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PAR JOSEPH LIOUVILLE

C. E. WEATHERBURN

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On the Lines of Equidistance of a Family of Surfaces;

Prof. C. E. WEATHERBURN, M. A., D. Sc.

1. *Introduction.* — This paper is concerned with some properties of the lines of equidistance of a singly infinite family of surfaces in Euclidean space. Such a family may be specified by an equation of the form $\varphi = \text{const.}$, where φ is a point-function in the space occupied by the surfaces. It is also determined if the unit vector $\mathbf{n}$ normal to a surface, is given as a point-function in the same space, satisfying the condition of normality

$$(1) \qquad \mathbf{n}.\operatorname{rot}\mathbf{n} = 0.$$

The two methods of representation are connected by the relation

$$(2) \qquad \mathbf{n} = \psi\nabla\varphi$$

where $\nabla\varphi$ is the gradient of φ, and ψ is the reciprocal of its magnitude, so that

$$(3) \qquad \psi^2 = \frac{1}{(\nabla\varphi)^2}.$$

The surfaces $\psi = \text{const.}$ are the *surfaces of equidistance* for the family $\varphi = \text{const.}$; and the curves in which the former intersect the latter are the *lines of equidistance* for the family. These constitute a congruence of curves, of which a singly infinite family lie on each surface $\varphi = \text{const.}$ The function ψ may be called the *distance function* for the original family of surfaces. For it is evident from (2) that the distance along the normal between consecutive surfaces φ and $\varphi + d\varphi$ has the value $\psi d\varphi$.

From (2) it also follows immediately, as we have shown ([1]) elsewhere, that *the direction of* rot $\mathbf{n}$ *at any point is that of the line of equidistance through the point.* For

$$\text{rot}\,\mathbf{n} = \text{rot}(\psi \nabla \varphi) = \nabla\psi \times \nabla\varphi.$$

Thus the direction of rot $\mathbf{n}$ is tangential to the surface $\psi = \text{const.}$ and also to the surface $\varphi = \text{const.}$, and the result follows.

Further, is we consider the curves which are the orthogonal trajectories of the family of surfaces $\varphi = \text{const.}$, and denote the curvature, the unit principal normal and the unit binormal by k, $\mathbf{p}$, $\mathbf{b}$ respectively, since $\mathbf{n}$ is a unit vector tangent to this curve, we have

$$-\mathbf{n} \times \text{rot}\,\mathbf{n} = \mathbf{n}.\nabla\mathbf{n} = k\mathbf{p}$$

and therefore

$$\text{rot}\,\mathbf{n} = \mathbf{n} \times (k\mathbf{p}) = k\mathbf{b}. \tag{4}$$

Thus *the magnitude of* rot $\mathbf{n}$ *is the curvature of the orthogonal trajectory of the family of surfaces.* Incidentally also we have the known property that the unit tangent to the curve of equidistance is the unit binormal to the orthogonal trajectory of the surfaces.

The reciprocal of k is the radius of curvature, ρ, of the orthogonal trajectory. In terms of this the unit tangent, $\mathbf{b}$, to a line of equidistance may be expressed

$$\mathbf{b} = \rho\,\text{rot}\,\mathbf{n}. \tag{5}$$

2. *Congruence of lines of equidistance.* — We have elsewhere ([2]) defined the *surface of striction*, or *orthocentric surface*, of a congruence of curves, as the locus of the points at which the two common normals to the curve and consecutive curves are perpendicular. It was there shown that this surface is given by the vanishing of the divergence of the unit tangent to the curves. Hence, for the congruence of lines of

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([1]) See art. 2 of a paper by the author *On Families of Surfaces*, recently communicated to the *Mathematalische Annalen*.

