} \frac{d}{dT}\left[A_i \frac{\partial x^i}{\partial u} + A_o \frac{\partial t}{\partial u}\right] du = o,$$

for arbitrary C_0 and C, the differentiation being along the integral curves. It is readily seen that the condition takes the form

$$(6.5)\qquad \int_{C'}\left[\left(X(A_i) + A_k \frac{\partial \xi^k}{\partial x_i}\right)\frac{\partial x^i}{\partial u} + \left(X(A_0) + A_k \frac{\partial \xi^k}{\partial t}\right)\frac{\partial t}{\partial u}\right] dt.$$

Now this is merely a line integral over a closed curve in the linearly simply connected region V_{n+1}. Its vanishing implies the existence of a single-valued function defined by

$$(6.6)\quad L(x, t) = \int_{(x_1, t_1)}^{(x,t)} \left(X(A_i) + A_k \frac{\partial \xi^k}{\partial x^i}\right) dx^i + \left(X(A_0) + A_k \frac{\partial \xi^k}{\partial t}\right) dt,$$

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(1) For this terminology see my paper (1.2), p. 737.

and possessing derivatives which satisfy (**6**.3), which is therefore a completely integrable system. Conversely, if the integrability conditions for (**6**.3) are satisfied, the integral (**6**.5) vanishes ([1]). This completes the proof.

From the integrability conditions for (**6**.3) we can obtain the conditions on the coefficients of Ω_1, and from these in turn, the conditions for an attached invariant J_1^*. These are found to be

$$(6.7)\qquad \begin{cases} \left(\dfrac{\partial A_i}{\partial x^k} - \dfrac{\partial A_k}{\partial x^i}\right)\xi^k = 0, & \dfrac{\partial}{\partial t}\left(\dfrac{\partial A_i}{\partial x^k} - \dfrac{\partial A_k}{\partial x^i}\right) = 0, \\ \left(\dfrac{\partial A_0}{\partial x^i} - \dfrac{\partial A_i}{\partial t}\right)\xi^i = 0, & \dfrac{\partial}{\partial t}\left(\dfrac{\partial A_0}{\partial x_i} - \dfrac{\partial A_i}{\partial t}\right) = 0. \end{cases}$$

From the first set of equations, and Theorem **5**.1 we conclude that the conditions for a relative complete invariant are:

$$(6.8)\qquad \left(\frac{\partial A_i}{\partial x^k} - \frac{\partial A_k}{\partial x^i}\right)\xi^k + \frac{\partial A_i}{\partial t} - \frac{\partial A_0}{\partial x^i} = 0.$$

THEOREM **6**.6. — *A necessary and sufficient condition that* $J_1^c = \int \Omega_1$ *be a complete invariant of* (**1**.1) *is that equations* (**6**.3) *admit the solution*

$$(6.9)\qquad L = A_k\xi^k + A_0.$$

Proof: That this function L satisfies (**6**.3) is an immediate consequence of (**6**.8). Conversely, if we suppose that (**6**.9) *is* a solution of (**6**,3), we obtain (**6**.8) at once.

This L is, up to an added constant, the function defined by the integral (**6**.6) when (**6**.8) hold.

Let us now suppose that we know an invariant J_1^c of (**1**.1), and let us regard the function L as a function of $x^1, \ldots, x^n, \xi^1, \ldots, \xi^n, t$, the x's and t entering merely through the coefficients $A_0, A_1, \ldots, A_n$. We observe that

$$\frac{\partial L}{\partial x^i} = \xi^k\frac{\partial A^k}{\partial x^i} + \frac{\partial A_0}{\partial x^i}, \quad \frac{\partial L}{\partial \xi^i} = A_i.$$

But

$$\frac{dA_i}{dt} = \xi^k\frac{\partial A_i}{\partial x^k} + \frac{\partial A_i}{\partial t} = \frac{\partial L}{\partial x^i},$$

