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On the fundamental formulæ of the geometry of the tensor submanifolds

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On the fundamental formulæ of the geometry of tensor submanifolds;

By P. DIENES.

General conventions.

I. The summation symbol Σ is suppressed if it applies to terms with two identical suffixes.

II. The first letters of Latin and Greek alphabets — as suffixes — vary from 1 to N, the midletters, $i, j, k, \ldots, o$; $\iota, \varkappa, \lambda, \ldots, o$, from 1 to m, the endletters $p, r, s, \ldots$; $\pi, \rho, \sigma, \ldots$ from $m+1$ to N.

III. Greek suffixes are used in initial mathematical axes and Latin suffixes in general geometrical axes.

IV. Structure tensors are denoted by Roman capitals.

1. *The idea of tensor geometry.* — All possible groups $(x^1, x^2, \ldots, x^N) \equiv (x^\alpha)$ of N real numbers, also called points, make up the N dimensional mathematical point manifold X_N. When x^α only varies, the other $N-1$ variables $x^1, \ldots, x^{\alpha-1}, x^{\alpha+1}, \ldots, x^N$ being kept constant, the points so obtained formes the α th parametric line of X_N. N different parametric lines pass through every point.

In the construction of every geometry, the directed interval $(x_0^\alpha, x_0^\alpha + \Delta x_0^\alpha)$, *i.e.* all the real numbers between x_0^α and $x_0^\alpha + \Delta x_0^\alpha$ in their natural order of magnitude, is made to correspond to our visual notion of the direction at (x_0^α) of the line we called the α th parametric line. These N directions at P are symbolised by the N contravariant base vectors $\underset{\alpha}{e}(P)$, attached to P. This (mathematical) system of local coordinate axes is completed by the N covariant (mathematical) base vectors $\overset{\alpha}{e}(P)$ representing the coordinate hyperplanes.

A tensor at P is, by definition, a polynomial of any finite degree homogeneous both in $\underset{\alpha}{e}(P)$ and in $\overset{\alpha}{e}(P)$. Linear forms in $\underset{\alpha}{e}$ or in $\overset{\alpha}{e}$ are also called contravariant and covariant *vectors* respectively.

In a change of variables $x^\alpha = x^\alpha(y^1, \ldots, y^N)$, the increments dx^α are transformed by $dx^\alpha = A^\alpha_{\alpha'}\, dy^{\alpha'}$ where $A^\alpha_{\alpha'} = \frac{\partial x^\alpha}{\partial y^{\alpha'}}$. Hence $\underset{\alpha}{e}\, dx^\alpha = \underset{\alpha}{e}\, A^\alpha_{\alpha'} dy^{\alpha'}$ and thus

$$(1) \qquad \underset{\alpha'}{e} = A^\alpha_{\alpha'} \underset{\alpha}{e}.$$

Putting $A^{\alpha'}_\alpha = \frac{\partial y^{\alpha'}}{\partial x^\alpha}$ so that

$$A^{\alpha'}_\alpha A^\alpha_{\beta'} = \delta^{\alpha'}_{\beta'}, \qquad A^{\alpha'}_\alpha A^\beta_{\alpha'} = \delta^\beta_\alpha,$$

we have for the inverse transformation

$$(2) \qquad \underset{\alpha}{e} = A^{\alpha'}_\alpha \underset{\alpha'}{e}.$$

We notice that $A^\alpha_{\alpha'}$ and $A^{\alpha'}_\alpha$ denote *different* sets of functions. For instance, $A^{\alpha=1}_{\alpha'=2}$ is, in general, different from $A^{\alpha'=1}_{\alpha=2}$ or from $A^{\alpha'=2}_{\alpha=1}$.

Covariant vectors and tensors might be transformed independently of contravariant tensors. For geometrical purposes, however (See Schouten, 1929, p. 415) it is convenient to put

$$(3) \qquad \underset{\alpha'}{e} = A^{\alpha'}_\alpha \underset{\alpha}{e}.$$

from which it follows that

$$(4) \qquad \overset{\alpha}{e} = A^\alpha_{\alpha'} \overset{\alpha'}{e}.$$

If we drop the condition that $dx^\alpha = A^\alpha_{\alpha'} dy^{\alpha'}$ is an exact differential, but still suppose that the rank of the determinant of $A^{\alpha'}_\alpha$ is N, formulae (1) — (4) determine a general *change of base* replacing the original mathematical axes by an arbitrary system of independent axes (base vectors) referred to as (general) *geometrical axes* (Schouten's nonholonomic parameters).

A manifold of tensors becomes a geometrical manifold or space if (a) we define local metric, i. e. length and angle, by means of a quadratic form $g_{\alpha\beta} dx^{\alpha} dx^{\beta}$ with arbitrary coefficients $g_{\alpha\beta}(x^{\alpha}) = g_{\beta\alpha}(x^{\alpha})$, ($b$) we dovetail the local geometrical tensor manifolds so constructed by a definition of equivalence between tensors at neighbouring points. Such a connexion replaces the congruence axioms. For further detail see Schouten 1924 and Dienes 1924, 1926.

2. *Tensor submanifolds.* — Tensor submanifolds are usually defined in three steps. (I) A point submanifold X_m of X_N is selected, (II) a pencil of tensors is assigned to the points of X_m, (III) metric and connexion are assigned to the selected pencil. In this Art. we shall discuss (I).

The points of X_m may be selected by $N - m$ equations

$$\Phi^{\sigma}(x^1, \ldots, x^N) = 0, \tag{1}$$

the Jacobian of Φ^{σ} having the rank m. To obtain a parametric representation of X_m we complete (1) by m arbitrary functions Φ^{λ} such that the Jacobian of the complete system Φ^{α} be different from zero and we put

$$u^{\alpha} = \Phi^{\alpha}(x^1, \ldots, x^N), \tag{2}$$

Solving for x^{α} we have

$$x^{\alpha} = \Psi^{\alpha}(u^1, \ldots, u^N), \tag{3}$$

and the points of X_m an characterized by the $N - m$ conditions $u^{\sigma} = 0$, so that

$$x^{\alpha} = \Psi^{\alpha}(u^1, \ldots, u^m, 0, \ldots, 0) \equiv \psi^{\alpha}(u^1, \ldots, u^m), \tag{4}$$

form a system of parametric equations for X_m.

We put

$$B^{\lambda}_{\alpha} = \frac{\partial \Phi^{\lambda}}{\partial x^{\alpha}}, \qquad C^{\sigma}_{\alpha} = \frac{\partial \Phi^{\sigma}}{\partial x^{\alpha}}, \qquad B^{\alpha}_{\lambda} = \frac{\partial \Psi^{\alpha}}{\partial u^{\lambda}}, \qquad C^{\alpha}_{\sigma} = \frac{\partial \Psi^{\alpha}}{\partial u^{\sigma}}, \tag{5}$$

and notice that

$$\begin{cases} B^{\lambda}_{\alpha} B^{\alpha}_{\mu} = \delta^{\lambda}_{\mu}, \quad B^{\lambda}_{\alpha} C^{\alpha}_{\sigma} = 0, \quad B^{\alpha}_{\lambda} C^{\sigma}_{\alpha} = 0, \\ C^{\sigma}_{\alpha} C^{\alpha}_{\tau} = \delta^{\sigma}_{\tau}, \quad B^{\lambda}_{\alpha} B^{\beta}_{\lambda} + C^{\sigma}_{\alpha} C^{\beta}_{\sigma} = \delta^{\beta}_{\alpha}. \end{cases} \tag{6}$$

If (x^α) is in X_m, the necessary and sufficient conditions that also $(x^\alpha + dx^\alpha)$ be in X_m are

$$(7) \qquad C^\sigma_\alpha dx^\alpha = du^\sigma = 0.$$

Since dx^α is the prototype of contravariant vectors, v^α is considered as a vector of X_m if and only if the N functions v^α satisfy the $N - m$ conditions

$$(8) \qquad C^\sigma_\alpha v^\alpha = 0.$$

The rank of the matrix (C^σ_α) being $N - m$, if $\underset{1}{v^\alpha}, \ldots, \underset{m}{v^\alpha}$ are m independant sets of solutions to (8), every other set of solutions is of the form $c_1 \underset{1}{v^\alpha} + \ldots + c_m \underset{m}{v^\alpha}$. It follows from $(6)_3$ that B^α_λ form m independent sets of solutions to (8), and thus every contravariant vector of X_m is a form in the m independent contravariant vectors

$$(9) \qquad \underset{\lambda}{e} = B^\alpha_\lambda \underset{\alpha}{e}.$$

The corresponding set of covariant vectors $\overset{\lambda}{e} = x^\lambda_\alpha \overset{\alpha}{e}$ is determined by the equations

$$(10) \qquad x^\lambda_\alpha B^\alpha_\mu = \delta^\lambda_\mu.$$

and, from $(6)_1$, B^λ_α are solutions of (10) so that every covariant vector of X_m is a form in the m independent covariant vectors

$$(11) \qquad \overset{\lambda}{e} = B^\lambda_\alpha \overset{\alpha}{e}.$$

Thus, *the tensors of X_m are the forms homogeneous in both systems of base vectors $\underset{\lambda}{e}$ and $\overset{\lambda}{e}$.* We add that the vectors

$$(12) \qquad \underset{\sigma}{e} = C^\alpha_\sigma \underset{\alpha}{e}, \qquad \overset{\sigma}{e} = C^\sigma_\alpha \overset{\alpha}{e},$$

form a complementary base called the *span* (Spannung) of X_m. *The tensors of the span are the forms in $\underset{\sigma}{e}$, $\overset{\sigma}{e}$.* The set of vectors $(\underset{\lambda}{e}, \underset{\sigma}{e}; \overset{\lambda}{e}, \overset{\sigma}{e})$ is a new mathematical base for X_N, called the *split base.*

The definition of the tensors of X_m hinges on (1). Now, there are dynamical systems called nonholonomic in which the points i. e. the independent variables, generalised coordinates, are subjected to no restriction, but the changes dx^α of the independent variables are restricted by linear relations, not necessarily exact. This means that in such a case step (I) in the construction of the submanifold is cancelled, and the gradients $C^\sigma_\alpha = \frac{\partial \Phi^\alpha}{\partial x^\sigma}$ in (7) are replaced by arbitrarily assigned functions denoted again by C^σ_α. This idea of a nonholonomic manifold has been introduced by Vranceanu (1928), and further developed by Schouten (1929*a* cf « Papers quoted » in fine), who denotes it by X^m_ν.

The admitted contravariant vectors (directions) of X^m_ν will again be determined by (8), i. e. any m independent sets of solution to (8), denoted again by B^α_λ, lead to a set of m contravariant base vectors (9). For C^α_φ we take a fixed set of independent solutions of

$$C^\sigma_\alpha x^\alpha_\varphi = \delta^\sigma_\varphi. \tag{13}$$

and putting for a moment $a^\alpha_{\gamma=\lambda} = B^\alpha_\lambda$, $a^\alpha_{\gamma=\sigma} = C^\alpha_\sigma$, the unique set of solutions to the equations

$$a^\alpha_\gamma x^\beta_\alpha = \delta^\beta_\gamma. \tag{14}$$

will be denoted by

$$x^{\beta=\lambda}_\alpha = B^\lambda_\alpha, \qquad x^{\beta=\sigma}_\alpha = C^\sigma_\alpha, \tag{15}$$

so that all the five equations (6) will be satisfied. In this case, (11) and (12) determine the remaining base vectors, and together with (9), lead to the initial mathematical axes in X^m_ν. These mathematical axes or base vectors, whose determination hinges on the originally given C^σ_α, may be replaced by arbitrary « geometrical » base vectors or axes $(\underset{i}{e}, \underset{p}{e}; \overset{i}{e}, \overset{p}{e})$ for which Latin suffixes will be used. By definition *forms in $\overset{\lambda}{e}, \underset{\lambda}{e}$, or in $\underset{i}{e}, \overset{i}{e}$ are tensors in the (admitted) facet; forms in $\underset{\sigma}{e}, \overset{\sigma}{e}$ or in $\underset{p}{e}, \overset{p}{e}$ those in its span.*

The difference between holonomic and non holonomic submanifolds, as manifested in the determination of the tensors of the submanifold, is that, in the nonholonomic case, B^α_λ, B^λ_α, C^α_σ, C^σ_α are not necessarily gradients. The determination of the tensors of the submanifold amounts to splitting the base of X_N into two complementary sets, of dimensions m and $N - m$ respectively. In the holonomic case this process is applied at the points of X_m only, whereas for X_N^m it applies at *every* point of X_N.