([2]) In a paper *On Congruence of Curves*, recently communicated to the *Tôhoku Math. Journal*.

equidistance, the surface of striction is found from the equation

$$0 = \operatorname{div}\mathbf{b} = \operatorname{div}(\rho \operatorname{rot}\mathbf{n}) = \nabla\rho . \operatorname{rot}\mathbf{n},$$

or

$$(6) \qquad \nabla k . \operatorname{rot}\mathbf{n} = 0.$$

Thus :

The surface of striction of the congruence of lines of equidistance is given by (6), *and is the locus of points at which these curves are tangent to a surface* $k = const.$

Since k is the magnitude of $\operatorname{rot}\mathbf{n}$, we may also interpret (6) by saying that the surface of striction is the locus of points at which the magnitude of $\operatorname{rot}\mathbf{n}$ is stationary for displacement along a line of equidistance.

Again, we defined the *limit surface* of a congruence ([1]) as the locus of points at which the two common normals to the curve and consecutive curves are coincident. At such points the foot of the normal is stationary for variation of the consecutive curve. In terms of the unit tangent, $\mathbf{b}$, this limit surface is given by ([2]),

$$(7) \qquad \operatorname{div}(\mathbf{b}\operatorname{div}\mathbf{b} + \mathbf{b} \times \operatorname{rot}\mathbf{b}) = 0.$$

Using the value of $\mathbf{b}$ given by (5) we find

$$\begin{aligned}\mathbf{b}\operatorname{div}\mathbf{b} + \mathbf{b} \times \operatorname{rot}\mathbf{b} &= k\nabla\rho + \rho^2 \operatorname{rot}\mathbf{n} \times \operatorname{rot}\operatorname{rot}\mathbf{n} \\ &= -\rho^2 \operatorname{rot}\mathbf{n} . (\nabla \operatorname{rot}\mathbf{n}).\end{aligned}$$

Hence *the equation of the limit surface* becomes

$$(8) \qquad \operatorname{div}[\rho^2 \operatorname{rot}\mathbf{n} . (\nabla \operatorname{rot}\mathbf{n})] = 0.$$

If $\frac{d}{ds}$ denotes differentiation in the direction of the line of equidistance, this may be expressed more concisely

$$(8') \qquad \operatorname{div}\left(\frac{1}{k}\frac{d}{ds}\operatorname{rot}\mathbf{n}\right) = 0.$$

([1]) *Loc. cit.*, art. 4.

([2]) See art. 6 of a paper *On Isometric Systems of Curves and Surfaces*, recently communicated to the *Amer Journ. of Math.*

Since the tangent to a line of striction is parallel to rot $\mathbf{n}$, it follows that

A necessary and sufficient condition that the lines of equidistance may constitute a normal congruence is that rot $\mathbf{n}$.*rot* rot $\mathbf{n}$ *vanish identically.*

If this condition is satisfied, the *surfaces orthogonal to these curves* have $\mathbf{b}$ as unit normal. Hence the *first curvature* (or mean curvature) of a surface of this family has the value (¹),

$$J = -\operatorname{div}\mathbf{b} = -\Delta\rho.\operatorname{rot}\mathbf{n} = \frac{1}{k^2}\nabla k.\operatorname{rot}\mathbf{n}$$

and the *second curvature* (or Gaussian curvature) is given by (²),

$$\begin{aligned} 2K &= \operatorname{div}(\mathbf{b}\operatorname{div}\mathbf{b} + \mathbf{b}\times\operatorname{rot}\mathbf{b}) \\ &= -\operatorname{div}[\rho^2\operatorname{rot}\mathbf{n}.(\nabla\operatorname{rot}\mathbf{n})] \\ &= -\operatorname{div}\left(\frac{1}{k}\frac{d}{ds}\operatorname{rot}\mathbf{n}\right) \end{aligned}$$

and vanishes at the limit surface of the normal congruence.

3. *Family of lines on any surface.* — The lines of equidistance on any one surface $\varphi = const.$ constitute a singly infinite family. In dealing with these it will be convenient to make use of the two-parametric differential invariants, whose theory the author has developed in a previous paper (³). We shall use a suffix 2 to indicate that the invariant is a two-parametric invariant for the surface $\varphi = const.$

First consider the *line of striction* of the family of curves of equidistance on the surface. We have elsewhere (⁴) considered some pro-

(¹) See the author's *Differential Geometry*, p. 226.

