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Differential Properties of Functions of a Complex Variable which are Invariant under Linear Transformations;

By E.-J. WILCZYNSKI.

PART I[1].

INTRODUCTION.

If $w = f(z)$ is a function of the complex variable z, other functions may be obtained from it by subjecting z, or w, or both variables to linear transformations. Clearly there will exist properties which are left unchanged by all such linear transformations, and it is evident that such properties are likely to be of considerable interest. It is the purpose of the present paper to show that this is actually the case.

In the first eleven articles we consider linear transformations of the independent variable only. Thus a given function $w = f(z)$ gives rise to a three-parameter family of functions

$$\bar{w} = f\left(\frac{\alpha z + \beta}{\gamma z + \delta}\right) = \bar{f}(z), \tag{1}$$

where α, β, γ, δ are arbitrary constants. It is quite evident that many of the most important properties of the function $f(z)$ are left unchanged by such linear transformations. If w is uniform, so is $\bar{w}$; to every singular point of w will correspond one of $\bar{w}$; the cross-ratio of four singular points, or of four zeros of the function will be

(1) La seconde partie du présent Mémoire paraîtra dans le premier fascicule du Journal, en 1913.

preserved; etc. Moreover, we may look upon the investigation of those properties which all of the functions (1) have in common, as an extension of the theory of invariants of an algebraic binary form to the case of transcendental forms or functions.

Let us denote the independent variable by Z, and let

$$w = f(Z) = a_0 + a_1(Z - z) + a_2(Z - z)^2 + \dots \tag{2}$$

be the expansion of $f(Z)$ in the neighborhood of $Z = z$. Those functions of the coefficients $a_0, a_1, a_2, \dots$ which are left invariant by all transformations of the three parameter group

$$\overline{Z} = \frac{\alpha Z + \beta}{\gamma Z + \delta}, \tag{3}$$

are called *invariants*. Since these invariants may be regarded as functions of w, $\frac{dw}{dz}$, $\frac{d^2 w}{dz^2}$, etc., and since their values change with z, we speak of them more specifically as *differential* invariants. Several complete systems of such invariants are obtained in Art. 1. In Art. 2 we introduce the notion of *integral invariant*. The most important integral invariant is

$$\varphi = \int \sqrt{\{w, z\}}\, dz, \tag{4}$$

where

$$\{w, z\} = \frac{w'''}{w'} - \frac{3}{2}\left(\frac{w''}{w'}\right)^2, \tag{5}$$

is the Schwarzian derivative of w with respect to z. The most important differential invariant is

$$\theta = \frac{\sqrt{\{w, z\}}}{w'}. \tag{6}$$

It turns out that, from our point of view, an analytic function is essentially determined when the relation between θ and φ is given. We speak of this relation as the *intrinsic equation* of the function $w = f(z)$. The actual determination of a function whose intrinsic equation is given, requires the integration of a linear differential equation of the second order, and may be reduced to the solution of an integral equation with a skew-symetric kernel.

In Art. 4 we introduce rational osculating functions of various orders, a notion very closely related to certain investigations due to Frobenius and Padé. The relations which exist between the poles of these functions, and also between their zeros, are very simple and elegant and seem to appear here for the first time. The *osculating logarithm*, introduced in Art. 5, serves to round out this theory in an essential fashion, and gives rise to further simple geometrical results. The osculating logarithm plays an essential role on account of the fact that every logarithmic function has an intrinsic equation of the form $\theta =$ constant.

The poles and zeros of the osculating rational functions and the singularities of the osculating logarithm are functions of the position of the point z at which the osculation takes place. If z is subjected to a linear transformation, each of these points is transformed by the same linear transformation, and we therefore speak of these points as *cogredients*. Special classes of functions may be defined by means of prescribed relations between certain ones of these cogredients. Numerous illustrations of this method are worked out in detail in Arts. 9, 10 and 11. One of the most interesting results obtained in this way is a new property of certain elliptic functions. Let

$$Q(z) = \frac{A_0 + A_1 z + A_2 z^2}{B_0 + B_1 z + B_2 z^2}, \tag{7}$$

be the rational quadratic function which osculates $w = F(Z)$ at $Z = z$, and let q_1 and q_2 be its poles. Let τ be the harmonic conjugate of z, the point of contact, with respect to q_1 and q_2. Then τ is the point which we call the *quadratic satellite* of z. We ask the question; what functions are those for which τ is a fixed point? It is evident that this will be so whenever $F(Z)$ is a quadratic function with coincident poles. But there is another case when τ is a fixed point; namely, when $w = F(Z)$ may be obtained by linear transformation from a Weierstrass ζ-function for which the invariant g_2 has the value zero, so that the cross-ratio of the branch points of the corresponding Riemann surface is equi-anharmonic.

The general expressions of the cogredients enable us to write down several new formulae which are valid for any analytic function. Each of these formulae gives an expression for $w=f(z)$ in the form of an integral whose integrand depens upon one or several of the cogredients. These formulae admit iteration and are likely to be very useful in a large number of investigations. Similar expressions have also been found for the integral invariant φ, the most important one in terms of the singularities of the osculating logarithm. These formulae for φ are developed in Art. 7. But we also give, in Art. 7, an independent definition of φ which is of still greater interest, because this definition is formulated entirely in terms of notions which remain invariant under linear transformations, and by means of a limit process which is the multiplicative analogon of the process used for defining a definite integral.

In Art. 8 we introduce the integral invariant φ as a new independent variable. Thus from $w=f(z)$, we have

$$z_1=\varphi=\int\sqrt{\{w,z\}}\,dz.$$

If we introduce $z_1=\varphi$ as independent variable, w becomes a function of z_1, and we may put

$$z_2=\int\sqrt{\{w,z_1\}}\,dz_1.$$

We may now introduce z_2 as independent variable and continue in this way. The resulting relations between the Schwarzian derivatives $\{w,z\}$, $\{w,z_1\}$, ..., $\{w,z_k\}$ take the form of continued fractions of a very simple and remarkable form. A number of new problems present themselves at once; in what cases will these continued fractions terminate? when will they be periodic? if they do not terminate, and k is allowed to grow beyond bound, will they converge? We have actually solved some of the simplest of these problems; but here, as elsewhere in this memoir, the new problems are too numérous and too far-reaching to make an immediate solution of all, or even of many of them, a possible undertaking.