The construction of a geometry for such tensor submanifold still requires a definition of connexion and metric in X_N^m (see Arts. 4 and 11). So far [see in particular Levi-Civitá (1917) and Schouten (1918, 1924, 1929)] the connexion in X_m or in X_N^m has been defined as the projection of the connexion in X_N. In this paper we shall consider the general case of an arbitrarily assigned connexion in X_N^m. Moreover, we shall establish all the fundamental formulae in general geometrical axes, and shall show that the fundamental formulae of the geometry of tensor submanifolds can all be obtained in a systematic way by the method of resolution of tensors. In this way, we shall obtain, in particular, the extension to X_N^m of the equation of Gauss and Codazzi, and that of Kühne, established for the holonomic case by Schouten (1924, p. 140) and Hlavaty (1926) respectively. Some other equations obtained by the same method seem to be new.

3. *Change of base. Resolution of tensors.* — In this paper we shall consider the mathematical axes of X_N and X_N^m i. e. the initial mathematical base vectors $\left(\underset{\alpha}{e}, \overset{\alpha}{e}\right)$ and $\left(\underset{\lambda}{e}, \underset{\sigma}{e}; \overset{\lambda}{e}, \overset{\sigma}{e}\right)$ as fixed once for all. The passage from these mathematical axes to general geometrical axes means to replace $\underset{\alpha}{e}, \underset{\lambda}{e}$ and $\underset{\sigma}{e}$ by N, m, $N - m$ arbitrary, but independent, linear combinations of $\underset{\alpha}{e}, \underset{\lambda}{e}$ and $\underset{\sigma}{e}$ respectively

$$(1) \qquad \underset{a}{e} = A_a^\alpha \underset{\alpha}{e}, \qquad \underset{i}{e} = B_i^\lambda \underset{\lambda}{e}, \qquad \underset{\rho}{e} = C_\rho^\sigma \underset{\sigma}{e}.$$

the corresponding inverse transformation is obviously given by

$$(2) \qquad \underset{\alpha}{e} = A_\alpha^a \underset{a}{e}, \qquad \underset{\lambda}{e} = B_\lambda^i \underset{i}{e}, \qquad \underset{\sigma}{e} = C_\sigma^\rho \underset{\rho}{e}$$

where

$$(3) \qquad A_\alpha^a A_b^\alpha = \delta_b^a, \qquad A_\alpha^a A_a^\beta = \delta_\alpha^\beta,$$

$$(4) \qquad B_\lambda^i B_j^\lambda = \delta_j^i, \qquad B_\lambda^i B_i^\mu = \delta_\lambda^\mu, \qquad C_\sigma^\rho C_r^\sigma = \delta_r^\rho, \qquad C_\sigma^\rho C_\rho^\varepsilon = \delta_\sigma^\varepsilon.$$

By convention, the transformation of the covariant base is given by

(5) $$\overset{a}{e}=A^a_\alpha\overset{\alpha}{e},\qquad \overset{i}{e}=B^i_\lambda\overset{\lambda}{e},\qquad \overset{p}{e}=C^p_\sigma\overset{\sigma}{e}.$$

(6) $$\overset{\alpha}{e}=A^\alpha_a\overset{a}{e},\qquad \overset{\lambda}{e}=B^\lambda_i\overset{i}{e},\qquad \overset{\sigma}{e}=C^\sigma_p\overset{p}{e}.$$

In a like manner we may pass from one system of geometrical axes

$$\left(\underset{a}{e},\overset{a}{e};\ \underset{i}{e},\overset{i}{e};\ \underset{p}{e},\overset{p}{e}\right)$$

to another

(7) $$\begin{cases} \underset{a'}{e}=A^a_{a'}\underset{a}{e}, \qquad \overset{a}{e}=A^a_{a'}\overset{a'}{e}, \\ \underset{i'}{e}=B^i_{i'}\underset{i}{e}, \qquad \overset{i'}{e}=B^{i'}_i\overset{i}{e}, \qquad \underset{p'}{e}=C^p_{p'}\underset{p}{e}, \qquad \overset{p'}{e}=C^{p'}_p\overset{p}{e} \end{cases}$$

where either $A^a_{a'}$, $B^i_{i'}$ and $C^p_{p'}$ or $A^{a'}_a$, $B^{i'}_i$, $C^{p'}_p$ can be arbitrarily given and then the other set is determined by either of the two equivalent sets of equations

(8) $$A^a_{a'}A^{a'}_b=\delta^a_b,\qquad B^i_{i'}B^{i'}_j=\delta^i_j,\qquad C^p_{p'}C^{p'}_r=\delta^p_r,$$

(9) $$A^a_{a'}A^{b'}_a=\delta^{b'}_{a'},\qquad B^i_{i'}B^{j'}_i=\delta^{j'}_{i'},\qquad C^p_{p'}C^{r'}_p=\delta^{r'}_{p'}.$$

Now, any set of geometrical base vectors $\left(\underset{i}{e},\ \underset{p}{e};\ \overset{i}{e},\ \overset{p}{e}\right)$ forms a special kind of base for X_N, *a split base.* If, for a moment, we put

$$\underset{a'=i}{e}=\underset{i}{e},\qquad \underset{a'=p}{e}=\underset{p}{e},\qquad \overset{a'=i}{e}=\overset{i}{e},\qquad \overset{a'=p}{e}=\overset{p}{e},$$

we have

(10) $$\underset{a}{e}=A^{a'}_a\underset{a'}{e}=B^i_a\underset{i}{e}+C^p_a\underset{p}{e},\qquad \ldots,$$

(11) $$\overset{a}{e}=A^a_{a'}\overset{a'}{e}=B^a_i\overset{i}{e}+C^a_p\overset{p}{e},\qquad \ldots,$$

Therefore, from (8) and (9), we have

(12) $$\begin{cases} B^a_iB^j_a=\delta^j_i, \qquad C^a_pC^r_a=\delta^r_p, \qquad B^a_iC^p_a=0, \qquad B_aC^p_a=0, \\ B^a_iB^i_b+C^a_pC^p_b=\delta^a_b. \end{cases}$$

The last two equations in the first row result from the fact that *i* and *p* never assume the same value.

We also notice that the result of two consecutive changes of base

$$\underset{a'}{e} = A^a_{a'}\underset{a}{e}, \quad \underset{i'}{e} = B^i_{i'}\underset{i}{e}, \quad \underset{p'}{e} = C^p_{p'}\underset{p}{e}; \quad \overset{a'}{e} = A^{a'}_a\overset{a}{e}, \quad \overset{i'}{e} = B^{i'}_i\overset{i}{e}, \quad \overset{p'}{e} = C^{p'}_p\overset{p}{e}$$

and

$$\underset{a''}{e} = A^{a'}_{a''}\underset{a'}{e}, \quad \underset{i''}{e} = B^{i'}_{i''}\underset{i'}{e}, \quad \underset{p''}{e} = C^{p'}_{p''}\underset{p'}{e}; \quad \overset{a''}{e} = A^{a''}_{a'}\overset{a'}{e}, \quad \overset{i''}{e} = B^{i''}_{i'}\overset{i'}{e}, \quad \overset{p''}{e} = C^{p''}_{p'}\overset{p'}{e}$$

is a change of base

$$\underset{a''}{e} = A^a_{a''}\underset{a}{e}, \quad \underset{i''}{e} = B^i_{i''}\underset{i}{e}, \quad \underset{p''}{e} = C^p_{p''}\underset{p}{e}, \quad \overset{a''}{e} = A^{a''}_a\overset{a}{e}. \quad \overset{i''}{e} = B^{i''}_i\overset{i}{e}. \quad \overset{p''}{e} = C^{p''}_p\overset{p}{e}.$$

where

$$(13) \qquad \begin{cases} A^a_{a''} = A^{a'}_{a''}A^a_{a'}, & B^i_{i''} = B^{i'}_{i''}B^i_{i'}, & C^p_{p''} = C^{p'}_{p''}C^p_{p'}. \\ A^{a''}_a = A^{a''}_{a'}A^{a'}_a; & B^{i''}_i = B^{i''}_{i'}B^{i'}_i. & C^{p''}_p = C^{p''}_{p'}C^{p'}_p; \end{cases}$$

Finally, expressing $\underset{i}{e}$ and $\underset{p}{e}$ in the base $\underset{a}{e}$ and $\underset{a'}{e}$, we get

$$\underset{i}{e} = B^a_i\underset{a}{e} = B^{a'}_i\underset{a'}{e}, \qquad \underset{p}{e} = C^a_p\underset{a}{e} = C^{a'}_p\underset{a'}{e}$$

which, from $\underset{a'}{e} = A^a_{a'}\underset{a}{e}$, gives

$$(14) \qquad B^{a'}_i A^a_{a'} = B^a_i. \qquad C^{a'}_p A^a_{a'} = C^a_p.$$

In this way we have established the following rules.

(A) If there is a summation (dummy) suffix in a product of two A'*s*, two B'*s* or two C'*s*, the product is equal to the A, B or C of the remaining suffixes.

(B) If there is a dummy suffix in a product of an A and a B or of an A and a C, the product is equal to the B or the C of the remaining suffixes.

(C) If there is dummy suffix in a product of a B and a C, the product is zero.

$$(D) \qquad B^{a'}_a + C^a_{a'} = A^a_{a'}; \qquad A^a_b = \delta^a_b; \qquad B^i_j = \delta^i_j. \qquad C^p_r = \delta^p_r.$$

In particular

$$B^a_b + C^a_b = A^a_b = \delta^a_b.$$

This mechanism of transformation factors is due to Schouten. We shall also follow his convention of writing one A, B or C only in products of A'*s*, B'*s* or C'*s*. For example

$$B^{a}_{i}B^{j}_{h}B^{h}_{k} \equiv B^{ajh}_{ihk}.$$

For tensor components we introduce the following notation. Every vector $v = v^{a}\underset{a}{e}$ or $w = w_{a}\overset{a}{e}$ can be thrown into the form

$$v = v^{a}B^{i}_{a}\underset{i}{e} + v^{a}C^{p}_{a}\underset{p}{e} \equiv \bar{v}^{i}\underset{i}{e} + \hat{v}^{p}\underset{p}{e},$$

$$w = w_{a}B^{a}_{i}\overset{i}{e} + w_{a}C^{a}_{p}\overset{p}{e} \equiv \bar{w}_{i}\overset{i}{e} + \hat{w}_{p}\overset{p}{e}.$$

The two resolutes

$$\bar{v}^{i} = v^{a}B^{i}_{a}, \qquad \hat{v}^{p} = v^{a}C^{p}_{a},$$

and

$$\bar{w}_{i} = v_{a}B^{a}_{i}, \qquad \hat{w}_{p} = w_{a}C^{a}_{p},$$

are called the projections of v^{a} and w_{a} on the facet and its span respectively.