[1] *See* [**10**], p. 142-150. This proof does not depend on Stokes' theorem.

because of (6.8). Thus, recalling that $\dot{x}^i = \xi^i$, we can write

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{x}^i} - \frac{\partial L}{\partial x^i} = 0. \tag{6.10}$$

These equations are equivalent to (6.8). If we regard J_1^c as given and seek to find the system (1.1) for which it is invariant, the equations take the Lagrangian form (6.10). As might be expected from the analogy with dynamics, there is associated with (1.1) a Hamiltonian system, with $2n$ independent variables, which admits the invariant J_1^c. To show this, we regard L as a function of $x^1, \ldots, x^n, t$, and A_0 as a function of $A_1, \ldots, A_n, x^1, \ldots, x^n, t$:

$$A_0 = L - A_k \xi^k.$$

From this standpoint the Hamiltonian system is

$$\frac{dx^i}{dt} = -\frac{\partial A_0}{\partial A_i} = \xi^i, \qquad \frac{dA_i}{dt} = \frac{\partial A_0}{\partial x^i} \tag{6.11}$$

and J_1^c is the usual complete relative invariant of Cartan.

II.

7. Determination of integral invariants. — There are many fundamental interrelations existing betwen the integral invariants and integrals of (1.1). These matters have been dealt with by Poincaré, Goursat, and Cartan, by a variety of methods. We propose to extend the known results to embrace the associated integral invariants. It will then be found that these latter constitute a tool of considerable value in adding to the body of existing theorems.

Our first concern is to show how the integral invariant may be found when (1.1) is regarded as a known system. The system (3.1) admits n distinct integrals

$$y^i(x^1, \ldots, x^n t) = C^i \qquad (i = 1, \ldots, n), \tag{7.1}$$

and a last integral of the form

$$\tau = t + C^{n+1}.$$

If we take $y^1, \ldots, y^n$, t as our new variables, the system (**3**.1) is reduced to the canonical form

$$(\mathbf{7}.2) \qquad \frac{dy^1}{0} = \ldots = \frac{dy^n}{0} = \frac{dt}{1} = d\tau.$$

If $I_p^* = \int \Omega_p$ is an invariant, it is a Poincaré invariant of (**3**.1) and if Ω_p is transformed into $\overline{\Omega}_p$ (*a* form in $y^1, \ldots, y^n$, t) by the change of variables, then $\int \overline{\Omega}_p$ is an invariant of (**7**.2). In order that this be so it is both necessary and sufficient (*cf.* theorem **3**.1 and equation **3**.5) that the coefficients in $\overline{\Omega}_p$ be independent of t. We can therefore assert that there will always exist invariants I_p^*, I_p, I_p^c of the system (**1**.1); indeed, if the solution of (**1**.1) is known, we can obtain *all* such invariants, for from (**1**.2) we can obtain (**7**.1) (1).

8. FIRST ORDER INTEGRAL INVARIANTS. — We have seen that when the the integrals of (**1**.1) are known, the integral invariants of the system can be constructed. The reverse problem, that of passing from

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(1) If the ξ'*s* contain t explicitly there may not be any attached invariants I_p^* or I_p. Thus, for instance, consider the system

$$\frac{dx^1}{t} = \frac{dx^2}{e^{x^1 x^2}} = dt.$$

It does not admit a non-zero attached invariant I_1, as may easily be verified.

There are no attached invariants I_n, but there may be attached invariants I_n^*, if the ξ'*s* are independent of t, for under these circumstances there always exist attached invariants I_p, if $p < n$ (*See* [3], p. 212-214, p. 236-237). To find all the attached invariants I_p^* of (**1**.1) when the ξ'*s* do not contain t, we reduce to canonical form

$$\frac{dz^1}{0} = \ldots = \frac{dz^{n-1}}{0} = \frac{dz^n}{1} = dt$$

where

$$z^i(x^1, \ldots, x^n) = C^i \qquad (i = 1, \ldots, n-1),$$
$$z^n(x^1, \ldots, x^n) = t + C^n$$

are integrals of (**1**.1). The attached invariants I_p of this system are characterized by the fact that $\overline{\omega}_p$ contains neither z^n nor $\delta_\sigma z^n$, and an attached invariant I_p^* is obtained from an attached I_p and an attached I_{p-1}.

integral invariants to integrals, is of considerable interest. In this paragraph we shall see what can be inferred from first order invariants.