In Art. 12 we enlarge our group by considering independent linear transformations of both dependent and independent variables. The corresponding invariant combinations are called *hyper invariants*. The integral invariant φ retains its invariantive property even under the enlarged group. The simplest differential hyperinvariant, denoted by I, is of the fifth order. The relation between I and φ, called the *hyperintrinsic equation*, defines a function $w = f(z)$ except for linear transformations of both variables. The simplest hyperintrinsic equation, namely I = const., defines the power functions

$$\frac{aw + b}{cw + d} = \left(\frac{\alpha z + \beta}{\gamma z + \delta}\right)^r,$$

where a, b, c, d, α, β, γ, δ, and r are constants. The problem immediately presents itself to determine a power function of this form which shall have the closest possible contact with a given analytic function at a given point. This problem is solved completely in Art. 13, and leads again to interesting geometric relations between the singularities of the osculating power function and the singularities of other osculants which have been introduced before.

We have presented in this introduction only a few of the most striking features of this new theory, just enough to indicate the general point of view. And even in the body of the paper we have purposely refrained at many points from developing the theory more in detail, because it was our desire to obtain merely a first general outlook over this new territory. There remains much to be done.

1. — The differential invariants of a function of a complex variable.

Let $w = f(z)$ be a function of the complex variable z defined, in the neighborhood of the origin, by means of its Taylor expansion

$$w = f(z) = a_0 + a_1 z + a_2 z^2 + \ldots + a_k z^k + \ldots. \tag{1}$$

If we transform the independent variable by putting

$$\bar{z} = \frac{az + b}{cz + d}, \qquad ad - bc \neq 0, \tag{2}$$

where a, b, c, d are arbitrary constants, w becomes a function $\bar{f}(\bar{z})$ of $\bar{z}$ and to properties of the function $f(z)$ in the neighborhood of $z = 0$ will correspond properties of the transformed function $\bar{f}(\bar{z})$

in the neighborhood of $\bar{z} = \frac{b}{d}$. Let us assume $d \neq 0$ and put $\frac{b}{d} = K$. We may then write (2) in the form

$$\bar{z} - K = \frac{\varepsilon z}{1 + \eta z}, \qquad \varepsilon \neq 0,$$

or

$$(3) \qquad z = \frac{\alpha z_1}{1 - \beta z_1}, \qquad z_1 = \bar{z} - K.$$

If $d = 0$, the transformation (2) may also be reduced to the same form (3) provided that z_1 in that case be interpreted to mean $\frac{1}{z}$, as is customary.

As a result of the substitution (3) the series (1) will become a power series in z_1, where z_1 is eiter equal to $\bar{z} - K$ or $\frac{1}{z}$. We proceed to determine the coefficients of this new expansion. We find from (3)

$$(4) \qquad z^k = \alpha^k z_1^k (1 - \beta z_1)^{-k} = \alpha^k z_1^k \left[1 + k\beta z_1 + \frac{k(k+1)}{2!} \beta^2 z_1^2 + \ldots \right].$$

If we substitute these values for z^k in (1) and denote by $\bar{a}_s$ the coefficient of z_1^s we find

$$(5) \quad \bar{a}_0 = a_0, \qquad \bar{a}_s = \sum_{k=1}^{s} \frac{k(k+1)\ldots(s-1)}{(s-k)!} a_k \alpha^k \beta^{s-k} \qquad (s = 1, 2, 3, \ldots).$$

The following special cases of (5) are especially important

$$(6) \quad \bar{a}_1 = \alpha a_1, \quad \bar{a}_2 = \alpha(a_2 \alpha + a_1 \beta), \quad \bar{a}_3 = \alpha[a_3 \alpha^2 + 2 a_2 \alpha \beta + a_1 \beta^2].$$

By an *absolute invariant* of the function $f(z)$ under the transformations of the group (2) or (3) we mean a function of the coefficients $a_0, a_1, a_2, \ldots$, which, as a result of the relations (5), is identically equal to the corresponding function of $\bar{a}_0, \bar{a}_1, \bar{a}_2, \ldots$. Evidently a_0 is such an invariant. In order to find the others we might use the method of infinitesimal transformations. Thus (3) represents an infinitesimal transformation if we put

$$\alpha = 1 + \lambda \delta t, \qquad \beta = \mu \delta t,$$

where δt is an infinitesimal. The corresponding infinitesimal transformations of the coefficients are given by

$$(7)\qquad \delta a_0 = 0, \qquad \delta a_k = [k a_k \lambda + (k-1) a_{k-1} \mu]\,\delta t, \qquad k > 0.$$

Therefore *every absolute invariant must be a solution of the two differential equations*

$$(8)\qquad \sum k a_k \frac{\partial I}{\partial a_k} = 0, \qquad \sum (k-1) a_{k-1} \frac{\partial I}{\partial a_k} = 0.$$

The first of these two equations is easy to interpret. Let us attribute to a_k the *weight* k, and let us agree that the weight of a product shall be the sum of the weights of it factors. Then, the first equation (8) merely asserts that *every absolute invariant is isobaric of weight zero*. We may therefore find the invariants by constructing isobaric functions of a given weight p, in such a way as to also satisfy the second equation of (8), and then divide by a_1^p. Isobaric functions of weight different from zero, which also satisfy the second equation of (8), shall be called *relative invariants*.

But it is easier to obtain a complets set of invariants by a different process. Let us assume $a_1 \neq 0$. We may then according to (6) determine α and β in a unique fashion so as to make $\overline{a_1} = 1$, $\overline{a_2} = 0$, namely by putting

$$(9)\qquad \alpha = \frac{1}{a_1}, \qquad \beta = -\frac{a_2}{a_1^2}.$$

We obtain in this way the *canonical expansion* (*of the first kind*)

$$(10)\qquad w = f(z) = A_0 + z_1 + A_3 z_1^3 + A_4 z_1^4 + \ldots$$

where, on account of (5), (6), and (9),

$$(11)\quad \left\{ \begin{aligned} & A_0 = a_0, \qquad A_1 = 1, \qquad A_2 = 0, \qquad A_3 = \frac{a_1 a_3 - a_2^2}{a_1^4}, \\ & A_s = \frac{1}{a_1^s}\left[a_s + \sum_{t=1}^{s-1} \frac{(-1)^t (s-1)(s-2)\ldots(s-t)}{t!} \frac{a_2^t a_{s-t}}{a_1^t}\right] \quad (s = 3, 4, 5, \ldots). \end{aligned} \right.$$

Another canonical form (the *second* kind) is obtained by putting

$$(12)\qquad \alpha = \frac{a_1}{\sqrt{a_2^2 - a_1 a_3}}, \qquad \beta = \frac{-a_2}{\sqrt{a_2^2 - a_1 a_3}}.$$

The resulting expansion is characterized by the relations

$$(13) \qquad \overline{a}_1 + \overline{a}_3 = 0, \qquad \overline{a}_2 = 0.$$

This form is very useful.