Tensors of the rank 3 have 2^{3} resolutes, and so on. For example, if $v = v^{a}_{b}\underset{a}{e}\overset{b}{e}$, we have

$$v = v^{a}_{b}\left(B^{i}_{a}\underset{i}{e} + C^{p}_{a}\underset{p}{e}\right)\left(B^{b}_{k}\overset{k}{e} + C^{b}_{r}\overset{r}{e}\right)$$

$$= v^{a}_{b}B^{ib}_{ak}\underset{i}{e}\overset{k}{e} + v^{a}_{b}B^{i}_{a}C^{b}_{r}\underset{i}{e}\overset{r}{e} + v^{a}_{b}C^{p}_{a}B^{b}_{k}\underset{p}{e}\overset{k}{e} + v^{a}_{b}B^{p}_{a}C^{b}_{r}\underset{p}{e}\overset{r}{e}.$$

The first and last resolutes, viz.

$$\bar{v}^{i}_{k} = v^{a}_{b}B^{ib}_{ak}, \qquad \ldots, \qquad \hat{v}^{p}_{r} = v^{a}_{b}C^{pb}_{ar}$$

lie in the facet and its span respectively so that we might call them *the facet and span projections* of v. The other two « cross resolutes » having one suffix in the facet, the other in the span show, however, that projection on two complementary facets is not, in general, equivalent to resolution.

4. *Connexion and differentiation.* — An essentially new idea in the modern conception of geometry is the construction of a finite space by dovetailing infinitesimal local geometries. This connexion is established by defining equivalence or parallelism between tensors attached to neighbouring points. If $Q(x^\alpha + \Delta x^\alpha)$ is a point in the admitted facet at $P(x^\alpha)$ so that

$$(1) \qquad \Delta x^\alpha = B^\alpha_k \Delta u^k$$

with arbitrary Δu^k, the quantities

$$(2) \qquad \Delta u^k = B^k_\alpha \Delta x^\alpha$$

are the coordinates of the points in the facet, and we put

$$(3) \qquad e_j(Q \| P) = e_j + l^i_{jk} e_i \Delta u^k, \qquad e^i(Q \| P) = e^i - l^i_{jk} e^j \Delta u^k,$$

$$(4) \qquad e_r(Q \| P) = e_r + \lambda^p_{rk} e_p \Delta u^k, \qquad e^p(Q \| P) = e^p - \lambda^p_{rk} e^r \Delta u^k.$$

If the point $R(x^\alpha + \Delta x^\alpha)$ is outside the facet so that

$$(5) \qquad \Delta x^\alpha = C^\alpha_s \Delta u^s,$$

the quantities

$$(6) \qquad \Delta u^s = C^s_\alpha \Delta x^\alpha$$

are its coordinates in the span near P, and we put

$$(7) \qquad e_j(R \| P) = e_j + s^i_{js} e_i \Delta u^s, \qquad e^i(R \| P) = e^i - s^i_{js} e^j \Delta u^s$$

$$(8) \qquad e_r(R \| P) = e_r + \sigma^p_{rs} e_p \Delta u^s, \qquad e^p(R \| P) = e^p - \sigma^p_{rs} e^r \Delta u^s$$

where (P) has been suppressed. The four sets of connexion parameters l^i_{jk}, λ^p_{rk}, s^i_{js}, σ^p_{rs} are arbitrarily assigned functions of x^α. Equations (3), (4), (7), (8) determine the equivalence also for tensors of any type and rank by the convention that the coefficients in the forms of the e's are not affected by the equivalence.

For example, if

$$v = v^j e_j \quad \text{and} \quad w = w_i e^i,$$

we have

$$w(R \| P) = w_i(R) e^i - w_i(R) s^i_{js} e_j \Delta u^s,$$

$$v(Q \| P) = v^j(Q) e_j + v^j(Q) l^i_{jk} e_i \Delta u^k.$$

thus, retaining linear terms only,

$$(9)\qquad \begin{cases} \delta w \equiv w(\mathrm{R}\|\mathrm{P}) - w(\mathrm{P}) = (\delta w_j - s^l_{js} w_l \Delta u^s) \overset{j}{e}, \\ \delta v \equiv v(\mathrm{Q}\|\mathrm{P}) - v(\mathrm{P}) = (\delta v^i + l^i_{jk} v^j \Delta u^k) \underset{i}{e}, \end{cases}$$

where

$$(10)\qquad \begin{cases} \delta v^i = v^i(\mathrm{Q}) - v^i(\mathrm{P}) = \partial_k v^i \Delta u^k, & \partial_k \equiv \mathrm{B}^\alpha_k \partial_\alpha. \\ \delta w_j = w_j(\mathrm{R}) - w_j(\mathrm{P}) = \partial_s w_j \Delta u^s, & \partial_s \equiv \mathrm{C}^\alpha_s \partial_\alpha. \end{cases}$$

Therefore we define tensor differentiation by putting

$$(11)\qquad \begin{cases} \nabla_k v^i = \partial_k v^i + l^i_{jk} v^j, & \nabla_k v_j = \partial_k v_j - l^i_{jk} v_i, \\ \nabla_k v^p = \partial_k v^p + \lambda^p_{rk} v^r, & \nabla_k v_r = \partial_k v_r - \lambda^p_{rk} v_p, \\ \nabla_s v^i = \partial_s v^i + s^i_{js} v^j, & \nabla_s v_j = \partial_s v_j - s^i_{js} v_i, \\ \nabla_s v^p = \partial_s v^p + \sigma^p_{rs} v^r, & \nabla_s v_r = \partial_s v_r - \sigma^p_{rs} v_p. \end{cases}$$

For tensors of higher rank we extend R. Lagrange's mixed differentiation (R. Lagrange 1922, 1926), used also by Bortolotti (1930) and Schouten and van Kampen (1930, D-symbolik) to the new types of differentiation by putting for example

$$(12)\qquad \nabla_k v^i_r = \partial_k v^i_r + l^i_{jk} v^j_r - \lambda^p_{rk} v^i_p, \qquad \nabla_s v^i_r = \partial_s v^i_r + s^i_{js} v^j_r - \sigma^p_{rs} v^i_p.$$

We shall also consider partially resolved tensors for which differentiation will be defined by formulae like

$$(13)\qquad \nabla_k v^a_r = \partial_k v^a_r + \Lambda^a_{bc} v^b_r \mathrm{B}^c_k - \lambda^p_{rk} v^a_p, \qquad \nabla_s v^i_b = \partial_s v^i_b + s^i_{js} v^j_b - \Lambda^a_{bc} v^i_a \mathrm{C}^c_s.$$

Finally, we put

$$(14)\qquad \nabla_c = \mathrm{B}^k_c \nabla_k + \mathrm{C}^s_c \nabla_s.$$

Λ^a_{bc} are the connexion parameters in X_N (geometrical axes).

In a change of base, the transformation of the connexion parameters is defined by the condition that tensor differential coefficients (or tensors differentials) are transformed as tensors, i. e. by the conditions

$$(15)\qquad \nabla_b \mathrm{A}^{a'}_a = 0, \qquad \nabla_k \mathrm{B}^{i'}_i = 0, \qquad \nabla_s \mathrm{B}^{i'}_i = 0, \qquad \nabla_k \mathrm{C}^{p'}_p = 0, \qquad \nabla_s \mathrm{C}^{p'}_p = 0.$$

These equations readily lead to the rules of transformation

$$(16)\quad \begin{cases} \Lambda^{a'}_{b'c'} = A^{a'bc}_{ab'c'}\Lambda^{a}_{bc} + A^{a'}_{b}\partial_{c'}A^{b}_{b'} \\ l^{i'}_{j'k'} = B^{i'jk}_{ij'k'}\, l^{i}_{jk} + B^{i'}_{j}\partial_{k'}B^{j}_{j'}, \qquad \lambda^{p'}_{r'k'} = C^{p'r}_{pr'}B^{k}_{k'}\lambda^{p}_{rk} + C^{p'}_{r}\partial_{k'}C^{r}_{r'}. \\ s^{i'}_{j's'} = B^{i'j}_{ij'}C^{s}_{s'}s^{i}_{js} + B^{i'}_{j}\partial_{s'}B^{j}_{j'}, \qquad \sigma^{p'}_{r's'} = C^{p'rs}_{pr's'}\sigma^{p}_{rs} + C^{p'}_{r}\partial_{s'}C^{r}_{r'}. \end{cases}$$

All these differentiations obey all the ordinary laws of tensor differentiation in X_N. In particular, contraction and any of the differentiations are interchangeable, and

$$(17)\qquad \nabla\delta^{a}_{b} = 0, \qquad \nabla\delta^{i}_{j} = 0, \qquad \nabla\delta^{p}_{r} = 0$$

where ∇ indicates ∇_c, ∇_k or ∇_s.

In the usual theory, $\nabla_s v^{a}$, $\nabla_a v^{i}$ etc. are not tensors. We have introduced the connexion parameters s and σ in order to make them tensors.

The projected connexions are obtained by requiring that

$$(18)\qquad \delta \underset{j}{e} = {}_{\iota}\Delta \underset{j}{e}, \qquad \delta \underset{r}{e} = {}_{\iota}\Delta \underset{r}{e}, \qquad '\delta \underset{j}{e} = {}_{n}\Delta \underset{j}{e}, \qquad '\delta \underset{r}{e} = {}_{n}\Delta \underset{r}{e}.$$

where Δ indicates that the following tensor is dealt with as a tensor of X_N, irrespective of the submanifold. These equations when explicited lead to the following specific values for the connexion parameters l, λ, s, σ,

$$(19)\quad \bar{l}^{i}_{jk} = B^{i}_{\alpha}\nabla_k B^{\alpha}_{j}, \qquad \bar{\lambda}^{p}_{rk} = C^{p}_{\alpha}\nabla_k C^{\alpha}_{r}. \qquad \bar{s}^{i}_{js} = B^{i}_{\alpha}\nabla_s B^{\alpha}_{j}. \qquad \bar{\sigma}^{p}_{rs} = C^{p}_{\alpha}\nabla_s C^{\alpha}_{r}.$$

where α can be replaced by a. A bracketed suffix like (j) after a differentiation symbol indicates that this suffix is disconsidered in the differentiation.

Proof of $(19)_1$. We see from $\underset{j}{e} = B^{a}_{j}\underset{a}{e}$ that the components of $\underset{j}{e}$ in $\left(\underset{a}{e}, \overset{a}{e}\right)$ are B^{a}_{j}, j being fixed. Thus

$$(20)\qquad \Delta\underset{j}{e} = \underset{j}{e}(Q\|P) - \underset{j}{e} = \nabla_c B^{a}_{\cdot j}\,.dx^{c}\,\underset{a}{e} = \nabla_c B^{a}_{j}\,.dx^{c}\left(B^{i}_{a}\underset{i}{e} + C^{p}_{a}\underset{p}{e}\right).$$

Hence from $dx^{c} = B^{c}_{k}\,dx^{k}$, the projection of the difference upon the facet is equal to

$${}_{\iota}\Delta\underset{j}{e} = B^{i}_{a}\nabla_k B^{a}_{\cdot(j)}\,.dx^{k}\,.\underset{i}{e}.$$

On the other hand, by definition, $\delta \underset{i}{e} = l^i_{jk} dx^k \underset{i}{e}$. Therefore, if we require that for every i and for every displacement in the facet, i. e. for every value of dx^k, we should have $(18)_1$, we obtain for l^i_{jk} the value given by $(19)_1$.

The argument is similar for the other equations. The special values (19) of l, λ, s, σ will be referred to as the connexion parameters of the projected connexion, or, briefly, as *the projected connexion parameters*, while the general arbitrarily assigned values will be called *intrinsic connexion parameters*.

5. *Fundamental tensors of the submanifold.* — Equations (4, 16) show that connexion parameters are not transformed as tensors. On the other hand, the equation

$$\Lambda^{a'b'c'}_{abc} \cdot \Lambda^{\alpha\beta\gamma}_{\alpha bc} \Gamma^{\alpha}_{\beta\gamma} = \Lambda^{\alpha\beta\gamma}_{\alpha b'c'} \Gamma^{\alpha}_{\beta\gamma}$$

proves that in any change of base the expression

$$\Lambda^{\alpha\beta\gamma}_{\alpha bc} \Gamma^{\alpha}_{\beta\gamma} \tag{1}$$

transforms as a tensor. If, however, a suffix of $\Gamma^{\alpha}_{\beta\gamma}$ is not a dummy suffix, the change of base requires the replacement of $\Gamma^{\alpha}_{\beta\gamma}$ by $\Lambda^{c'}_{a'b'}$, say, and our equality is destroyed. Similarly, if we try to change dummy suffixes $\alpha\beta\gamma$ into $a'b'c'$. On the other and, the equation

$$B^i_\alpha \nabla_k B^\alpha_j = B^i_a \nabla_k B^a_j. \tag{2}$$

resulting from $B^\alpha_j = \Lambda^\alpha_a B^a_j$ and $\nabla_k \Lambda^\alpha_a = 0$, shows that the expression (2) is unaltered by changing the dummy suffix α into a. The definition of (1) requires a specified system of axes, the value of (2) is the same in every system. We shall say that (1) is a *relative tensor*, and in this paper relative tensors will be all defined in the mathematical frame, while (2) will be called an *absolute tensor*. We shall see that torsion and curvature are not absolute tensors.