(²) *Differential Geometry*, p. 261, or art. 2 of a paper *On Families of Curves and Surfaces*, recently communicated to the *Quarterly Journal of Pure and Applied Math.*

(³) *On Differential Invariants in Geometry of Surfaces*, etc. (*Quarterly Journal of Math.*, vol. 50, 1925, p. 230-269).

(⁴) *Some new Theorems in Geometry of a Surface* (*The Mathematical Gazette*, vol. 13, Jan. 1926, p. 1-6).

perties of a line of striction, and have shown that it is given by the vanishing of the two-parametric divergence of the unit tangent to the curves. Now since for any vector, $\mathbf{V}$, perpendicular to $\mathbf{n}$,

$$\operatorname{div} \mathbf{V} = \frac{1}{\psi} \operatorname{div}_2 (\psi \mathbf{V}),$$

and $\operatorname{div} \operatorname{rot} \mathbf{n}$ vanishes identically, it follows that

$$\operatorname{div}_2 (\psi k \mathbf{b}) = 0$$

and therefore

$$b . (\psi \nabla_2 k + k \nabla_2 \psi) + \psi k \operatorname{div}_2 \mathbf{b} = 0.$$

The second term is zero, because $\mathbf{b}$ is parallel to the curve $\psi = \text{const}$. Also, since $\mathbf{b}$ is the unit tangent to the curve, $\operatorname{div}_2 \mathbf{b}$ vanishes on the line of striction, and the équation of this line may be expressed in the form

$$\mathbf{b} . \nabla_2 k = 0,$$

or, since b is tangent to the surface $\varphi = \text{const.}$,

$$(9) \qquad \mathbf{b} . \nabla k = 0.$$

Thus the line of striction is the locus of points at which k is stationary for displacement along a line of equidistance. Hence,

The line of striction of the curves of equidistance on any surface is the intersection of that surface with the surface of striction of the congruence.

The lines of equidistance will be a family of parallels provided ([1]) $\operatorname{div}_2 \mathbf{b}$ vanishes identically. It follows then, as we have pointed out elsewhere ([2]), that

A necessary and sufficient condition that the lines of equidistance on any surface may be a family of parallels, is that k be constant along each such curve.

The *geodesic torsion* (i. e. the torsion of the geodesic tangent), τ, of

([1]) *On Families of Curves and Surfaces*, art. 7.
([2]) *On Families of Surfaces*, art. 2.

a line of equidistance is given by [1]

$$\tau = \mathbf{b} \,.\, \mathrm{rot}_2 \mathbf{b} = (\rho \,\mathrm{rot}\,\mathbf{n}) \,.\, \mathrm{rot}_2 (\rho \,\mathrm{rot}\,\mathbf{n})$$
$$= \rho^2 (\mathrm{rot}\,\mathbf{n}) \,.\, \mathrm{rot}_2 \,\mathrm{rot}\,\mathbf{n}.$$

Hence *the curves of equidistance will be lines of curvature provided*

$$(\mathrm{rot}\,\mathbf{n}) \,.\, \mathrm{rot}_2 \,\mathrm{rot}\,\mathbf{n} = 0.$$

The *geodesic curvature*, k_g, of a line of equidistance is given by [2].

$$k_g = \mathbf{n} \,.\, \mathrm{rot}_2 \mathbf{b} = \mathbf{n} \,.\, \mathrm{rot}\,\mathbf{b},$$

since the normal resolute is the same for these two invariants. Thus

$$k_g = \mathbf{n} \,.\, \mathrm{rot}(\rho \,\mathrm{rot}\,\mathbf{n}) \tag{10}$$

and *the lines of equidistance will be geodesics provided*

$$\mathbf{n} \,.\, \mathrm{rot}(\rho \,\mathrm{rot}\,\mathbf{n}) = 0.$$

The *second curvature*, K, of a surface $\varphi =$ const. may be neatly expressed in terms of the two-parametric invariants of the functions ψ and ρ on this surface. For, since $\mathbf{p}$, $\mathbf{b}$ are unit tangents to orthogonal curves on the surface, we have [3]

$$K = \mathrm{div}_2 (\mathbf{p} \,\mathrm{div}\,\mathbf{p} + \mathbf{b} \,\mathrm{div}\,\mathbf{b}).$$