We shall discuss the convergence of (10) later. At present we only wish to emphasize the fact that this canonical expansion (10) is *unique*. That is, to every series of the form (1), in which a_1 is not equal to zero, there corresponds a uniquely defined canonical expansion of form (10). From this it follows that *the coefficients of the canonical expansion are absolute invariants of $f(z)$*.

In fact, suppose we consider any function

$$g(\overline{z}) = b_0 + b_1\overline{z} + b_2\overline{z}^2 + \ldots, \qquad b_1 \neq 0,$$

obtainable from $f(z)$ by a transformation of form (2). By means of the transformation

$$\overline{z} = \frac{\frac{1}{b_1}\overline{z}_1}{1 + \frac{b_2}{b_1^2}\overline{z}_1},$$

$g(\overline{z})$ would be transformed into *its* canonical form

$$(14) \qquad \mathrm{B}_0 + \overline{z}_1 + \mathrm{B}_3\overline{z}_1^3 + \ldots,$$

where B_0, B_3, etc., are related to b_0, b_1, b_2, ... in the same way as are A_0, A_3, ..., to a_0, a_1, a_2, ... But the expansions (10) and (14) must be identical on account of the uniqueness of the canonical form, so that

$$\mathrm{B}_s = \mathrm{A}_s,$$

showing that A_0, A_3, A_4, ... are actually absolute invariants. It is not difficult, moreover, to verify this fact by applying the infinitesimal transformations (7) to the expression (11) for A_s. For we find $\delta\mathrm{A}_s = 0$, which again proves that A_s is an invariant.

The invariants A_0, A_3, A_4, ..., A_s, ... are independent functions of the coefficients a_0, a_1, a_2, ... — For, taken in this order, each of the invariants involves a coefficient a_s wich does not appear in any of the earlier invariants.

Every absolute invariant is a function of $A_0, A_3, A_4, \ldots$ — For, let $I(a_0, a_1, a_2, a_3, \ldots)$ be an absolute invariant. Then we shall have

$$(15) \qquad I(\bar{a}_0, \bar{a}_1, \bar{a}_2, \bar{a}_3, \ldots) = I(a_0, a_1, a_2, a_3, \ldots,$$

whenever $\bar{a}_0, \bar{a}_1, \bar{a}_2, \ldots$ are connected with $a_0, a_1, a_2, \ldots$ by means of the relations (5). In particular, (15) will be verified if we choose the values (9) for α and β, as a result of which choice, the expansion assumes its canonical form. Therefore we shall have

$$I(a_0, a_1, a_2, a_3, \ldots) = I(A_0, 1, 0, A_3, A_4, \ldots)$$

showing that I is a function of $A_0, A_3, A_4, \ldots$ as stated.

We may summarize our last three theorems by saying, that *the invariants* $A_0, A_3, A_4, \ldots$ *form a complete system of independent absolute invariants whenever* a_1 *is not equal to zero.*

We shall ordinarily maintain the hypothesis $a_1 \neq 0$. If a_1 is equal to zero we study, in place of the expansion (1), the Taylor expansion of $f(z)$ for some point $z = k$ where $k \neq 0$. Of course k may always be chosen in infinitely many ways so as to ensure that the coefficient of $z - k$ in the new expansion will not be equal to zero, provided that w is not a constant. It is possible, however, to replace the invariants $A_0, A_3, A_4, \ldots$ by a new system adapted to the case

$$a_1 = a_2 = \ldots = a_{p-1} = 0, \qquad a_p \neq 0,$$

by an obvious extension of the method which we have used for the case $a_1 \neq 0$.

Since we have

$$\bar{a}_1 = \alpha a_1,$$

a_1 is a relative invariant of weight 1. Consequently $a_1^s A_s$ is a relative invariant of weight s. According to (11), the last term of $a_1^s A_s$ (corresponding to $t = s - 1$) contains

$$\frac{a_2^{s-1} a_1}{a_1^{s-1}} = \frac{a_2^{s-2}}{a_1^{s-2}}$$

as a factor, and no other term of $a_1^s A_s$ contains a higher power of a_1 in its denominator.

Therefore $a_1^{2s-2} A_s$ *is a relative invariant of weight* $2s-2$, *which is an integral rational function of* $a_1, a_2, \ldots, a_s$. *Moreover this invariant is a linear function of* $a_3, a_4, \ldots, a_s$. *The only coefficients which appear raised to a higher power are* a_1 *and* a_2.

For this reason we shall occasionally refer to the invariants A_s as the *linear invariants* of the function $f(z)$. It is now easy to see that these linear invariants form a complete system in a more specific sense than that mentioned above. *Every absolute invariant, which is a rational function of the coefficients of* $f(z)$, *will be a rational function of the linear invariants.*

The invariant A_3 is of special importance. We have

$$a_k = \frac{1}{k!}\left(\frac{d^k w}{dz^k}\right)_{z=0},$$

so that A_3 is equal to the value wich

$$\frac{1}{6}\frac{\{w, z\}}{(w')^2}$$

assumes for $z = 0$, if we write

$$(16) \quad w' = \frac{dw}{dz}, \qquad w'' = \frac{d^2 w}{dz^2}, \qquad w''' = \frac{d^3 w}{dz^3}, \qquad \{w, z\} = \frac{w'''}{w'} - \frac{3}{2}\left(\frac{w''}{w'}\right)^2,$$

so that $\{w, z\}$ represents the Schwarzian derivative of w with respect to z.

We shall frequently think of an expansion of the function $W = f(Z)$ in the neighborhood of a point z, not at the origin, where $f(Z)$ is analytic,

$$W = f(Z) = a_0 + a_1(Z - z) + a_2(Z - z)^2 + \ldots,$$

and we shall then write

$$(17) \qquad a_k = \frac{1}{k!}\left(\frac{d^k W}{dZ^k}\right)_{Z=z} = \frac{1}{k!}\frac{d^k w}{dz^k} = \frac{1}{k!} w^{(k)}.$$

We may then think of $a_0, a_1, a_2, \ldots$, as well as $w, w', w'', \ldots$ as functions of z and we shall have

$$(18) \qquad A_3 = \frac{a_1 a_3 - a_2^2}{a_1^4} = \frac{1}{6}\frac{\{w, z\}}{(w')^2} = \frac{1}{6}\theta^2$$

where

$$\theta = \frac{\sqrt{\{w, z\}}}{w'}, \tag{19}$$

a specific determination of the square root being chosen in any particular case.