Another distinction between tensors will be the following. It is known that the Riemann-Christoffel tensor transforms as a tensor in a mathematical change of variables only, while in a general change of base it loses its tensor character. On the other hand, (1) and (2) transform as tensors in any change of base (axes). We shall say that (1) and (2) are *geometrical tensors*, while the Riemann-Christoffel tensor is a *Ricci tensor* only.

Finally, if a Ricci or geometrical tensor contains metric or connexion (i. e. structure) parameters, il will be called a structure tensor. For example (2) is an absolute geometrical structure tensor. The projection factors, B^{a}_{j}, say, transform as tensors, for

$$A^{a'}_{a} B^{j}_{j'} . B^{a}_{j} = B^{a'}_{j'},$$

but they are not structure tensors. We wight call them *formal* tensors.

Table I gives a list of important tensors.

TABLE I.

$D^{i'}_{jk} = B^{i'}_{\alpha} \nabla_k B^{\alpha}_{j}$,	$d^{i'}_{jk} = B^{\alpha}_{j} \partial_k B^{i'}_{\alpha} - B^{\cdot}_{j} \partial_k B^{i'}_{\cdot}$,
$E^{p'}_{rk} = C^{p'}_{\alpha} \nabla_k C^{\alpha}_{r}$,	$e^{p'}_{rk} = C^{\alpha}_{r} \partial_k C^{p'}_{\alpha} - C^{\cdot}_{r} \partial_k C^{p'}_{\cdot}$,
$F^{p'}_{jk} = C^{p'}_{\alpha} \nabla_k B^{\alpha}_{j} = C^{p'}_{\alpha} \nabla_k B^{\alpha}{}_{j}$,	$f^{p'}_{jk} = B^{\alpha}_{j} \partial_k C^{p'}_{\alpha}$,
$G^{i'}_{rk} = B^{i'}_{\alpha} \nabla_k C^{\alpha}_{r} = B^{i'}_{\alpha} \nabla_k C^{\alpha}{}_{r}$,	$g^{i'}_{rk} = C^{\alpha}_{r} \partial_k B^{i'}_{\alpha}$,
$H^{i'}_{js} = B^{i'}_{\alpha} \nabla_s B^{\alpha}_{j}$,	$h^{i'}_{js} = B^{\alpha}_{j} \partial_s B^{i'}_{\alpha} - B^{\cdot}_{j} \partial_s B^{i'}_{\cdot}$,
$I^{p'}_{rs} = C^{p'}_{\alpha} \nabla_s C^{\alpha}_{r}$,	$i^{p'}_{rs} = C^{\alpha}_{r} \partial_s C^{p'}_{\alpha} - C^{\cdot}_{r} \partial_s C^{p'}_{\cdot}$,
$J^{p'}_{js} = C^{p'}_{\alpha} \nabla_s B^{\alpha}_{j} = C^{p'}_{\alpha} \nabla_s B^{\alpha}{}_{j}$.	$j^{p'}_{js} = B^{\alpha}_{j} \partial_s C^{p'}_{\alpha}$.
$K^{i'}_{rs} = B^{i'}_{\alpha} \nabla_s C^{\alpha}_{r} = B^{i'}_{\alpha} \nabla_s C^{\alpha}{}_{r}$,	$k^{i'}_{rs} = C^{\alpha}_{r} \partial_s B^{i'}_{\alpha}$.

The tensors in the first column are absolute geometrical structure tensors (α may be replaced by a, a', etc.), those in the second column are relative geometrical formal tensors. F and G are Schouten's — H and — L, the alternated [1] f is Schouten's Z.

The set of equations

$$(3) \begin{cases} l^{i}_{jk} = \bar{l}^{i}_{jk} - D^{i}_{jk}, \quad \lambda^{p}_{rk} = \bar{\lambda}^{p}_{rk} - E^{p}_{rk}, \quad s^{i}_{js} = \bar{s}^{i}_{js} - H^{i}_{js}, \quad \sigma^{p}_{rs} = \bar{\sigma}^{p}_{rs} - I^{p}_{rs}, \\ \underset{j}{e}(Q \parallel P) - \underset{j}{l} = \left(\bar{l}^{i}_{jk} \underset{i}{e} + F^{p}_{jk} \underset{p}{e}\right) dx^k, \quad \underset{r}{e}(Q \parallel P) - \underset{r}{e} = \left(G^{i}_{rk} \underset{i}{e} + \bar{\lambda}^{p}_{rk} \underset{p}{e}\right) dx^k, \\ \underset{j}{e}(R \parallel P) - \underset{j}{e} = \left(\bar{s}^{i}_{js} \underset{i}{e} + J^{p}_{js} \underset{p}{e}\right) dx^s, \quad \underset{r}{e}(R \parallel P) - \underset{r}{e} = \left(K^{i}_{sr} \underset{i}{e} + \bar{\sigma}^{p}_{rs} \underset{p}{e}\right) dx^s, \end{cases}$$

---

(1) *Alternation* is defined by $v_{[ab]} = \frac{1}{2}(v_{ab} - v_{ba})$. *averaging* (Mischung) by $v_{(ab)} = \frac{1}{2}(v_{ab} + v_{ba})$. If there is a suffix between a and b which is not affected by the process, we put it between two bars (Schouten, 1924).

where the point Q $(x^\alpha + dx^\alpha)$ lies in the facet (tangent plane) at P (x^α) and R $(x^\alpha + 'dx^\alpha)$ lies outside the facet so that $dx^\alpha = B^\alpha_k dx^k$, $'dx^\alpha = C^\alpha_s dx^s$, shows that D, E, H, I characterise the difference between projected and intrinsic connexion (*connexion tensors*) while F, G, J, K characterise the parts of the connexion in X_n which is ignored in the definition of the connexions (projected or intrinsic) in X^m_N and in its span (*deviation tensors*).

We notice that in the holonomic case no facet (split base) need be specified at R. Equations (2, 5), however define a split base at R, a natural though highly arbitrary continuation of that at P.

In Table II we give five fundamental tensors whose alternated forms with respect to the low suffixes are the *torsion tensors*.

TABLE II.

$$T^a_{bc} = A^a_{bc} + A^\alpha_b \partial_c A^a_\alpha,$$
$$T^i_{jk} = \Gamma^i_{jk} + B^\alpha_j \partial_k B^i_\alpha,$$
$$T^p_{rk} = \lambda^p_{rk} + C^\alpha_r \partial_k C^p_\alpha,$$
$$T^i_{js} = s^i_{js} + B^\alpha_j \partial_s B^i_\alpha,$$
$$T^p_{rs} = \sigma^p_{rs} + C^\alpha_r \partial_s C^p_\alpha,$$

These tensors are relative geometrical structure tensors. It is known that the transform of an infinitesimal square, when we pass from mathematical to geometrical axes, is not closed. The vector closing the transformed square is determined in the usual way by the alternated forms of the T tensors. The four torsions of the submanifold correspond to squares in the facet, span and in both.

We also give the formulae

$$(4)\qquad \begin{cases} \nabla_k B^a_j = D^a_{jk} + F^a_{jk}, & \nabla_k C^a_r = E^a_{rk} + G^a_{rk}, \\ \nabla_s B^a_j = H^a_{js} + J^a_{js}, & \nabla_s C^a_r = I^a_{rs} + K^a_{rs}, \\ \nabla_k B^i_a = - D^i_{ak} - G^i_{ak}, & \nabla_k C^r_a = - E^r_{ak} - F^r_{ak}, \\ \nabla_s B^i_a = - H^i_{as} - K^i_{as}, & \nabla_s C^r_a = - I^r_{as} - J^r_{as}, \end{cases}$$

where

$$D^a_{jk} = B^a_i D^i_{jk}, \qquad F^a_{jk} = C^a_p F^p_{jk}, \qquad \ldots.$$

Proof of $(4)_1$:

$$\nabla_k B^a_i = \delta^a_b \nabla_k B^b_i = (B^a_b + C^a_b) \nabla_k B^b_i = B^a_j . B^j_b \nabla_k B^b_i + C^a_r . C^r_b \nabla_k B^b_i,$$

Proof of $(4)_5$:

$$\nabla_k B^i_a = B^b_a \nabla_k B^i_b + C^b_a \nabla_k B^i_b,$$

whose first term is equal to

$$B^j_a . B^b_j \nabla_k B^i_b = - B^j_a . B^i_b \nabla_k B^b_j,$$

and whose second term is equal to $C^r_a . C^b_r \nabla_k B^i_b = - C^r_a . B^i_b \nabla_k C^b_r$.

The proof is similar for the other relations.

Combining $\nabla_b = B^k \nabla_k + C^s_b \nabla_s$ with formulae of (4) we obtain formulae like

(5) $\quad \nabla_b B^a_j = D^a_{jb} + F^a_{jb} + H^a_{jb} + J^a_{jb}, \qquad \nabla_b C^a_r = E^a_{rb} + G^a_{rb} + I^a_{rb} + K^a_{rb}.$

and

(6) $$\begin{cases} B^i_a \nabla_b B^a_j = D^i_{jb} + H^i_{jb}, \\ C^p_a \nabla_b C^a_r = E^p_{rb} + I^p_{rb}, \\ C^p_a \nabla_b C^a_j = F^p_{jb} + J^p_{jb}, \\ B^i_a \nabla_b C^a_r = G^i_{rb} + K^i_{rb}. \end{cases}$$

6. *Curvature tensors.* — Another group of fundamental tensors is that of the curvature tensors; they are relative geometrical structure tensors. To simplify writing we put,

(1) $$m^i_{jc} = l^i_{jk} B^k_c + s^i_{js} C^s_c, \qquad \mu^p_{rc} = \lambda^p_{rk} B^k_c + \sigma^p_{rs} C^s_c.$$

TABLE III.