THEOREM **8**.1. — *If* $I_1^* = \int \Omega_1$ *is an invariants of* (**1**.1),

$$A_k \xi^k + A_0 = \text{const.}$$

is an integral. If $A_k \frac{\partial \xi^k}{\partial t} = 0$, $A_k \xi^k$ *and* A_0 *yield two distinct integrals. If* I_1^* *is attached to the trajectories of* (**1**.1), $A_0 = $ *const. is an integral independent of* t.

Proof : $\Omega_1 = A_i \delta x^i + A_0 \delta t$. By (**3**.5) and (**3**.7) we have

$$X(A_k \xi^k + A_0) = 0$$

and if $A_k \frac{\partial \xi^k}{\partial t} = 0$ then

$$X(A_k \xi^k) = 0, \qquad X(A_0) = 0$$

separately. The last assertion of the theorem is a consequence of the fact that $A_k \xi^k = 0$ for attached invariants.

We may regard integrals of (**1**.1) as absolute Poincaré invariants of zero order; if the integrals are independent of t they may be regarded as *attached* invariants of zero order. With this convention the foregoing theorem is merely a combination of theorems **3**.2, **3**.3 and **4**.1 for the case $p = 1$.

THEOREM **8**.2. — *If* $I_1^c = \int \Omega_1$ *is an invariant such that*

$$\frac{\partial A_i}{\partial x^k} - \frac{\partial A_k}{\partial x^i} = 0$$

then $A_i dx^i + A_0 dt$ *is an exact differential form* du, *and it yields an integral* $u(x^1, \ldots, x^n, t) = $ *const.*

Proof : We know that $A_0 = - A_k \xi^k$; from this and the hypothesis it follows that

$$\frac{\partial A_i}{\partial t} - \frac{\partial A_0}{\partial x_i} = 0$$

whence we infer that the line integral

$$u(x, t) = \int A_i dx^i + A_0 dt$$

defines a single-valued function. It provides an integral, for

$$X(u) = A_i \xi^i + A_0 = 0$$

THEOREM **8**.3. — *If* $J_1^* = \int \Omega_1$ *is known, the line integral*

$$\int \left\{ \left(\frac{\partial A_i}{\partial x^k} - \frac{\partial A_k}{\partial x^i} \right) \xi^k + \frac{\partial A_i}{\partial t} - \frac{\partial A_0}{\partial x_i} \right\} dx^i + \left(\frac{\partial A_0}{\partial x^k} - \frac{\partial A_k}{\partial t} \right) \xi^k dt = \text{const.}$$

is an integral of (**1**.1). *This integral vanishes identically if the* J_1^* *is a* J_1^c, *and it reduces to the integral* $A_k \xi^k + A_0 = \text{const.}$ *of theorem* **8**.1 *if* $J_1^* = I_1^*$.

Proof : By theorem **6**.4 we pass from J_1^* to an invariant I_1^c. It may be verified that this latter invariant satisfies the conditions of theorem **8**.2, and yields the line integral as given. The last remarks of the theorem require no comment in view of (**3**.5), (**3**.7) and (**6**.8).

THEOREM **8**.4. — *If* $J_1^c = \int \Omega_1$ *is known, and if*

$$\xi^i \frac{\partial A_i}{\partial t} + \frac{\partial A_0}{\partial t} = 0$$

then $A_0 = \text{const.}$ *is an integral of* (**1**.1).

The proof of this follows at once from (**6**.8).