Since A_3, A_4, A_5, ... are the coefficients of z_1^3, z_1^4, ... in the canonical expansion, it is apparent that A_4, A_5, ... *may be obtained from* A_3 *by repeated differentiation with respect to* z_1.

But we may exhibit another and more convenient differentiation process which accomplishes the same purpose. If we make any linear transformation of the independent variable,

$$\bar{z} = \frac{\alpha z + \beta}{\gamma z + \delta}, \qquad \alpha\delta - \beta\gamma \neq 0, \tag{20}$$

we find

$$\frac{d\bar{z}}{dz} = \frac{\alpha\delta - \beta\gamma}{(\gamma z + \delta)^2}, \tag{21}$$

and

$$\frac{dw}{d\bar{z}} = \frac{dw}{dz} : \frac{d\bar{z}}{dz} = \frac{dw}{dz} \frac{(\gamma z + \delta)^2}{\alpha\delta - \beta\gamma}. \tag{22}$$

From our knowledge of the fact that θ^2 is an absolute invariant and from (22), we conclude

$$\{w, \bar{z}\} = \frac{\{w, z\}}{\left(\frac{d\bar{z}}{dz}\right)^2} = \{w, z\} \frac{(\gamma z + \delta)^4}{(\alpha\delta - \beta\gamma)^2}, \tag{23}$$

a familiar formula which may also be verified directly.

More generally we make the following remark. *If* I *is a relative invariant of weight p, the transformation* (20) *will transform* I *into* $\bar{\text{I}}$ *where*

$$\bar{I} = I : \left(\frac{d\bar{z}}{dz}\right)^p = I \frac{(\gamma z + \delta)^{2p}}{(\alpha\delta - \beta\gamma)^p}. \tag{24}$$

If I is an absolute invariant, we have $p = 0$, and therefore

$$\frac{d\bar{I}}{dw} = \frac{dI}{dw}.$$

Thus *we may take* w, θ, $\frac{d\theta}{dw}$, $\frac{d^2\theta}{dw^2}$, $\cdots$ *as a fundamental system of invariants.*

II. — Integral invariants.

All of the invariants obtained so far are functions of the values, w, w', w'', ..., which $W = f(Z)$ and its derivatives assume for a given value z of the complex variable Z, the function $f(Z)$ being analytic in the neighborhood of this point. We shall henceforth speak of these invariants as *differential* invariants. But equation (23) of Art. I enables us to define a new kind of invariant which we shall call an *integral invariant* of the function. Equation (23) may be written as follows

$$(25) \qquad \sqrt{\{\bar{w}, \bar{z}\}}\, d\bar{z} = \sqrt{\{w, z\}}\, dz,$$

if the square roots are properly determined. Let us select a curve C of finite length in the z plane. This curve may be open or closed but it should be so chosen that $\sqrt{\{w, z\}}$ is analytic in the neighorhood of each of its points. Les $\overline{C}$ be the curve in the $\bar{z}$ plane obtained from C by the transformation (20). Then we shall have

$$(26) \qquad \int_{\overline{C}} \sqrt{\{\bar{w}, \bar{z}\}}\, d\bar{z} = \int_{C} \sqrt{\{w, z\}}\, dz.$$

Thus, *the value of the integral*

$$(27) \qquad \varphi = \int_{C} \sqrt{\{w, z\}}\, dz$$

remains unchanged if the independent variable and the path of integration are transformed simultaneously by the same linear transformation.

We may express this by saying that φ is an integral invariant. More specifically we shall speak of φ as a *simple* integral invariant because the function under the integral sign depends only upon w, w', w'', ... The integrand does not itself involve integration.

If I is any absolute differential invariant, the integral

(28) $$\int_C I\, d\varphi = \int_C I\sqrt{\{w, z\}}\, dz,$$

will be an integral invariant. Conversely, if w is not a linear function of z, any simple integral invariant may be expressed in this form.

The truth of the direct statement is apparent. To prove the converse let

$$J = \int_C F(z, w, w', w'', \ldots)\, dz$$

be a simple integral invariant. If w is not a linear function of z, the Schwarzian $\{w, z\}$ will not be equal to zero identicaly, and we may introduce φ as a new independent variable in place of z. Then J becomes

$$J = \int_C G\left(z, w, \frac{dw}{d\varphi}, \ldots\right) d\varphi.$$

Since J is invariant for all possible curves C, we must have

$$d\bar{J} = dJ.$$

Since we have also

$$d\bar{\varphi} = d\varphi,$$

we find

$$\frac{d\bar{J}}{d\bar{\varphi}} = \frac{dJ}{d\varphi},$$

wich show that $\frac{dJ}{d\varphi}$ or $\log\left(z, w, \frac{dw}{d\varphi}, \ldots\right)$ is an absolute differential invariant.

There exists an integral invariant even simpler than φ, namely $\int dw$, and by an argument analogous to that just completed we can show that any other simple integral invariant may be expressed in the form $\int I\, dw$ where I is an absolute differential invariant. In fact this form remains valid even if w is a linear function of z, provided it is not a constant.

But the absolute differential invariant of lowest order, excepting w itself, is θ. Therefore

$$(29) \qquad \int_C \theta\, dw = \int_C \theta w'\, dz = \int_C \sqrt{\{w, z\}}\, dz \; .$$

is the simple integral invariant of lowest order, excepting only the integral $\int dw$. This latter integral however is related so simply to the values of w at the end points of the path of integration, that we prefer not to think of it as an integral at all. With this understanding we may claim that φ *is the simplest integral invariant of* $w = f(z)$ *for linear transformations of the independent variable.*

III. — The intrinsic equation.

Let w be given as an analytic function of z. We put

$$\theta = \frac{\sqrt{\{w, z\}}}{w'}, \qquad \varphi = \int_{z_0}^{z} \sqrt{\{w, z\}}\, dz = \int_{z_0}^{z} \theta w'\, dz = \int_{w_0}^{w} \theta\, dw,$$

the square root being properly specified, and the integration being performed over a path C at all of whose points $\sqrt{\{w, z\}}$ is analytic. If we regard φ as a function of its upper limit z, and eliminate z, we obtain a relation of the form

$$(30) \qquad \theta = \mathrm{F}(\varphi),$$

where F is an analytic function of φ. We shall speak of (30), which expresses the relation between the simplest differential and integral invariants of the function $w = f(z)$, as the *intrinsic equation* of $w = f(z)$, a terminology which corresponds to that used by Cesàro in his intrinsic geometry.