$$C^a_{bcd}(\Lambda) = \partial_{[d} \Lambda^a_{|b|c]} + \Lambda^a_{e[d} \Lambda^e_{|b|c]} + \Lambda^\alpha_{[c} \partial_{d]} \Lambda^e_\alpha . \Lambda^a_{be}.$$

$$C^i_{jcd}(m) = \partial_{[d} m^i_{|j|c]} + m^i_{k[d} m^k_{|j|c]} + \Lambda^\alpha_{[c} \partial_{d]} \Lambda^e_\alpha . m^i_{je}.$$

$$C^p_{rcd}(\mu) = \partial_{[d} \mu^p_{|r|c]} + \mu^p_{q[d} \mu^q_{|r|c]} + \Lambda^\alpha_{[c} \partial_{d]} \Lambda^e_\alpha . \mu^p_{re}.$$

$$C^i_{jkt}(l) = \partial_{[t} l^i_{|j|k]} + l^i_{m[t} l^m_{|j|k]} + B^\alpha_{[k} \partial_{t]} B^m_\alpha . l^i_{jm} + f^s_{[kt]} s^i_{js},$$

$$C^i_{jst}(s) = \partial_{[t} s^i_{|j|s]} + s^i_{m[t} s^m_{|j|s]} + C^\alpha_{[s} \partial_{t]} C^r_\alpha . s^i_{jr} + k^m_{[st]} l^i_{jm},$$

$$2C^i_{jkt}(l, s) = \partial_t l^i_{jk} - \partial_k s^i_{jt} + s^i_{mt} l^m_{jk} - l^i_{mk} s^m_{jt}$$
$$+ B^\alpha_k \partial_t B^m_\alpha . l^i_{jm} - C^\alpha_t \partial_k C^p_\alpha . s^i_{jp} + f^s_{kt} s^i_{js} - g^m_{th} l^i_{jm},$$
$$C^i_{jtk} = - C^i_{jkt}.$$

$$C^p_{rkl}(\lambda) = \partial_{[l} \lambda^p_{|r|k]} + \lambda^p_{q[l} \lambda^q_{|r|k]} + B^\alpha_{[k} \partial_{l]} B^m_\alpha . \lambda^p_{rm} + f^s_{[kl]} \sigma^p_{rs},$$

$$C^p_{rst}(\sigma) = \partial_{[t} \sigma^p_{|r|s]} + \sigma^p_{q[t} \sigma^q_{|r|s]} + C^\alpha_{[s} \partial_{t]} C^q_\alpha . \sigma^p_{rq} + k^m_{[st]} \lambda^p_{rm},$$

$$2C^p_{rkt}(\lambda, \sigma) = \partial_t \lambda^p_{rk} - \partial_k \sigma^p_{rt} + \sigma^p_{qt} \lambda^q_{rk} - \lambda^p_{qk} \sigma^q_{rt}$$
$$+ B^\alpha_k \partial_t B^m_\alpha . \lambda^p_{rm} + C^\alpha_t \partial_k C^s_\alpha . \sigma^p_{rs} + f^s_{kt} \sigma^p_{rs} - g^m_{th} \lambda^p_{rm},$$
$$C^p_{rtk} = - C^p_{rkt}.$$

The geometrical significance of the last eight tensors in X_ν^m and its span is similar to that of the first in X_ν. In Art. 10 we shall express the 16 resolutes of C^a_{bcd} in terms of the last eight and in the absolute tensors D, E, F, G, H, I, J, K. The second and third tensors can be expressed in terms of the last eight :

(2) $$C^i_{jcd}(m) = B^{kl}_{cd} C^i_{jkl}(l) + C^{st}_{cd} C^i_{jst}(s) + B^k_{[c} C^t_{d]} C^i_{jkt}(l, s).$$

(3) $$C^p_{rcd}(\mu) = B^{kl}_{ce} C^p_{rkl}(\lambda) + C^{st}_{cd} C^p_{rst}(\sigma) + B^k_{[c} C^t_{d]} C^p_{rkt}(\lambda, \sigma).$$

Proof of (2). $C^i_{jcd}(m)$ is the alternation with respect to c, d of

(4) $$\partial_d l^i_{jk} . B^k_c + l^i_{jk} \partial_d B^k_c + \partial_d s^i_{js} . C^s_c - s^i_{js} \partial_d C^s_c$$
$$+ (l^i_{ml} B^l_d + s^i_{mt} C^t_d)(l^m_{jk} B^k_c + s^m_{js} C^s_c) - A^\alpha_b \partial_d A^\alpha_c . (l^i_{jk} B^k_b + s^i_{js} C^s_b)$$
$$= \underset{1}{B^{kl}_{cd} \partial_l l^i_{jk}} + \underset{2}{B^k_c C^t_d \partial_t l^i_{jk}} + \underset{3}{B^l_d \partial_l B^k_c . l^i_{jk}} + \underset{4}{C^t_d \partial_t B^k_c . l^i_{jk}} + \underset{5}{C^t_c B^k_d \partial_k s^i_{jt}}$$
$$+ \underset{6}{C^{st}_{cd} \partial_t s^i_{js}} + \underset{7}{B^l_d \partial_l C^s_c . s^i_{js}} + \underset{8}{C^t_d \partial_t C^s_c . s^i_{js}} + \underset{9}{B^{kl}_{cd} l^i_{ml} l^m_{jk}} + \underset{10}{C^{st}_{cd} s^i_{mt} s^m_{js}}$$
$$+ \underset{11}{B^k_c C^t_d l^m_{jk} s^i_{mt}} + \underset{12}{C^t_c B^k_d l^i_{mk} s^m_{jt}} - \underset{13}{B^{kl}_{\alpha d} \partial_l A^\alpha_c . l^i_{jk}} - \underset{14}{B^k_\alpha C^t_d \partial_t A^\alpha_c . l^i_{jk}}$$
$$- \underset{15}{C^s_\alpha B^l_d \partial_l A^\alpha_c . s^i_{js}} - \underset{16}{C^{st}_{\alpha d} \partial_t A^\alpha_c . s^i_{js}}.$$

The group 3 + 13 is equal to

$$A^\alpha_c B^l_d \partial_l B^k_\alpha . l^i_{jk} = B^{l\alpha}_{dc} \partial_l B^k_\alpha . l^i_{jk} + B^l_d C^\alpha_c \partial_l B^k_\alpha . l^i_{jk} =$$
(5) $$B^{kl}_{cd} . B^\alpha_k \partial_l A^m_\alpha . l^i_{jm} + C^t_c B^k_d . C^\alpha_t \partial_k B^m_\alpha . l^i_{jm}, \quad \ldots \qquad 3 + 13,$$

Similarly, 7 + 15 is equal to $A^\alpha_c B^l_d \partial_e C^s_\alpha . s^i_{js} =$

(6) $$B^{kl}_{cd} . B^\alpha_k \partial_l C^s_\alpha . s^i_{js} + C^t_c B^k_d . C^\alpha_t \partial_k C^p_\alpha . s^i_{jp}, \quad \ldots \qquad 7 + 15.$$

Thus, after alternation, 1 + 9 and the first terms of (5) and (6) give $B^{kl}_{cd} C^i_{jkl}(l)$.

Moreover, 8 + 16 is equal to

$$A^\alpha_c C^t_d \partial_t C^s_\alpha . s^i_{js} =$$
(7) $$B^k_c C^t_d . B^\alpha_k \partial_t C^s_\alpha . s^i_{js} + C^{st}_{cd} . C^\alpha_s \partial_t C^r_\alpha . s^i_{jr}, \quad \ldots \qquad 8 + 16,$$

and 4 + 14 is equal to

$$A^\alpha_c C^t_\alpha \partial_t B^k_\alpha . l^i_{jk} =$$
(8) $$B^k_c C^t_d . B^\alpha_k \partial_t B^m_\alpha . l^i_{jm} + C^{st}_{cd} . C^s_\alpha \partial_t B_\alpha . l^i_{jk}, \quad \ldots \qquad 4 + 14$$

After alternation, 6 + 10 and the last two terms of (7) and (8) give $C^{st}_{cd} C^i_{jst}(s)$.

The remaining terms are 2, 5, 11, 12, $(5)_2$, $(6)_2$, $(7)_1$, $(8)_1$. We have to exchange c, d and subtract the terms so obtained (and divide by 2). This difference can be written in the form

$$
\begin{aligned}
& B^k_c C^t_d [\underset{2}{\partial_t l^i_{jk}} - \underset{\bar{5}}{\partial_k s^i_{jt}} + \underset{11}{l^m_{jk} s^i_{mt}} - \underset{\overline{12}}{s^m_{jt} l^i_{mk}} \\
& \qquad + \underset{(8)_1}{B^\alpha_k \partial_t B^m_\alpha . l^i_{jm}} - \underset{(6)_2}{C^\alpha_t \partial_k C^\rho_\alpha . s^i_{j\rho}} + \underset{(7)_1}{j^s_{kt} s^i_{js}} - \underset{(5)_2}{g^m_{tk} l^i_{jm}}] \\
& + C^t_c B^k_d [\underset{5}{\partial_k s^i_{jt}} - \underset{2}{\partial_t l^i_{jk}} + \underset{12}{s^m_{jt} l^i_{mk}} - \underset{\overline{11}}{l^m_{jk} s^i_{mt}} \\
& \qquad + \underset{(6)_2}{C^\alpha_t \partial_k C^\rho_\alpha . s^i_{j\rho}} - \underset{(8)_1}{B^\alpha_k \partial_t B^m_\alpha . l^i_{jm}} + \underset{(5)_2}{g^m_{tk} l^i_{jm}} - \underset{(7)_1}{j^s_{kt} s^i_{js}}],
\end{aligned}
$$

where the bar indicates that the corresponding term has been obtained by exchanging c, d and taking the negative sign. This proves (2). A similar argument proves (3).

7. *Resolution of* T^a_{bc}. — Since the torsion tensor of X_N is the alternated form of T^a_{bc}, we give the resolutes of the latter.

(1)	$B^{ibc}_{ajk} T^a_{bc}$	$= B^{i\beta\gamma}_{\alpha jk} \Gamma^\alpha_{\beta\gamma}$	$= D^i_{jk} + T^i_{jk}(l)$,
(2)	$C^{\rho b}_{ar} B^c_k T^a_{bc}$	$= C^{\rho\beta}_{\alpha r} B^\gamma_k \Gamma^\alpha_{\beta\gamma}$	$= E^\rho_{rk} + T^\rho_{rk}(\lambda)$,
(3)	$C^\rho_a B^{bc}_{jk} T^a_{bc}$	$= C^\rho_\alpha B^{\beta\gamma}_{jk} \Gamma^\alpha_{\beta\gamma}$	$= F^\rho_{jk} + f^\rho_{jk}$,
(4)	$B^i_a C^b_r B^c_k T^a_{bc}$	$= B^i_\alpha C^\beta_r B^\gamma_k \Gamma^\alpha_{\beta\gamma}$	$= G^i_{rk} - g^i_{rk}$,
(5)	$B^{ib}_{aj} C^c_s T^a_{bc}$	$= B^{i\beta}_{\alpha j} C^\gamma_s \Gamma^\alpha_{\beta\gamma}$	$= H^i_{js} + T^i_{js}(s)$,
(6)	$C^{\rho bc}_{ars} T^a_{bc}$	$= C^{\rho\beta\gamma}_{\alpha rs} \Gamma^\alpha_{\beta\gamma}$	$= I^\rho_{rs} - T^\rho_{rs}(\sigma)$,
(7)	$C^\rho_a B^b_j C^c_s T^a_{bc}$	$= C^\rho_\alpha B^\beta_j C^\gamma_s \Gamma^\alpha_{\beta\gamma}$	$= J^\rho_{js} - j^\rho_{js}$,
(8)	$B^i_a C^{bc}_{rs} T^a_{bc}$	$= B^i_\alpha C^{\beta\gamma}_{rs} \Gamma^\alpha_{\beta\gamma}$	$= K^i_{rs} - k^i_{rs}$.

Proof of (1). From

$$T^a_{bc} = \Lambda^a_{bc} + \Lambda^\alpha_b \partial_c \Lambda^a_\alpha \qquad \text{and} \qquad \Lambda^a_{bc} = \Lambda^{a\beta\gamma}_{\alpha bc} \Gamma^\alpha_{\beta\gamma} - \Lambda^\alpha_b \partial_c \Lambda^a_\alpha,$$

we obtain the first equality. Moreover

$$
\begin{aligned}
B^{ibc}_{ajk}(\Lambda^a_{bc} + \Lambda^\alpha_b \partial_c \Lambda^a_\alpha) &= B^{ibc}_{ajk} \Lambda^a_{bc} - B^{i\alpha}_{aj} \partial_k \Lambda^a_\alpha \\
&= B^{ibc}_{ajk} \Lambda^a_{bc} + B^i_a \partial_k B^a_j + B^\alpha_j \partial_k B^i_\alpha - l^i_{jk} + l^i_{jk},
\end{aligned}
$$

which proves the second equality of (1) by remarking that, in the definition of D^i_{jk}, the dummy suffix α may be replaced by a. The proof is similar for the other lines.