In concluding this paragraph we shall make a few remarks about a result due to Koenigs [1]. He showed that, given an invariant $I_1 = \int \omega_1$, the $A's$ and $\xi's$ being independent of t, the reduction of ω_1 to a canonical form results in the transformation of (**1**.1) to Hamiltonian form. The exact details of this process depend on the « class » of the form ω_1. The result may be used in an obvious manner to reduce (**3**.1) to Hamiltonian form when an I_1^* is known. When

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(1) *See* [6], p. 875-878; also [3], p. 226.

the I_1 is attached to the trajectories of (**1**.1) the resulting Hamiltonian system is one for which $H=0$, and several integrals of the system are determined. Regarding I_1^c as an invariant attached to the trajectories of (**3**.1) we obtain the following theorem :

THEOREM **8**.5. — *If* $I_1^c=\int \Omega_1$ *is an invariant of* (**1**.1) *the corresponding reduction to Hamiltonian form of* (**3**.1) *is one for which* $H=0$, *and it leads* $2p+1$ *or* $2p$ *integrals of* (**3**.1), *according as* Ω_1 *is of class* $2p+1$ *or* $2p$ *in the variables* $x^1, \ldots, x^n, t$. *These integrals, being independent of* τ, *are also integrals of* (**1**.1). *In general there will be* n *of them.*

9. MULTIPLIERS AND INTEGRALS. — In this paragraph we shall investigate certain general relations between integral invariants, integrals, and multipliers of (**1**.1). The results obtained are generalizations of the work of Koenigs and De Donder. We begin by discussing absolute Poincaré invariants of arbitrary order; the use of associated invariants then gives us immediate extensions.

In working with an invariant $I_{n-r}(0 \leq r < n)$ it is useful to adopt a different notation for the coefficients $A_{\alpha_1 \ldots \alpha_{n-r}}$. In any particular set of indices there are r numbers, say $\alpha_1, \ldots, \alpha_r$ of the set $1, 2, \ldots, n$ which are missing. Let us define

$$(\mathbf{9}.1) \qquad B_{\alpha_1 \ldots \alpha_r}=(-1)^{(\alpha_1, \ldots, \alpha_r)} A_{1 \ldots \alpha_1-1\, \alpha_1+1 \ldots \alpha_r-1\, \alpha_r+1 \ldots n}$$

where

$$(\mathbf{9}.2) \qquad (\alpha_1, \ldots, \alpha_r)=\sum_{i=1}^{r}(\alpha_i-i)=\sum_{i=1}^{r}\alpha_i-\frac{r(r+1)}{2}$$

and $\alpha_1<\alpha_2<\ldots<\alpha_r$. If the indice $\alpha_1, \ldots, \alpha_r$ are not in their natural order we define

$$B_{\alpha_1 \ldots \alpha_r}=\pm B_{\beta_1 \ldots \beta_r}$$

where $\beta_1 \ldots \beta_r$ is the set $\alpha_1 \ldots \alpha_r$ in its natural order, and the plus or minus sign is chosen according as an even or odd number of inversions is required to bring $\alpha_1, \ldots, \alpha_r$ to the form $\beta_1, \ldots, \beta_r$. Lastly, we define $B_{\alpha_1, \ldots, \alpha_r} \equiv 0$ if any two α's coincide.

We wish to re-write coditions (3.5) in terms of the $B's$. To do this we write

$$(9.3) \quad (-1)^{(\alpha_1 \dots \alpha_r)} \left\{ X(A_{1 \dots \alpha_1 - 1 \alpha_1 + 1 \dots n}) + A_{k2 \dots \alpha_1 - 1 \alpha_1 + 1 \dots n} \frac{\partial \xi^k}{\partial x^1} + \dots + A_{1 \dots \alpha_r - 1 \alpha_r + 1 \dots n - 1 k} \frac{\partial \xi^k}{\partial x^n} \right\} = 0.$$