On substitution of the values [4]

$$\mathbf{b} = \rho \,\mathrm{rot}\,\mathbf{n}, \qquad \mathbf{p} = -\rho \nabla_2 \log\psi,$$

this reduces to

$$K = \mathrm{div}_2 (\nabla_2 \log\rho + \theta \nabla_2 \log\psi), \tag{11}$$

where θ is given by

$$\theta = \rho^2 \nabla_2^2 \log\psi = \frac{\nabla_2^2 \log\psi}{(\nabla_2 \log\psi)^2}.$$

The lines of equidistance and their orthogonal trajectories on the

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[1] *Mathematical Gazette* (*loc. cit.*, p. 4).
[2] *Ibid.*, p. 4.
[3] *Ibid.*, p. 5.
[4] *On Families of Surfaces*, art. 2.

surface $\varphi =$ const. will constitute an *isometric system* of curves provided θ is a function of ψ only (1). In this case, if we write

$$F = \int \frac{\theta}{\psi} d\psi,$$

the equation (11) becomes

$$(12) \qquad K = \text{div}_2 \, \text{grad}_2 (\log \rho + F) = \nabla_2^2 (F + \log \rho).$$

Then, also, any orthogonal system of curves cutting these at a variable angle, ω, will also be isometric provided (2)

$$\nabla_2^2 \omega = 0.$$

4. *Lamé Family of surfaces.* — The necessary and sufficient condition that a family of surfaces may form part of a triply orthogonal system, may be expressed in terms of the distance function, ψ, in different ways. We have shown in another paper that, in terms of two-parametric differential invariants, one manner of expression is by the équation (3)

$$\text{div}(\mathbf{n} \times \nabla \mathbf{n} . \nabla \psi) = 0,$$

or

$$(13) \qquad \text{div}(\mathbf{n} \times \overline{\nabla} \psi) = 0.$$

in which we have dropped the suffix 2, *the invariants being now all two-parametric*, and the function $\overline{\nabla} \psi$, defined by (4)

$$\overline{\nabla} \psi = -(\nabla \mathbf{n}) . \nabla \psi$$

may be interpreted geometrically as follows. If the two-parametric gradient, $\nabla \psi$, is resolved into components in the principal directions for the surface, and these components are multiplied by the principal

(1) *On Isometric Systems of Curves and Surfaces*, art. 1.

(2) *Ibid.*, art. 2.

(3) See Art. 4 of a paper by the author *On Lamé Families of Surfaces*, recently communicated to the *Annals of Mathematics*.

(4) In a paper by the author *On small Deformation of Surfaces*, etc. (*Quarterly Journal of Math.*, vol. 50, 1925, p. 277).

curvatures in these directions, the resultant of these is the vector $\overline{\nabla}\psi$. Now, since the two-parametric rotation of $\mathbf{n}$ vanishes identically [1], we may write (13) as

$$\mathbf{n} \cdot \operatorname{rot} \overline{\nabla}\psi = 0. \tag{14}$$

Then, since both $\nabla\psi$ and its rotation are tangential to the surface, it follows that $\overline{\nabla}\psi$ is the gradient of some scalar function [2], or else is zero; and conversely. Hence the theorem :

A necessary and sufficient condition that a family of surfaces may form part of a triply orthogonal system is that the vector $\overline{\nabla}\psi$ may be the two-parametric gradient of some scalar function.

For instance, in the case of a family of *parallel surfaces*, ψ is constant over each surface, and both $\nabla\psi$ and $\overline{\nabla}\psi$ vanish identically. Similarly for a *family of planes* the principal curvatures are zero, so that $\overline{\nabla}\psi$ again vanishes, and (14) is satisfied. In the case of a *family of spheres* the principal curvatures are equal and constant for each surface. And when the two components of $\nabla\psi$ are multiplied by the same constant, c, the resultant vector is the gradient of $c\psi$. Each of these families is therefore a Lamé family, as is well known.

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(1) *Ibid.*, p. 240.
(2) *Ibid.*, p. 258.