If we denote the resolutes of T^a_{bc}, *i.e.*, the lefthand sides of (1)—(8) by $\overline{T}^i_{jk}$, $\overline{T}^p_{rk}$ etc., and the second column by $\overline{\Gamma}^i_{jk}$ etc., we obtain the following list of formulae for the resolutes of the torsion tensor $T^a_{[bc]}$

(9) $$\overline{T}^i_{[jk]}(\Lambda) = \overline{\Gamma}^i_{[jk]} = D^i_{[jk]} + T^i_{[jk]}(l),$$

(10) $$2\overline{T}^p_{[rk]}(\Lambda) = 2\overline{\Gamma}^p_{[rk]} = E^p_{rk} + T^p_{rk}(\lambda) - J^p_{kr} - j^p_{kr},$$

(11) $$\overline{T}^p_{[jk]}(\Lambda) = \overline{\Gamma}^p_{[rk]} = F^p_{[rk]} + f^p_{[jk]},$$

(12) $$2\overline{T}^i_{[rk]}(\Lambda) = 2\overline{\Gamma}^i_{[rk]} = G^i_{rk} + g^i_{rk} - H^i_{kr} - T^i_{ks}(s),$$

(13) $$\overline{T}^i_{[kr]} = \overline{T}_{.rk},$$

(14) $$\overline{T}^p_{[rs]}(\Lambda) = \overline{\Gamma}^p_{[rs]} = I^p_{[rs]} + T^p_{[rs]}(\sigma),$$

(15) $$\overline{T}^p_{[kr]} = -\overline{T}^p_{[kk]},$$

(16) $$\overline{T}^i_{[rs]} = \overline{\Gamma}^i_{[rs]} = K^i_{[rs]} + k^i_{[rs]}.$$

8. *Inversion formulae for differentiation of scalars and vectors.* — If f is a function of the mathematical variables x^α, we have

(1) $$\partial_d \partial_c = A^b_\alpha \partial_d A^\alpha_c . \partial_b f,$$

and, from $\partial_l B^\rho_k = (B^\rho_\alpha + C^\rho_\alpha)\partial_l B^\alpha_k = B^\rho_\alpha \partial_l B^\alpha_k - f^\alpha_{kl}$,

(2) $$\partial_l \partial_k f = (B^\alpha_\rho \partial_l B^\rho_{k]} - f^\alpha_{[kl]}) \partial_\alpha f.$$

In a like manner,

(3) $$\partial_t \partial_s f = \partial_t C^\alpha_{s]} . \partial_\alpha f = (C^\alpha_\beta \partial_t C^\beta_{s]} - k^\alpha_{[st]}) \partial_\alpha f;$$

(4) $$\partial_l \partial_k f - \partial_k \partial_t f = (\partial_t B^\alpha_k - \partial_k C^\alpha_t) \partial_\alpha f$$
$$= (B^\alpha_\beta \partial_t B^\beta_k - f^\alpha_{kt} - C^\alpha_\beta \partial_k C^\beta_t + g^\alpha_{tk}) \partial_\alpha f.$$

For every vector or tensor v, where we have suppressed the suffixes to make the statement general, $\nabla_k v = B^c_k \nabla_c v$, $\nabla_s v = C^c_s \nabla_c v$, thus

$$\nabla_l \nabla_k v = \nabla_l B^c_k . \nabla_c v + B^{cd}_{kl} \nabla_d \nabla_c v,$$
$$\nabla_t \nabla_s v = \nabla_t C^c_s . \nabla_c v + C^{cd}_{st} \nabla_d \nabla_c v.$$

Therefore, from (4, 4),

(5) $$\nabla_l \nabla_k v = (D^c_{[kl]} + F^c_{[kl]}) \nabla_c v + B^{cd}_{kl} \nabla_{[d} \nabla_{c]} v,$$

(6) $$\nabla_t \nabla_s v = (I^c_{[st]} + K^c_{[st]}) \nabla_c v + C^{cd}_{st} \nabla_{[d} \nabla_{c]} v,$$

(7) $$\nabla_{[t} \nabla_{k]} v = \frac{1}{2}(H^c_{kt} + J^c_{kt} - E^c_{tk} - G^c_{tk}) \nabla_c v + B^c_k C^d_t \nabla_{[d} \nabla_{c]} v.$$

We have to calculate $\nabla_{[d}\nabla_{c]}v$ only.

For vectors we have

$$(8) \qquad \nabla_{[d}\nabla_{c]}v^i = -T^b_{[cd]}\nabla_b v^i + C^i_{jcd}(m)v^j,$$

$$(9) \qquad \nabla_{[d}\nabla_{c]}v^p = -T^b_{[cd]}\nabla_b v^p + C^p_{rcd}(\mu)v^r,$$

$$(10) \qquad \nabla_{[d}\nabla_{c]}v_j = -T^b_{[cd]}\nabla_b v_j - C^i_{jcd}(m)v_i,$$

$$(11) \qquad \nabla_{[d}\nabla_{c]}v_r = -T^b_{[cd]}\nabla_b v_r - C^p_{rcd}(\mu)v_p.$$

Proof of (8). From $\nabla_c v^i = \partial_c v^i + m^i_{jc}v^j$, we have

$$\nabla_d\nabla_c v^i = \partial_d\nabla_c v^i - \Lambda^b_{cd}\nabla_b v^i + m^i_{md}\nabla_c v^m$$
$$= \underset{1}{\partial_d\partial_c v^i} + \underset{2}{\partial_d m^i_{jc}.v^j} + \underset{3}{m^i_{jc}\partial_d v^j} - \underset{4}{\Lambda^b_{cd}\nabla_b v^i} + \underset{5}{m^i_{jd}\partial_c v^j} + \underset{6}{m^i_{md}m^m_{jc}v^j}.$$

From (1)

$$\partial_{[d}\partial_{c]}v^i = \Lambda^b_\alpha\partial_{[d}\Lambda^\alpha_{c]}.\Big(\underset{1.1}{\nabla_b v^i} - \underset{1.2}{m^i_{jb}v^j}\Big).$$

The group $3+5$ is symetric in c, d and thus cancels out in the alternation. From $4+1.1$, the coefficient of

$$\nabla_b v^i \quad \text{is} \quad -\Lambda^b_{[cd]} + B^b_\alpha\partial_{[d}\Lambda^\alpha_{c]} = -T^b_{[cd]}.$$

Finally, in $2+6+1.2$ the coefficient of v^j is $C^i_{jcd}(m)$, this proves (8). The proof is similar for the other three equations.

Substituting one of the expressions (8), (11) into one of the formulae (5), (7) we obtain the corresponding inversion formula. For instance, substituting (8) into (5), written down for $v = v^j$, and noticing that

$$B^{cd}_{kl}T^b_{cd} = B^{bcd}_{ekl}T^e_{cd} + C^b_e B^{cd}_{kl}T^e_{cd} = B^b_m(D^m_{kl} + T^m_{kl}) + C^b_s(F^s_{kl} + f^s_{kl}).$$

we obtain

$$(12) \qquad \nabla_{[l}\nabla_{k]}v^i = -(T^b_{[kl]} + f^b_{[kl]})\nabla_b v^i + C^i_{jkl}(l)v^j,$$

since the last two terms of (6,2) cancel out.

If we make use of the equations $v^i = B^i_a v^a$, $v^a = B^a_j v^j$, we have

$$\nabla_b v^i = \nabla_b B^i_a . B^a_j v^j + B^i_a\nabla_b v^a,$$

where

$$B^a_j\nabla_b B^i_a = -B^i_a\nabla_b B^a_j = -B^{mi}_{ba}\nabla_m B^a_j - C^s_b B^i_a\nabla_s B^a_j$$
$$= -B^m_b D^i_{jm} - C^s_b H^i_{js}.$$

Thus we obtain

$$(13) \quad \nabla_{[l}\nabla_{k]}v^i = -(T^b_{[kl]} + f^b_{[kl]})B^i_a\nabla_b v^a + [C^i_{jkl}(l) - T^m_{[kl]}D^i_{jm} + f^s_{[kl]}H^i_{js}]v^j,$$

which, in the case of projected connexion, $D = 0$, $H = 0$, reduces to Schouten's formula (67) (Schouten 1929).

Combining (9) and (6) and noticing that

$$C^{cd}_{st} T^b_{cd} = B^b_e C^{cd}_{st} T^e_{cd} + C^{bcd}_{est} T^e_{cd} = B^b_m (K^m_{st} + k^m_{st}) + C^b_p [F^p_{st} + T^p_{st}(\sigma)],$$

we obtain

$$(14) \qquad \nabla_t \nabla_s v^p = -[T^b_{[st]}(\sigma) + k^b_{[st]}] \nabla_b v^p - C^p_{rst}(\sigma) v^r.$$

Combining (8) and (6) we get

$$(15) \qquad \nabla_t \nabla_s v^i = -[T^b_{[st]}(\sigma) + k^b_{[st]}] \nabla_b v^i - C^i_{jst}(s) v^j$$

and so on.

9. *Inversion formulae for tensors.* — As typical examples we give the formulae

$$(1) \qquad \nabla_{[d} \nabla_{c]} v^a_j = -T^b_{[cd]} \nabla_b v^a_j + C^a_{bcd}(\Lambda) v^b_j - C^i_{jcd}(m) v^a_i;$$

$$(2) \qquad \nabla_{[d} \nabla_{c]} v^a_r = -T^b_{[cd]} \nabla_b v^a_r + C^a_{bcd}(\Lambda) v^b_r - C^p_{rcd}(\mu) v^a_p.$$

Proof of (1). From

$$\nabla_c v^a_j = \partial_c v^a_j + \Lambda^a_{bc} v^b_j - m^i_{jc} v^a_i$$

we have

$$\begin{aligned} \nabla_d \nabla_c v^a_j &= \partial_d \nabla_c v^a_j + \Lambda^a_{ed} \nabla_c v^e_j - \Lambda^b_{cd} \nabla_b v^a_j - m^k_{jd} \nabla_c v^a_k \\ &= \underset{1}{\partial_d \partial_c v^a_j} + \underset{2}{\partial_d \Lambda^a_{bc} v^b_j} + \underset{3}{\Lambda^a_{bc} \partial_d v^b_j} - \underset{4}{\partial_d m^i_{jc} v^a_i} - \underset{5}{m^i_{jc} \partial_d v^a_i} \\ &\quad + \underset{6}{\Lambda^a_{ed} \partial_c v^e_j} + \underset{7}{\Lambda^a_{ed} \Lambda^e_{bc} v^b_j} - \underset{8}{\Lambda^a_{ed} m^i_{jc} v^e_i} - \underset{9}{\Lambda^b_{cd} \nabla_b v^a_j} \\ &\quad - \underset{10}{m^k_{jd} \partial_c v^a_k} - \underset{11}{m^k_{jd} \Lambda^a_{bc} v^b_k} + \underset{12}{m^k_{jd} m^i_{kc} v^a_i}. \end{aligned}$$

The groups $3 + 6$, $5 + 10$, $8 + 11$ are symetric in c, d and thus cancel out in the alternation. Moreover

$$\partial_{[d} \partial_{c]} v^a_j = \Lambda^b_\alpha \partial_{[d} \Lambda^\alpha_{c]} (\nabla_b v^a_j - \Lambda^a_{eb} v^e_j + m^i_{jb} v^a_i);$$

so that collecting terms we obtain (1).

The proof is similar for (2) and other analogous formulae.