In each summation on k, k takes on the value of the place it fills, and the values $\alpha_1, \dots, \alpha_r$. The coefficient of $\frac{\partial \xi^{\alpha_i}}{\partial x^k}$ is $A_{1 \dots k-1 \alpha_i k+1 \dots n}$, in which the indices $k, \alpha_1, \dots, \alpha_{i-1} \alpha_{i+1}, \dots, \alpha_r$ are missing. With a little calculation, recalling the skew-symmetry of the $A's$ and $B's$, we find that this coefficient, after taking account af the factor $(-1)^{(\alpha_1, \dots, \alpha_r)}$, is $-B_{\alpha_1 \dots \alpha_{i-1} k \alpha_{i+1} \dots \alpha_r}$. It is then easy to see that conditions (9.3) take the form

$$(9.4) \qquad X(B_{\alpha_1 \dots \alpha_r}) + B_{\alpha_1 \dots \alpha_r} \sigma - \sum_{i=1}^{r} B_{\alpha_1 \dots \alpha_{i-1} k \alpha_{i+1} \dots \alpha_r} \frac{\partial \xi^{\alpha_i}}{\partial x^k} = 0$$

where

$$(9.5) \qquad \sigma = \sum_{k=1}^{n} \frac{\partial \xi^k}{\partial x^k}.$$

Let us now introduce a multiple differential operator

$$(9.6) \qquad D(\varphi_1, \dots, \varphi_r) = B_{\alpha_1 \dots \alpha_r} \frac{\partial(\varphi_1, \dots, \varphi_r)}{\partial(x^{\alpha_1} \dots x^{\alpha_r})},$$

$\varphi_1, \dots, \varphi_r$ being arbitrary functions of $x^1, \dots, x^n, t$. We shall demonstrate the following theorem :

Theorem 9.1. — *In order that I_{n-r} be an invariant of* (1.1) *it is both necessary and sufficient that*

$$(9.7) \quad XD(\varphi_1 \dots \varphi_r) - \sum_{i=1}^{r} D(\varphi_1 \dots \varphi_{i-1} X(\varphi_i) \varphi_{i+1} \dots \varphi_r) = -\sigma D(\varphi_1 \dots \varphi_r)$$

be an identity in $\varphi_1 \dots \varphi_r$.

Proof : Let

$$\Delta_{\alpha_1\ldots\alpha_r} = \frac{\partial(\varphi_1\ldots\varphi_r)}{\partial(x^{\alpha_1}\ldots x^{\alpha_r})}$$

be the Jacobian determinant in (9.6), and denote its elements by

$$a_{pq}^{\alpha_1\ldots\alpha_r} = \frac{\partial\varphi_p}{\partial x^{\alpha_q}}$$

while the corresponding minor is written

$$A_{pq}^{\alpha_1\ldots\alpha_r} = \frac{\partial(\varphi_1\ldots\varphi_{p-1}\varphi_{p+1}\ldots\varphi_r)}{\partial(x^{\alpha_1}\ldots x^{\alpha_{q-1}}x^{\alpha_{q+1}}\ldots x^{\alpha_r}}.$$

Then

$$\Delta_{\alpha_1\ldots\alpha_r} = \sum_{p=1}^{r}(-)^{p+q}\, a_{pq}^{\alpha_1\ldots\alpha_r} A_{pq}^{\alpha_1\ldots\alpha_r}$$

for any q, with a similar relation for summation on p, and

$$\frac{\partial\Delta_{\alpha_1\ldots\alpha_r}}{\partial a_{dq}^{\alpha_1\ldots\alpha_r}} = (-1)^{p+q} A_{pq}^{\alpha_1\ldots\alpha_r}.$$