The importance of this notion for our theory appears in the following theorem.

All functions, which can be obtained from each other by linear transformation of the independent variable, have the same intrinsic equation. Conversely, if the intrinsic equation is given, *there will correspond to it infinitely many functions all of which may be obtained from one of them, $w = f(z)$, by adding a constant and making a linear transformation of z.*

The truth of the direct theorem is obvious, since θ and φ are absolute invariants for all linear transformations of z.

To prove the converse we observe that

$$\frac{d\varphi}{dw} = \theta = F(\varphi), \tag{31}$$

so that

$$w = \int_{\varphi_0}^{\varphi} \frac{d\varphi}{F(\varphi)} + k. \tag{32}$$

Consequently w is determined as *a* function of φ except for an additive constant.

From the definition of θ we find

$$\frac{\{w, z\}}{(w')^2} = \theta^2.$$

On the other hand we have

$$\{w, z\} = -(w')^2 \{z, w\} \quad (^1), \tag{33}$$

so that

$$\{z, w\} = -\theta^2 = -[F(\varphi)]^2. \tag{34}$$

In the right member of this equation we may replace φ by the function of w which results for it from (32), giving $\{z, w\}$ as a known function of w, say

$$\{z, w\} = G(w),$$

and $G(w)$ will be independent of k.

If z is any solution of this equation, the most general solution will be

$$\bar{z} = \frac{\alpha z + \beta}{\gamma z + \delta},$$

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(1) Cayley, *Collected Mathematical Papers*, vol. XI, p. 152.

where α, β, γ, δ, are arbitrary constants. Thus if $w = f(z)$ is one function which corresponds to the given intrinsic equation,

$$w = f\left(\frac{\alpha z + \beta}{\gamma z + \delta}\right) + k$$

will be the most general function of the same kind.

The following method of proving the same theorem is more elegant and leads to some further results. Since (32) furnishes an expression for as a w function of φ by means of a simple quadrature, we are naturally led to think of z also as a function of φ. Now Cayley first proved the formula

$$(35) \qquad \{s, x\} = \left(\frac{d\mathfrak{X}}{dx}\right)^2 [\{s, \mathfrak{X}\} - \{x, \mathfrak{X}\}] \text{ (1)},$$

which shows how to transform the independent variable of a Schwarzian. If we put $s = w$, $x = z$, $\mathfrak{X} = \varphi$, we find

$$\{w, z\} = \{w, z\}[\{w, \varphi\} - \{z, \varphi\}].$$

If w is not a linear function $\{w, z\}$ is not equal to zero, and we conclude

$$(36) \qquad \{w, \varphi\} - \{z, \varphi\} = 1.$$

We may write

$$(37) \qquad \{z, \varphi\} = \{w, \varphi\} - 1,$$

and our theorem now follows immediately from the fact that (32) gives the right member of (37) as a known function of φ which is independent of k, and that all solutions z of (37) are linear functions of any particular solution.

But we may actually indicate the analytic nature of this problem a little more closely. If $0 = F(\varphi)$ is the given intrinsic equation, we find from (32)

$$(38) \qquad \{w, \varphi\} = -\left[\frac{d^2 \log F}{d\varphi^2} + \frac{1}{2}\left(\frac{d \log F}{d\varphi}\right)^2\right] = 1,$$

(1) *Loc. cit*, p. 152.

where I is now a known function of φ. Of course the most general solution of (38) is a linear function of w.

Consider the linear homogeneous differential equation

$$(39) \qquad \frac{d^2W}{d\varphi^2} + \frac{1}{2} IW = 0,$$

and let W_1, W_2 be two of its linearly independent solutions. Then $\frac{W_2}{W_1}$ will be a solution of (38). Conversely if w is given as function of φ, as in the present case, we may find W_1 and W_2. Let $\varphi = \varphi_0$ be a value of φ for which I is analytic and let

$$(40) \qquad \begin{cases} W_1 = 1, & \frac{dW_1}{d\varphi} = 0, \qquad \text{for } \varphi = \varphi_0, \\ W_2 = 0, & \frac{dW_2}{d\varphi} = 1, \qquad \text{for } \varphi = \varphi_0; \end{cases}$$

moreover let the constant k equation (32) be equal to zero.

Then there exist in the φ-plane a circle of non-vanishing radius around φ_0, such that for all points in this circle

$$W_1 \frac{dW_2}{d\varphi} - W_2 \frac{dW_1}{d\varphi} = 1,$$

whence follows

$$\frac{dw}{d\varphi} = \frac{d}{d\varphi}\left(\frac{W_2}{W_1}\right) = \frac{1}{W_1^2},$$

if we put

$$w = \frac{W_2}{W_1},$$

and therefore

$$W_1 = \pm \frac{1}{\sqrt{\frac{dw}{d\varphi}}}, \qquad W_2 = \pm \frac{w}{\sqrt{\frac{dw}{d\varphi}}}.$$

Since we have

$$\frac{dw}{d\varphi} = \frac{1}{F(\varphi)},$$

we may write

$$(41) \qquad W_1 = \sqrt{F(\varphi)}, \qquad W_2 = \sqrt{F(\varphi)} \int_{\varphi_0}^{\varphi} \frac{d\varphi}{F(\varphi)}.$$

In order to determine z as a function of φ, we must integrate the differential equation

$$\{z, \varphi\} = I - 1.$$

Any solution of this equation may be equated to

$$(42) \qquad z = \frac{Z_2}{Z_1},$$

where Z_1, Z_2 are independent solutions of

$$(43) \qquad \frac{d^2Z}{d\varphi^2} + \frac{1}{2}(I - 1)Z = 0.$$

We may write this equation in the form

$$(44) \qquad \frac{d^2Z}{d\varphi^2} + \frac{1}{2}IZ = \frac{1}{2}Z$$

and integrate it as though the right member were a known function of φ, that is, as though (44) were a non-homogeneous linear differential equation whose right member is given. We find in this way that any solution of (44) satisfies the equation

$$Z = c_1 W_1 + c_2 W_2 + W_1 \int_{\varphi_0}^{\varphi} \frac{1}{2} Z(-W_2)\, d\psi + W_2 \int_{\varphi_0}^{\varphi} \frac{1}{2} Z W_1\, d\psi,$$

where W_1 and W_2 are given by (41), or if we put the arguments into evidence,

$$(45) \quad Z(\varphi) = c_1 W_1(\varphi) + c_2 W_2(\varphi) + \frac{1}{2}\int_{\varphi_0}^{\varphi} [W_1(\psi) W_2(\varphi) - W_1(\varphi) W_2(\psi)] Z(\psi)\, d\psi.$$