If we combine (1) and (2) with (**8**,5) and (**8**,6) we get

$$(3)\qquad \nabla_{l}\nabla_{k}\, v^a_j = -[T^b_{|kl|}(l) + f^b_{|kl|}]\nabla_b v^a_j + B^{cd}_{kl} C^a_{bcd}(\Lambda) v^b_j - C^i_{jkl}(l) v^a_i,$$

$$(4)\qquad \nabla_{l}\nabla_{k}\, v^a_r = -[T^b_{|kl|}(l) + f^b_{|kl|}]\nabla_b v^a_r + B^{cd}_{kl} C^a_{bcd}(\Lambda) v^b_r - C^p_{rkl}(\lambda) v^a_p,$$

$$(5)\qquad \nabla_{t}\nabla_{s}\, v^a_j = -[T^b_{|st|}(\sigma) + k^b_{|st|}]\nabla_b v^a_j + C^{cd}_{st} C^a_{bcd}(\Lambda) v^b_j - C^i_{jst}(s) v^a_i,$$

$$(6)\qquad \nabla_{t}\nabla_{s}\, v^a_r = -[T^b_{|st|}(\sigma) + k^b_{|st|}]\nabla_b v^a_r + C^{cd}_{st} C^a_{bcd}(\Lambda) v^b_r - C^p_{rst}(\sigma) v^a_p;$$

Combining (1) and (2) with (**8**,7) and noticing that

$$B^c_k C^d_t T^b_{cd} = B^{bc}_{ek} C^d_t T^e_{cd} + C^b_e B^c_k C^d_t T^e_{cd} = B^b_m [H^m_{kt} + T^m_{kt}(s)] + C^b_p (J^p_{kt} + j^p_{kt}),$$
$$C^c_t B^d_k T^b_{cd} = B^b_e C^c_t B^d_k T^e_{cd} + C^{bc}_{et} B^d_k T^e_{cd} = B^b_m (G^m_{tk} + g^m_{tk}) + C^b_p [E^p_{tk} + T^p_{tk}(\lambda)],$$

we obtain

$$(7)\qquad \nabla_{t}\nabla_{k}\, v^a_j = -\frac{1}{2}[T^b_{kt}(s) + j^b_{kt} - T^b_{tk}(\lambda) - g^b_{tk}]\nabla_b v^a_j$$
$$- B^c_k C^d_t C^a_{bcd}(\Lambda) v^b_j - C^i_{jkt}(l,s) v^a_i,$$

$$(8)\qquad \nabla_{t}\nabla_{k}\, v^a_r = -\frac{1}{2}[T^b_{kt}(s) + j^b_{kt} - T^b_{tk}(\lambda) - g^b_{tk}]\nabla_b v^a_r$$
$$+ B^c_k C^d_t C^a_{bcd}(\Lambda) v^b_j - C^i_{jkt}(l,s) v^a_i.$$

10. *Resolution of the curvature tensor.* — The inversion formulae (**9**,3) — (**9**,8) lead to explicit relations between the resolutes of the curvature tensor $C^a_{bcd}(\Lambda)$ of X_N and the various curvature tensors attached to X^m_N. To obtain these relations we have to replace v^a_j by B^a_j (and i by m, say), v^a_r by C^a_r (and p by q, say) and multiply by B^i_a or C^p_a.

For example, replacing v by B and i by m in (**9**,3) and multiplying by B^i_a we obtain

$$B^{ibcd}_{ajkl} C^a_{bcd}(\Lambda) = C^i_{jkl}(l) + [T^b_{|kl|}(l) + f^b_{|kl|}] B^i_a \nabla_b B^a_j + B^i_a \nabla_{l}\nabla_{k} B^a_j.$$

The last two terms can be expressed in the fundamental tensors, by (**5**,6) and by noticing that the last term in

$$B^i_a \nabla_l \nabla_k B^a_j = D^i_{jk|l} - \nabla_l B^i_a . \nabla_k B^a_j$$

is equal to

$$\nabla_l B^i_a . (B^a_b + C^a_b) \nabla_k B^b_j = B^a_m \nabla_l B^i_a . B^m_b \nabla_k B^b_j + C^a_s \nabla_l B^i_a . C^s_b \nabla_k B^b_j$$
$$= - D^i_{ml} D^m_{jk} - G^i_{sl} F^s_{jk}.$$

Thus we obtain

$$(1)\quad B^{ibcd}_{ajkl} C^a_{bcd} = C^i_{jkl} + F^s_{j|k} G^i_{l|s} + T^m_{|kl|} D^i_{jm} + f^s_{|kl|} H^i_{js} + D^i_{j|k|l|} + D^i_{m|l} D^m_{|j|k|}$$

which, in the case of projected connexion ($D=0$, $E=0$, $H=0$, $I=0$) reduces to

$$(2) \qquad B^{ibcd}_{ajkl}\,C^a_{bcd}(A) = C^i_{jkl}(l) + F^s_{j[k}G^i_{l]s}.$$

In the holonomic case, $C^a_{bcd}(A)$ and C^i_{jkl} reduce to the usual (*but halved*) curvature tensors. Since $-F$ and $-G$ are Schouten's H and L, (2) reduces to Schouten's generalisation of the Gauss equation. Therefore, (1) and (2) are the generalised forms of the Gauss equation for non holonomic tensor submanifolds; (1) for general intrinsic connexion, (2) for the special case of projected connexion.

We give the complete list of relevant formulae for the resolution of $C^a_{bcd}(A)$:

$$(3) \qquad B^{ibcd}_{ajkl}C^a_{bcd} = C^i_{jkl}(l) + [T^b_{[kl]}(l) + f^b_{[kl]}]B^i_a\nabla_b B^a_j + B^i_a\nabla_{[l}\nabla_{k]}B^a_j.$$

$$(4) \qquad C^{pb}_{ar}B^{cd}_{kl}C^a_{bcd} = C^p_{rkl}(\lambda) + [T^b_{[kl]}(l) + f^b_{[kl]}]C^p_a\nabla_b C^a_r + C^p_a\nabla_{[l}\nabla_{k]}C^a_r.$$

$$(5) \qquad C^p_a B^{bcd}_{jkl}C^a_{bcd} = \qquad [T^b_{[kl]}(l) + f^b_{[kl]}]C^p_a\nabla_b B^a_j + C^p_a\nabla_{[l}\nabla_{k]}B^a_j.$$

$$(6) \qquad B^i_a C^b_r B^{cd}_{kl}C^a_{bcd} = \qquad [T^b_{[kl]}(l) + f^b_{[kl]}]B^i_a\nabla_b C^a_r + B^i_a\nabla_{[l}\nabla_{k]}C^a_r.$$

$$(7) \qquad B^{ib}_{aj}C^{cd}_{st}C^a_{bcd} = C^i_{jst}(s) + [T^b_{[st]}(\sigma) + h^b_{[st]}]B^i_a\nabla_b B^a_j + B^i_a\nabla_{[t}\nabla_{s]}B^a_j.$$

$$(8) \qquad C^{pbcd}_{arst}C^a_{bcd} = C^p_{rst}(\sigma) + [T^b_{[st]}(\sigma) + h^b_{[st]}]C^p_a\nabla_b C^a_r + C^p_a\nabla_{[t}\nabla_{s]}C^a_r.$$

$$(9) \qquad C^p_a B^b_j C^{cd}_{st}C^a_{bcd} = \qquad [T^b_{[st]}(\sigma) + h^b_{[st]}]C^p_a\nabla_b B^a_j + C^p_a\nabla_{[t}\nabla_{s]}B^a_j.$$

$$(10) \qquad B^i_a C^{bcd}_{rst}C^a_{bcd} = \qquad [T^b_{[st]}(\sigma) + h^b_{[st]}]B^i_a\nabla_b C^a_r + B^i_a\nabla_{[t}\nabla_{s]}C^a_j.$$

$$(11) \qquad B^{ibc}_{ajk}C^d_t C^a_{bcd} = C^i_{jkt}(l, s) + \frac{1}{2}[T^b_{kt}(s) + j^b_{kt} - T^b_{tk}(\lambda) - g^b_{tk}]B^i_a\nabla_b B^a_j$$
$$+ B^i_a\nabla_{[t}\nabla_{k]}B^a_j.$$

$$(12) \qquad C^{pb}_{ar}B^c_k C^d_t C^a_{bcd} = C^p_{rkt}(\lambda, \sigma) + \frac{1}{2}[T^b_{kt}(s) + j^b_{kt} - T^b_{tk}(\lambda) - g^b_{tk}]C^p_a\nabla_b C^a_r$$
$$+ C^p_a\nabla_{[t}\nabla_{k]}C^a_r.$$

$$(13) \qquad C^p_a B^{bc}_{jk}C^d C^a_{bcd} = \qquad \frac{1}{2}[T^b_{kt}(s) + j^b_{kt} - T^b_{tk}(\lambda) - g^b_{tk}]C^p_c\nabla_b B^a_j$$
$$+ C^p_a\nabla_{[t}\nabla_{k]}B^a_j.$$

$$(14) \qquad B^i_a C^b_r B^c_k C^d_t C^a_{bcd} = \qquad \frac{1}{2}[T^b_{kt}(s) + j^b_{kt} - T^b_{tk}(\lambda) - g^b_{tk}]B^i_a\nabla_b C^a_r$$
$$+ B^i_a\nabla_{[t}\nabla_{k]}C^a_r.$$

The terms containing derivatives of projection factors can be expressed in the fundamental tensors by (5.6) and by the formulae

(15) $B^i_a \nabla_l \nabla_k B^a_j = \nabla_l D^i_{jk} + D^m_{jk} D^i_{ml} + F^q_{jk} G^i_{ql}$,

(16) $C^p_a \nabla_l \nabla_k C^a_r = \nabla_l E^p_{rk} + E^q_{rk} E^p_{ql} + G^m_{rk} F^p_{ml}$;

(17) $C^p_a \nabla_l \nabla_k B^a_j = \nabla_l F^p_{jk} + D^m_{jk} F^p_{ml} + F^r_{jk} E^p_{rl}$;

(18) $B^i_a \nabla_l \nabla_k C^a_r = \nabla_l G^i_{rk} + E^q_{rk} G^i_{ql} + G^m_{rk} D^i_{ml}$.

(19) $B^i_a \nabla_l \nabla_s B^a_j = \nabla_l H^i_{js} + H^m_{js} H^i_{ml} + J^q_{js} K^i_{ql}$.

(20) $C^p_a \nabla_l \nabla_s C^a_r = \nabla_l I^p_{rs} + I^q_{rs} I^p_{ql} + K^m_{rs} J^p_{ml}$;

(21) $C^p_a \nabla_l \nabla_s B^a_j = \nabla_l J^p_{js} + H^m_{js} J^p_{ml} + J^q_{js} I^p_{ql}$.

(22) $B^i_a \nabla_l \nabla_s C^a_r = \nabla_l K^i_{rs} + I^q_{rs} K^i_{ql} + K^m_{rs} H^i_{ml}$.

(23) $B^i_a \nabla_l \nabla_k B^a_j = \nabla_l D^i_{jk} + D^m_{jk} H^i_{ml} + F^q_{jk} K^i_{ql}$;

(24) $C^p_a \nabla_l \nabla_k C^a_r = \nabla_l E^p_{rk} + E^q_{rk} I^p_{ql} + G^m_{rk} J^p_{ml}$;

(25) $C^p_a \nabla_l \nabla_k B^a_j = \nabla_l F^p_{jk} + D^m_{jk} J^p_{ml} + F^q_{jk} I^p_{ql}$;

(26) $B^i_a \nabla_l \nabla_k C^a_r = \nabla_l G^i_{rk} + E^q_{rk} K^i_{ql} + G^m_{rk} H^i_{ml}$.

(27) $B^i_a \nabla_k \nabla_l B^a_j = \nabla_k H^i_{jl} + H^m_{jl} D^i_{mk} + J^q_{jl} G^i_{qk}$.

(28) $C^p_a \nabla_k \nabla_l C^a_r = \nabla_k I^p_{rl} + I^q_{rl} E^p_{qk} + K^m_{rl} F^p_{mk}$;

(29) $C^p_a \nabla_k \nabla_l B^a_j = \nabla_k J^p_{jl} + H^m_{jl} F^p_{mk} + J^q_{jl} E^p_{qk}$;

(30) $B^i_a \nabla_k \nabla_l C^a_r = \nabla_k K^i_{rl} + I^q_{rl} G^i_{qk} + K^m_{rl} D^i_{mk}$.