Now

$$X(\Delta_{\alpha_1\ldots\alpha_r}) = (-1)^{p+q} A_{pq}^{\alpha_1\ldots\alpha_r} X(a_{pq}^{\alpha_1\ldots\alpha_r}$$

and

$$X(a_{pq}^{\alpha_1\ldots\alpha_r}) = \xi^k \frac{\partial}{\partial x^k}\frac{\partial\varphi_p}{\partial x^{\alpha_q}} + \frac{\partial}{\partial t}\frac{\partial\varphi_p}{\partial x^{\alpha_q}} = \frac{\partial}{\partial x^{\alpha_q}} X(\varphi_p) - \frac{\partial\varphi_p}{\partial x^k}\frac{\partial\xi^k}{\partial x^{\alpha_q}}$$

But

$$\sum_{p=1}^{r}(-1)^{p+q} A_{pq}^{\alpha_1\ldots\alpha_r}\frac{\partial\varphi_p}{\partial X^k} = \Delta_{\alpha_1\ldots\alpha_{q-1}\ldots k\alpha_{q+1}\alpha_r} \qquad (\text{any } q)$$

and

$$\sum_{q=1}^{r}(-1)^{p+q} A_{pq}^{\alpha_1\ldots\alpha_r}\frac{\partial}{\partial x^{\alpha_q}} X(\varphi_p) = \frac{\partial(\varphi_1\ldots\varphi_{p-1}X(\varphi_p)\varphi_{p+1}\ldots\varphi_r)}{\partial(x^{\alpha_1}\ldots x^{\alpha_r}} \qquad (\text{any } p).$$

Therefore, finally

$$(9.8)\quad X(\Delta_{\alpha_1\ldots\alpha_r}) = \sum_{p=1}^{r}\left[\frac{\partial(\varphi_1\ldots\varphi_{p-1}X(\varphi_p)\varphi_{p+1}\ldots\varphi_r)}{\partial(x^{\alpha_1}\ldots x^{\alpha_r})} - \Delta_{\alpha_1\ldots\alpha_{p-1}k\alpha_{p+1}\ldots\alpha_r}\frac{\partial\xi^k}{\partial x^{\alpha_p}}\right]$$

Let us now compute $XD(\varphi_1 \ldots \varphi_r)$, using (**9**.4).

$$\begin{aligned}
XD(\varphi_1\ldots\varphi_r) = {} & \Delta_{\alpha_1\ldots\alpha_r} X(B_{\alpha_1\ldots\alpha_r}) + B_{\alpha_1\ldots\alpha_r} X(\Delta_{\alpha_1\ldots\alpha_r}) \\
= {} & \Delta_{\alpha_1\ldots\alpha_r}\left[-\sigma B_{\alpha_1\ldots\alpha_r} + \sum_{i=1}^{r} B_{\alpha_1\ldots\alpha_{i-1}k\alpha_{i+1}\ldots\alpha_r}\frac{\partial\xi^{\alpha_i}}{\partial x^k}\right] \\
& + B_{\alpha_1\ldots\alpha_r}\left[\sum_{p=1}^{r}\left\{\frac{\partial(\varphi_1\ldots\varphi_{p-1}X(\varphi_p)\varphi_{p+1}\ldots\varphi_r)}{\partial(x^{\alpha_1}\ldots x^{\alpha_r})}\right.\right. \\
& \left.\left. - \Delta_{\alpha_1\ldots\alpha_{p-1}k\alpha_{p+1}\ldots\alpha_r}\frac{\partial\xi^k}{\partial x^{\alpha_p}}\right]\right\},
\end{aligned}$$

$$XD(\varphi_1\ldots\varphi_r) = -\sigma D(\varphi_1\ldots\varphi_r) + \sum_{p=1}^{r} D[\varphi_1\ldots\varphi_{p-1}X(\varphi_p)\varphi_{p+1}\ldots\varphi_r]$$

as was to be shown. Conversely, if (**9**.7) is an identity, (**9**.4) follows at once, with the aid of (**9**.8) This proves the theorem.