If Z_1 *and* Z_2 *denote those solutions of* (43) *which are defined by the same initial conditions* (40) *which determine* W_1 *and* W_2, Z_1 *and* Z_2 *will satisfy the two integral equations*

$$(46) \quad Z_k(\varphi) = W_k(\varphi) + \frac{1}{2}\int_{\varphi_0}^{\varphi} [W_1(\psi) W_2(\varphi) - W_1(\varphi) W_2(\psi)] Z_k(\psi)\, d\psi, \qquad (k = 1, 2)$$

and we shall have

$$Z = \frac{\alpha Z_1 + \beta Z_2}{\gamma Z_1 + \delta Z_2}$$

as the most general expression for the independent variable z which corresponds to the given intrinsic equation.

The common kernel of the two integral equations (46) may be written

$$(47) \qquad K(\varphi, \psi) = \sqrt{F(\varphi)F(\psi)} \int_{\psi}^{\varphi} \frac{d\rho}{F(\rho)},$$

so that

$$(48) \qquad K(\varphi, \psi) = -K(\psi, \varphi).$$

Therefore *the kernel is skew symmetric.* Thus, *the general problem of finding the functions which correspond to a given intrinsic equation has been reduced to a problem in linear integral equations of the Volterra type with a skew symmetric kernel.*

We shall actually determine the intrinsic equations of several important classes of functions later. For our present purposes it suffices to note a few very simple cases.

If w is a linear function of z, but not a constant, θ is identically equal to zero. Thus, *the intrinsic equation of linear functions is* $\theta = 0$.

Let us consider the function

$$w = a \log z, \qquad a \neq 0,$$

we find

$$w' = a z^{-1}, \qquad w'' = -a z^{-2}, \qquad w''' = 2 a z^{-3},$$

so that

$$\{w, z\} = \frac{1}{2} z^{-2},$$

and

$$\theta = \frac{1}{a\sqrt{2}}.$$

The same value of θ will, of course, be obtained if we replace z by a linear function of z.

Therefore, *the intrinsic equation of functions of the form*

$$(49) \qquad w = \frac{1}{k\sqrt{2}} \log \frac{\alpha z + \beta}{\gamma z + \delta},$$

where k, α, β, γ, δ *are constants, is* $\theta = k \neq 0$.

We shall henceforth speak of functions of the form (49) as *logarithmic functions*.

Finally let

$$w = e^{az} + b, \qquad a \neq 0,$$

where a and b are constants. We find

$$\{w, z\} = -\frac{1}{2}a^2, \qquad \theta = \frac{i}{\sqrt{2}} e^{-az} = \frac{i}{\sqrt{2}} \frac{1}{w-b},$$

$$\varphi = \int_{w_0}^{w} \theta \, dw = \frac{i}{\sqrt{2}} \log \frac{w-b}{w_0-b},$$

whence

$$w - b = (w_0 - b) e^{-i\sqrt{2}\varphi}$$

and therefore

$$\theta = \frac{i}{\sqrt{2}(w_0 - b)} e^{i\sqrt{2}\varphi}.$$

Therefore, *the intrinsic equation of any exponential function of the form*

$$(50) \qquad w = e^{\frac{\alpha z + \beta}{\gamma z + \delta}} + l,$$

where l, α, β, γ, δ are constants, is

$$(51) \qquad \theta = k e^{i\sqrt{2}\varphi},$$

where k is a constant, which may be equated to unity if the lower limit of the integral invariant φ be selected accordingly.

IV. — Rational osculants.

In order to be able to interpret the invariants already found, and for the purpose of obtaining still others of a more fundamental character, we now introduce a device suggested by differential geometry. In the metric theory of plane curves, for instance, certain osculating curves of a simple character are introduced, such as the tangent, the osculating circle, etc. The properties of these osculants, and their relations to each other constitute the subject matter of differential geometry. But the kind of osculants considered in any part of differential geometry depends essentially upon the group of transformations which, in that branch of geometry, is regarded as fundamental. Thus the osculating circle plays no part in projective differential geometry, since a projective transformation does not, in general, transform a circle into a circle.

Our present investigations are in the domain of the theory of functions of a complex variable, and we are interested in those properties of such functions which remain unchanged under any linear transformation of the independent variable. Now obviously *a rational function of degree n will be transformed into another rational function of the same degree by any linear transformation of the independent variable.*

Again, let us consider two analytic functions

$$w_1 = a_0 + a_1 z + a_2 z^2 + \ldots + a_k z^k + a_{k+1} z^{k+1} + \ldots,$$
$$w_2 = b_0 + b_1 z + b_2 z^2 + \ldots + b_k z^k + b_{k+1} z^{k+1} + \ldots,$$

which are so related that

$$a_0 = b_0, \quad a_1 = b_1, \quad \ldots, \quad a_k = b_k, \quad a_{k+1} \neq b_{k+1}.$$

We shall say that *the two functions have k^{th} order contact at the point $z = 0$*. The point $z = 0$ may be called the *point of contact*. Let us now subject z to any linear transformation. Formulae (5) show that we shall have

$$\bar{a}_0 = \bar{b}_0, \quad \bar{a}_1 = \bar{b}_1, \quad \ldots, \quad \bar{a}_k = \bar{b}_k, \quad \bar{a}_{k+1} \neq \bar{b}_{k+1}.$$

Consequently, the order of contact between two functions remains unchanged under linear transformations of the independent variable.

Therefore, the following problem clearly belongs to our field. *To determine the rational function of degree α which has contact of the highest possible order with a given analytic function at a given point.* We shall henceforth speak of this function as the *osculating rational function of degree α.*

This problem may be solved very easily by explicit formulae. It is included in a more general problem studied by several mathematicians, especially by Frobenius [1], and by Padé [2], namely the following : to determine the rational function, whose numerator is of degree α and whose denominator is of degree β, such that the power series expansion of this fraction at a specified point (say at $z = 0$) shall agree with a given power series

$$a_0 + a_1 z + a_2 z^2 + \ldots$$

as far as possible, that is, ordinarily to $\alpha + \beta + 1$ terms. The fractions obtained in this way may be arranged in a double entry table usually known as Padé's table. We are only interested in those fractions of Padé's table for which $\alpha = \beta$, that is, those which would be located normally in the principal diagonal. The reason for this is apparent. To specify the degrees of numerator and denominator separately is equivalent to the requirement that a certain number of zeros or poles of the rational function shall be at infinity. But in our present theory the point at infinity is in no way distinguished by special properties.