Substituting (3)—(14) we obtain the final forms of the resolutes indicated by a bar,

(31) $\overline{C}^i_{jkl}(\Lambda) = C^i_{jkl}(l) + F^q_{j|k} G^i_{|q|l|} + T^m_{|kl|} D^i_{jm} + f^q_{|kl|} H^i_{jq} + D^i_{j|k,l|} + D^m_{j|k} D^i_{|m|l}$;

(32) $\overline{C}^p_{rkl}(\Lambda) = C^p_{rkl}(\lambda) + G^m_{r|k} F^p_{|m|l|} + T^m_{|kl|} E^p_{rm} + f^p_{|kl|} I^p_{rq} + E^p_{r|k,l|} + E^q_{r|k} E^p_{|q|l|}$;

(33) $C^p_{jkl}(\Lambda) =$ $F^q_{j|k} E^p_{|q|l|} + T^m_{|kl|} F^p_{jm} + f^q_{|kl|} J^p_{jq} + F^p_{j|k,l|} + D^m_{j|k} F^p_{|m|l|}$.

(34) $\overline{C}^i_{rkl}(\Lambda) =$ $G^m_{r|k} D^i_{|m|l|} + T^m_{|kl|} G^i_{rm} + f^q_{|kl|} K^i_{rq} + G^i_{r|k,l|} + E^q_{r|k} G^i_{|q|l|}$.

(35) $C^i_{jst}(\Lambda) = C^i_{jst}(s) + J^q_{j|s} K^i_{|q|t|} + T^q_{|st|} H^i_{jq} + k^m_{|st|} D^i_{jm} + H^i_{j|s,t|} + H^m_{j|s} H^i_{|m|t|}$.

(36) $\overline{C}^p_{rst}(\Lambda) = C^p_{rst}(\sigma) + K^m_{r|s} J^p_{|m|t|} + T^q_{|st|} I^p_{rq} + k^m_{|st|} E^p_{rm} + I^p_{r|s,t|} + I^q_{r|s} I^p_{|q|t|}$.

(37) $C^p_{jst}(\Lambda) =$ $J^q_{j|s} I^p_{|q|t|} + T^q_{|st|} J^p_{jq} + k^m_{|st|} F^p_{jm} + J^p_{j|s,t|} + H^m_{j|s} J^p_{|m|t|}$.

(38) $\overline{C}^p_{rst}(\Lambda) =$ $K^m_{r|s} H^i_{|m|t|} + T^q_{|st|} K^i_{rq} + k^m_{|st|} G^i_{rm} + K^i_{r|s,t|} + I^q_{r|s} K^i_{|q|t|}$.

(39) $$2C^i_{jkl}(\Lambda) = 2C^i_{jkl}(l, s) + F^q_{jk} K^i_{ql} - J^q_{jl} G^i_{qk} + [T^m_{kl}(s) - g^m_{lk}] D^i_{jm}$$
$$+ [f^q_{kl} - T^q_{lk}(\lambda)] H^i_{jq} + D^i_{jk,l} - H^i_{jl|k} + D^m_{jk} H^i_{ml} - H^m_{jl} D^i_{mk}.$$

(40) $$2\overline{C}^p_{rkl}(\Lambda) = 2C^p_{rkl}(\lambda, \sigma) + G^m_{rk} J^p_{ml} - K^m_{rl} F^p_{mk} + [T^m_{kl}(s) - g^m_{lk}] E^p_{rm}$$
$$+ [f^q_{kl} - T^q_{lk}(\lambda)] I^p_{rq} + E^p_{rk,l} - I^p_{rl|k} + E^q_{rk} I^p_{ql} - I^q_{rl} E^p_{qk};$$

(41) $$2\overline{C}^p_{jkl}(\Lambda) = F^q_{jk} I^p_{ql} - J^q_{jl} E^p_{qk} + [T^m_{kl}(s) - g^m_{lk}] F^p_{jm}$$
$$- [f^q_{jl} - T^q_{lk}(\lambda)] J^p_{jq} - F^p_{jk,l} - J^p_{jl|k} + D^m_{jk} J^p_{ml} - H^m_{jl} F^p_{mk};$$

(42) $$2C^i_{rkl}(\Lambda) = G^m_{rk} H^i_{ml} - K^m_{rl} D^i_{mk} + [T^m_{kl}(s) - g^m_{lk}] G^i_{rm}$$
$$+ [f^q_{kl} - T^q_{lk}(\lambda)] K^i_{rq} + G^i_{rk|l} - K^i_{rl|k} + E^q_{rk} K^i_{ql} - I^q_{rl} G^i_{qk}.$$

From $C^a_{bdc} = - C^a_{bcd}$, the remaining four resolutes are given by the negative values of (39) — (42).

Equation (31) is the extension of Schouten's form of Gauss'equation, (32) that of Hlavaty's and Schouten's form (Hlavaty 1926, Schouten and van Kampen 1930) of the Kühne-Ricci equation, (33) and (34) are the extension of Schouten's form of Codazzi's equations, to non holonomic tensor submanifolds with independent (intrinsic) fourfold connexion, in general geometrical axes. The other relations seem to be new.

In the case of projected connexion, *i. e.* when $D = 0$, $E = 0$, $H = 0$, $I = 0$, the formulae become quite simple even for nonholonomic submanifolds. This simplicity is due to the introduction of fourfold connexion with the corresponding fundamental tensors.

11. *Resolution of metric tensors.* — From $v^a = \bar{v}^i B^a_i + \acute{v}^p C^a_p$, the squared length of v^a is resolved into four parts :

$$(1) \qquad g_{ab} v^a v^b = g_{ab} B^{ab}_{ij} \bar{v}^i \bar{v}^j + g_{ab} B^a_i C^b_r \bar{v}^i \acute{v}^r + g_{ab} B^a_p B^b_j \hat{v}^p \bar{v}^j + g_{ab} C^{ab}_{pr} \hat{v}^p \hat{v}^r.$$

If v^a is in the admitted facet, i. e. if $\acute{v}^p = 0$, we put

$$(2) \qquad |v|^2 = \bar{g}_{ij} v^i v^j.$$

where

$$(3) \qquad \bar{g}_{ij} = g_{ab} B^{ab}_{ij}.$$

In this way, the length of v measured in the facet by $\bar{g}_{ij}$ will be equal to its original length in X_N.

In a like manner, if we put

$$(4) \qquad \hat{g}_{pr} = g_{ab} C^{ab}_{pr},$$

the length of a vector v^a in the span $(\bar{v}^i = 0)$, measured in the span by $\hat{g}_{pr}$ will be equal to its original length in X_N.

We also put

$$(5) \qquad g_{ip} = g_{ab} B^a_i C^b_p,$$

and notice that $g_{pi} = g_{ip}$ follows from the symmetry of g_{ab}. Thus we can write (1) in the form

$$g_{ab} v^a v^b = \bar{g}_{ij} \bar{v}^i \bar{v}^j + 2 g_{ip} \bar{v}^i \hat{v}^p + \hat{g}_{pr} \hat{v}^p \hat{v}^r. \tag{6}$$

The angle (v, w) between v^a and w^a is defined by putting

$$|v| |w| \cos(v, w) = g_{ab} v^a w^b. \tag{7}$$

Since the components v^a of $\underset{i}{e}$ and $\underset{p}{e}$ are B^a_i and C^a_p respectively, we have

$$\left|\underset{i}{e}\right| \left|\underset{p}{e}\right| \cos\left(\underset{i}{e}, \underset{p}{e}\right) = g_{ab} B^a_i C^b_p = g_{ip}. \tag{8}$$

Therefore, $g_{ip} = 0$ if (a) $\underset{i}{e}$ or $\underset{p}{e}$ has a zero length, (b) $\underset{i}{e}$ and $\underset{p}{e}$ are orthogonal. If the latter is true for every i and p, we obtain from (6) the pythagorian relation

$$|v|^2 = |\bar{v}|^2 + |\hat{v}|^2. \tag{9}$$

Similar considerations lead to the corresponding formulae for covariant vectors with arbitrary g^{ab}. We obtain, however, more significant relations if we suppose as usual that g^{ab} are determined by the equations

$$g^{ab} g_{ac} = \delta^b_c. \tag{10}$$

We also put

$$g^{ij} = B^{ij}_{ab} g^{ab}, \qquad g^{ir} = B^i_a C^j_b g^{ab}, \qquad g^{pr} = C^{pr}_{ab} g^{ab}. \tag{11}$$

All these metric parameters are absolute geometrical structure tensors. To keep, however, to general usage, we do not denote them by capitals.

We give a list of fundamental tensors derived from a gives g_{ab}.

TABLE IV.

$Q_{abc} = \nabla_c g_{ab}$,		$Q^{ab}_c = \nabla_c g^{ab}$,	
$Q_{ijk} = \nabla_k g_{ij}$,	$W_{ijs} = \nabla_s g_{ij}$,	$Q^{ij}_k = \nabla_k g^{ij}$,	$W^{ij}_s = \nabla_s g^{ij}$,
$P_{irk} = \nabla_k g_{ir}$,	$V_{irs} = \nabla_s g_{ir}$,	$P^{ir}_k = \nabla_k g^{ir}$,	$V^{ir}_s = \nabla_s g^{ir}$,
$O_{prk} = \nabla_k g_{pr}$,	$U_{prs} = \nabla_s g_{pr}$,	$O^{pr}_k = \nabla_k g^{pr}$,	$U^{pr}_s = \nabla_s g^{pr}$.

In the resolution of metric tensors we restrict ourselves to Q_{abc}, since, from (10),

$$g_{ac} Q^{ab}_{\cdot d} + g^{ab} Q_{acd} = 0. \tag{12}$$

The relations between the resolutes of Q_{abc} and the metric tensors of the submanifold are given by the following formulae, where, again, resolutes are indicated by a bar,

$$\overline{Q}_{ijk} = Q_{ijk} - 2 g_{m(j} D^{m}_{i).k} - 2 g_{q(j} F^{q}_{i).k}, \tag{13}$$

$$\overline{Q}_{irk} = P_{irk} - g_{mr} D^{m}_{ik} - g_{qr} F^{q}_{ik} - g_{qi} E^{q}_{r.k} - g_{mi} G^{m}_{rk}, \tag{14}$$

$$\overline{Q}_{prk} = O_{prk} - 2 g_{q(r} E^{q}_{p).k} - 2 g_{m(r} G^{m}_{p).k}, \tag{15}$$

$$\overline{Q}_{ijs} = W_{ijs} - 2 g_{m(j} H^{m}_{i).s} - 2 g_{q(j} J^{q}_{i).s}, \tag{16}$$

$$\overline{Q}_{irs} = V_{irs} - g_{mr} H^{m}_{is} - g_{qr} J^{q}_{is} - g_{qi} I^{q}_{rs} - g_{mi} K^{m}_{rs}, \tag{17}$$

$$\overline{Q}_{prs} = U_{prs} - 2 g_{q(r} I^{q}_{p).s} - 2 g_{m(r} K^{m}_{p).s}. \tag{18}$$

The remaining two resolutes are obtained from $Q_{bac} = Q_{abc}$.

Proof of (13). From

$$Q_{ijk} = \nabla_k g_{ij} = \nabla_k (B^{ab}_{ij}) . g_{ab} + B^{ab}_{ij} \nabla_k g_{ab}$$
$$= (B^{b}_{j} \nabla_k B^{a}_{i} + B^{a}_{i} \nabla_k B^{b}_{j}) g_{ab} + B^{abc}_{ijk} Q_{abc}.$$

(3,4) readily leads to (13). The proof is similar in the other cases.

We notice that, for an orthogonal span, P and V vanish, but not the other four tensors. In the special case of projected connexion and orthogonal space, formulae (13) — (18) become

$$\overline{Q}_{ijk} = Q_{ijk}, \tag{19}$$

$$\overline{Q}_{irk} = - g_{qr} F^{q}_{ik} - g_{mi} G^{m}_{rk}, \tag{20}$$

$$\overline{Q}_{prk} = O_{prk}, \tag{21}$$

$$\overline{Q}_{ijs} = W_{ijs}, \tag{22}$$

$$\overline{Q}_{irs} = - g_{qk} J^{q}_{is} - g_{mi} K^{m}_{rs}, \tag{23}$$

$$\overline{Q}_{prs} = U_{prs}. \tag{24}$$

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