Corollary [1]. — If $\varphi_1, \ldots, \varphi_n$ are integrals (distinct) of the system (**1**.1), and I_{n-n} is an invariant, then $D(\varphi_1, \ldots, \varphi_n)$ is a multiplier of the system.

This is clear as soon as we observe that the condition for a multiplier M is

$$X(M) + \sigma M = 0.$$

The knowledge of p distinct integrals of (**1**.1) always enables us to construct an invariant I_p. For suppose that $\varphi_1, \ldots, \varphi_p$ are distinct integrals of (**1**.1); we then define the coefficients of I_p

$$A_{\alpha_1\ldots\alpha_p} = \frac{\partial(\varphi_1, \ldots, \varphi_p)}{\partial(x^{\alpha_1}, \ldots, x^{\alpha_p})}.$$

Then, since $X(\varphi_i) = 0$, $i = 1, 2, \ldots, p$, equations (**9**.8) reduce to precisely (**3**.5), thus proving the assertion.

Theorem **9**.**1** and its corollary have their analogues when we consider an invariant I^*_{n-r}. The coefficients of the form Ω_{n-r} may be

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[1] When $r = 0$ the single coefficient M in I_n is a multiplier, as is well known. For $r = 1$ this theorem was demonstrated by Koenigs (8), p. 25-27. For further remarks on this case see [3], p. 223-224.

described by $n+1$ indices, among which appear the index zero. More precisely, we define

$$(\mathbf{9}.9)\qquad \begin{cases} C_{\alpha_1\ldots\alpha_r 0} = (-1)^{(\alpha_1\ldots\alpha_r)+n-r} A_{1\ldots\alpha_1-1\,\alpha_1+1\ldots\alpha_r-1\,\alpha_r+1\ldots n}, \\ C_{\alpha_1\ldots\alpha_{r+1}} = (-1)^{(\alpha_1\ldots\alpha_{r+1})} A_{1\ldots\alpha_1-1\,\alpha_1+1\ldots\alpha_{r+1}-1\,\alpha_{r+1}+1\ldots n0} \end{cases}$$

when $\alpha_1, \ldots, \alpha_{r+1}$, are integers from 1 to n in natural order; for other than natural order we use skew-symmetry to define the coefficients. Then, if $\varphi_1, \ldots, \varphi_{r+1}$ are arbitrary functions, we define the operator

$$(\mathbf{9}.10)\quad D^*(\varphi_1\ldots\varphi_{r+1}) = C_{\alpha_1\ldots\alpha_{r+1}}\frac{\partial(\varphi_1\ldots\varphi_{r+1})}{\partial(x^{\alpha_1}\ldots x^{\alpha_{r+1}})} + \sum_{\sigma=1}^{r+1} C_{\alpha_1\ldots\alpha_{\sigma-1}0\alpha_{\sigma+1}\ldots\alpha_{r-1}}\frac{\partial(\varphi_1\ldots\varphi_{r+1})}{\partial(x^{\alpha_1}\ldots t\ldots x^{\alpha_{r+1}})}.$$

As a consequence of theorems **3**.1 and **9**.1 we infer that this operator satisfies an identity of the form **9**.7, and that if $\varphi_1, \ldots, \varphi_{r+1}$ are distinct integrals of (**1**.1), $D^*(\varphi_1, \ldots, \varphi_{r+1})$ is a multiplier of the system. This is not all, however. For, knowing I^*_{n-r} we know the following invariants

$$I_{n-r} = \int \omega_{n-r}, \qquad I_{n-(r+1)} = \int \omega^{(n-r)}_{n-r-1} + (\omega_{n-r}, \xi)_{n-r},$$
$$I_{n-r+1} = \int \omega'_{n-r}, \qquad I^*_{n-r+1} = \int \Omega'_{n-r}.$$