The formulae of Frobenius give us immediately the following results.

Given the expansion of a function, analytic in the neighborhood of $z = 0$,

$$w = a_0 + a_1 z + a_2 z^2 + \ldots + a_k z^k + \ldots.$$

The osculating rational function of degree α will be

$$\frac{T_\alpha}{U_\alpha} \tag{52}$$

where

$$T_\alpha = \begin{vmatrix} a_1, & a_2, & \ldots, & a_\alpha, & a_0 z^\alpha \\ a_2, & a_3, & \ldots, & a_{\alpha+1}, & a_1 z^\alpha + a_0 z^{\alpha-1} \\ \ldots, & \ldots, & \ldots, & \ldots, & \ldots\ldots\ldots \\ a_{\alpha+1}, & a_{\alpha+2}, & \ldots, & a_{2\alpha}, & a_\alpha z^\alpha + a_{\alpha-1} z^{\alpha-1} + \ldots + a_0 \end{vmatrix} \tag{53}$$

(1) *Crelle's Journal*, vol. 90, 1881.

(2) *Annales scientifiques de l'École Normale supérieure*, 3e série, 1892, vol. 9.

and

$$(54)\qquad U_\alpha = \begin{vmatrix} a_1, & a_2, & \dots, & a_\alpha, & z^\alpha \\ a_2, & a_3, & \dots, & a_{\alpha+1}, & z^{\alpha-1} \\ \dots, & \dots, & \dots, & \dots, & \dots \\ a_{\alpha+1}, & a_{\alpha+2}, & \dots, & a_{2\alpha}, & 1 \end{vmatrix}.$$

Moreover this rational function will actually be of degree α *and in its lowest terms, if and only if the determinant*

$$(55)\qquad c_\alpha = \begin{vmatrix} a_1, & a_2, & \dots, & a_\alpha \\ a_2, & a_3, & \dots, & a_{\alpha+1} \\ \dots, & \dots, & \dots, & \dots \\ a_\alpha, & a_{\alpha+1}, & \dots, & a_{2\alpha-1} \end{vmatrix}$$

is different from zero.

The following relation between two successive rational osculants should also be noted

$$(56)\qquad \frac{T_{\alpha+1}}{U_{\alpha+1}} - \frac{T_\alpha}{U_\alpha} = \frac{c_{\alpha+1}^2 z^{2\alpha+1}}{U_\alpha U_{\alpha+1}} \quad (^1).$$

One very obvious remark may be made at once. *The quantities* c_α, *defined by* (55), *are relative invariants of the function* $w = f(z)$ *under linear transformations of the independent variable.*

To prove this, let us suppose

$$c_1 \neq 0, \qquad c_2 \neq 0, \qquad c_{\alpha-1} \neq 0, \qquad c_\alpha = 0.$$

Then each of the osculants

$$\frac{T_1}{U_1}, \quad \frac{T_2}{U_2}, \quad \dots, \quad \frac{T_{\alpha-1}}{U_{\alpha-1}},$$

is actually of the degree indicated and is not reductible to a lower degree, but according to (56) we shall have

$$\frac{T_\alpha}{U_\alpha} = \frac{T_{\alpha-1}}{U_{\alpha-1}}.$$

In other words, the rational osculating function of degree α, which has $2\alpha^{\text{th}}$ order contact with w, is not in its lowest terms but reduces

(1) Frobenius, *loc. cit.*, p. 6, équation (16).

to $\frac{T_{\alpha-1}}{U_{\alpha-1}}$. That is, the osculating rational function of degree $\alpha-1$ has not only contact of order $2(\alpha-1)$ with $w=f(z)$ but contact of order at least 2α. We shall say that it *hyperosculates* the function $w=f(z)$. But hyperosculation is obviously a property invariant under linear transformation. Therefore the condition $c_\alpha=0$ is an invariant equation, and c_α must be an invariant, as stated above.

Since the sum of the indices in any term of the expanded determinant is α^2, *the weight of c_α is equal to α^2, and the effect of the infinitesimal transformation* (7) *upon c_α is given by*

$$(57) \qquad \delta c_\alpha = \alpha^2 \lambda c_\alpha \delta t.$$

The direct proof of (57) is somewhat complicated and may be omitted. The invariants

$$c_1 = a_1, \qquad c_2 = a_1 a_3 - a_2^2,$$

have already made their appearance in Art. 1. Moreover *the conditions*

$$(58) \qquad c_{\alpha-1} \neq 0, \qquad c_\alpha = 0, \qquad c_{\alpha+1} = 0, \qquad c_{\alpha+2} = 0, \qquad \ldots$$

obviously imply that $w=f(z)$ is a rational function of degree $\alpha-1$.

This also follows from the fact that the coefficients of the power series will, in this case, possess a scale of relation.

Of course the invariants $c_1, c_2, c_3, \ldots$ do not form a complete system. For, as we advance from c_α to $c_{\alpha+1}$, two new coefficients, $a_{2\alpha}$ and $a_{2\alpha+1}$, are introduced at the same time. It is easy however to form a complete system from these invariants by adding to them the invariants

$$(59) \qquad d_\alpha = a_1 c'_\alpha - 2\alpha^2 a_2 c_\alpha.$$

That these quantities are invariants is easily seen as follows. Since c_α is a relative invariant of weight α^2, $\frac{c_\alpha}{a_1^{\alpha^2}}$ is an absolute invariant. Therefore (See Art. 1) its derivative is a relative invariant. But this derivative is

$$\frac{a_1^{\alpha^2} c'_\alpha - \alpha^2 a_1^{\alpha^2-1} c_\alpha a'_1}{a_1^{2\alpha^2}} = \frac{a_1 c'_\alpha - \alpha^2 a'_1 c_\alpha}{a_1^{\alpha^2+1}}.$$

Since a_1 is a relative invariant, the numerator of this fraction is also a relative invariant; but this numerator reduces to d_α since

$$a'_1 = w'' = 2\left(\frac{1}{2} w''\right) = 2a_2.$$

Of course the weight of d_α is $\alpha^2 + 2$.