Corresponding to the $r+1$ distinct combinations of $\varphi_1, \ldots, \varphi_{r+1}$, r at a time, the operator D gives us $r+1$ multipliers. If ω'_{n-r} and Ω'_{n-r} are not identically zero, their coefficients may be used to define new operators D' and $D^{*\prime}$, operating on $r-1$ and r functions, respectively. It may be verified that these operators have the form

$$D'(\varphi_1\ldots\varphi_{r-1}) = (-1)^{n+r}\sum_{k=1}^{n}\frac{\partial B_{\alpha_1\ldots\alpha_{r-1}k}}{\partial x^k}\,\frac{\partial(\varphi_1\ldots\varphi_{r-1})}{\partial(x^{\alpha_1}\ldots x^{\alpha_{r-1}})},$$

$$D^{*\prime}(\varphi_1\ldots\varphi_r) = (-1)^{n+r}\left(\sum_{k=1}^{n}\frac{\partial C_{\alpha_1\ldots\alpha_r k}}{\partial x^k} + \frac{\partial C_{\alpha_1\ldots\alpha_r 0}}{\partial t}\right)\frac{\partial(\varphi_1\ldots\varphi_r)}{\partial(x^{\alpha^1}\ldots x^{\alpha_r})} + \sum_{\sigma=1}^{r}\sum_{k=1}^{n}\frac{\partial B_{\alpha_1\ldots\alpha_{\sigma-1}k\,\alpha_{\sigma+1}\ldots\alpha_r}}{\partial x^k}\,\frac{\partial(\varphi_1\ldots\varphi_r)}{\partial(x^{\alpha_1}\ldots x^{\alpha_{\sigma-1}}\,t\,x^{\alpha_{\sigma+1}}\ldots x^{\alpha_r})}.$$

In the most general case D' and $D^{*\prime}$ will lead to $\frac{r(r+1)}{2}$ and $r+1$ new multipliers, respectively. The operator based on $I_{n-(r+1)}$ yields no new multipliers, it gives the same one that we obtained from D^*. The quotients of these multipliers, taken by pairs, yield integrals of(1.1), wich need not, however, all be functionally independent. The situation may be summarized in a theorem :

Theorem 9.2. — *Let an invariant I^*_{n-r} and $r+1$ distinct integrals of(1.1) be given. Then* :

1° If ω'_{n-r} and Ω'_{n-r} do not vanish, $\frac{(r+2)(r+3)}{2}$ multipliers of (1.1) are determined. In the most favorable case this determines $\frac{(r+1)(r+4)}{2}$ new integrals of the system, provided that the total number of integrals thus obtained does not exceed n.

2° If $\Omega'_{n-r} \not\equiv 0$, and $\omega'_{n-r} \equiv 0$, then we get $2r+3$ multipliers and $2r+2$ integrals.

3° If $\Omega'_{n-r} \equiv \omega'_{n-r} \equiv 0$, we get $2r+2$ multipliers and $r+1$ integrals.

The case where $r=0$ is of particular interest. The single coefficient $A_{12\ldots n} = (-1)^n C_0$ of ω_n is a multiplier, and so also is the single coefficient

$$\sum_{i=1}^{n} \frac{\partial C_i}{\partial x_i} + \frac{\partial C_0}{\partial t},$$

of Ω'_n. The operator $D^*(\varphi)$ is defined

$$D^*(\varphi) = C_i \frac{\partial \varphi}{\partial x_i} + C_0 \frac{\partial \varphi}{\partial t},$$

and $D^*(\varphi)$ is a multiplier whenever φ is an integral. The two multipliers first given determine an integral, however :

$$\mu = \frac{\sum_i \frac{\partial C_i}{\partial x_i} + \frac{\partial C_0}{\partial t}}{C_0},$$

and in general $D^*(\mu)$ will be a multiplier wich may be combined with one of the others to yield a second integral. Thus the knowledge of I_n^* is equivalent to knowing two distinct integrals, if $n \geqq 2$.

A great many theorems may be proved, dealing with integrals and integral invariants of associated type. Those stated above will at least serve to indicate the possibilities.

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