We can find a simple general expression for c'_α in terms of $a_1, a_2, \ldots, a_{2\alpha}$. To do this we observe in the first place that

$$a'_k = (k+1)a_k \qquad (k = 0, 1, 2, \ldots), \tag{60}$$

since

$$a'_k = \frac{1}{k!} w^{(k+1)} = (k+1)\frac{1}{(k+1)!} w^{(k+1)}.$$

Now let us write c_α as follows

$$c_\alpha = \begin{vmatrix} a_1^{(1)}, & a_2^{(2)}, & \ldots, & a_{k-1}^{(k-1)}, & a_k^{(k)}, & a_{k+1}^{(k+1)}, & \ldots, & a_\alpha^{(\alpha)} \\ a_2^{(1)}, & a_3^{(2)}, & \ldots, & a_k^{(k-1)}, & a_{k+1}^{(k)}, & a_{k+2}^{(k+1)}, & \ldots, & a_{\alpha+1}^{(\alpha)} \\ \ldots, & \ldots, & \ldots, & \ldots, & \ldots, & \ldots, & \ldots, & \ldots \\ a_\alpha^{(1)}, & a_{\alpha+1}^{(2)}, & \ldots, & a_{\alpha+k-2}^{(k-1)}, & a_{\alpha+k-1}^{(k)}, & a_{\alpha+k}^{(k+1)}, & \ldots, & a_{2\alpha-1}^{(\alpha)} \end{vmatrix}$$

where the upper indices, which indicate the column to which each element belongs, have been introduced merely for the sake of fixing a notation for the cofactors. We actually have

$$a_k^{(\lambda)} = a_k^{(\mu)} = a_k,$$

and we shall denote by $A_k^{(\lambda)}$ the cofactor of $a_k^{(\lambda)}$, so that

$$c_\alpha = \sum_{i=1}^{\alpha} a_i^{(1)} A_i^{(1)} = \sum_{i=1}^{\alpha} a_{i+1}^{(2)} A_{i+1}^{(2)} = \ldots = \sum_{i=1}^{\alpha} a_{\alpha+i}^{(\alpha)} A_{\alpha+i}^{(\alpha)}.$$

We find

$$c'_\alpha = \sum_{k=1}^{\alpha-1} \begin{vmatrix} a_1^{(1)}, & a_2^{(2)}, & \ldots, & a_{k-1}^{(k-1)}, & a_{k+1}, & a_{k+1}^{(k+1)}, & \ldots, & a_\alpha^{(\alpha)} \\ \ldots, & \ldots, & \ldots, & \ldots, & \ldots, & \ldots, & \ldots, & \ldots \\ a_\alpha^{(1)}, & a_{\alpha+1}^{(2)}, & \ldots, & a_{\alpha+k-2}^{(k-1)}, & \alpha a_{k+\alpha}, & a_{k+\alpha}^{(k+1)}, & \ldots, & a_{2\alpha-1}^{(\alpha)} \end{vmatrix}$$

$$+ \begin{vmatrix} a_1, & a_2, & \ldots, & a_{\alpha-1}, & (\alpha+1)a_{\alpha+1} \\ \ldots, & \ldots, & \ldots, & \ldots, & \ldots \\ a_\alpha, & a_{\alpha+1}, & \ldots, & a_{2\alpha-2}, & 2\alpha a_{2\alpha} \end{vmatrix},$$

or

$$c'_\alpha = \sum_{k=1}^{\alpha-1} \left[a_{k+1} A_k^{(k)} + 2 a_{k+2} A_{k+1}^{(k)} + \ldots + \alpha a_{k+\alpha} A_{k+\alpha-1}^{(k)} \right]$$
$$+ (\alpha+1) a_{\alpha+1} A_\alpha^{(\alpha)} + (\alpha+2) a_{\alpha+2} A_{\alpha+1}^{(\alpha)} + \ldots + 2\alpha a_{2\alpha} A_{2\alpha-1}^{(\alpha)}.$$

Thus we have found

$$\begin{aligned}
c'_\alpha = {} & a_2 A_1^{(1)} + 2 a_3 A_2^{(1)} + \ldots + (\alpha-1) a_\alpha A_{\alpha-1}^{(1)} + \alpha a_{\alpha+1} A_\alpha^{(1)} \\
& + a_3 A_2^{(2)} + 2 a_4 A_3^{(2)} + \ldots + (\alpha-1) a_{\alpha+1} A_\alpha^{(2)} + \alpha a_{\alpha+2} A_{\alpha+1}^{(2)} \\
& \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\
& + a_\alpha A_{\alpha-1}^{(\alpha-1)} + 2 a_{\alpha+1} A_\alpha^{(\alpha-1)} + \ldots + (\alpha-1) a_{2\alpha-2} A_{2\alpha-3}^{(\alpha-1)} + \alpha a_{2\alpha-1} A_{2\alpha-2}^{(\alpha-1)} \\
& + a_{\alpha+1} A_\alpha^{(\alpha)} + 2 a_{\alpha+2} A_{\alpha+1}^{(\alpha)} + \ldots + (\alpha-1) a_{2\alpha-1} A_{2\alpha-2}^{(\alpha)} + \alpha a_{2\alpha} A_{2\alpha-1}^{(\alpha)} \\
& + \alpha \left[a_{\alpha+1} A_\alpha^{(\alpha)} + a_{\alpha+2} A_{\alpha+1}^{(\alpha)} + \ldots\ldots\ldots\ldots + a_{2\alpha} A_{2\alpha-1}^{(\alpha)} \right].
\end{aligned}$$

The sum of the first α terms, in the first column

$$a_2 A_1^{(1)} + \ldots + a_{\alpha+1} A_\alpha^{(\alpha)} = 0.$$

The same thing is true of the sum of the first α terms in all of the other columns, except the last. Consequently

$$\begin{aligned}
c'_\alpha = {} & \alpha \left[a_{\alpha+1} A_\alpha^{(1)} + a_{\alpha+2} A_{\alpha+1}^{(2)} + \ldots + a_{2\alpha} A_{2\alpha-1}^{(\alpha)} \right] \\
& + \alpha \left[a_{\alpha+1} A_\alpha^{(\alpha)} + a_{\alpha+2} A_{\alpha+1}^{(\alpha)} + \ldots + a_{2\alpha} A_{2\alpha-1}^{(\alpha)} \right],
\end{aligned}$$

so that we find finally

$$c'_\alpha = 2\alpha \begin{vmatrix} a_1, & a_2, & \ldots, & a_\alpha \\ a_2, & a_3, & \ldots, & a_{\alpha+1} \\ \ldots, & \ldots, & \ldots, & \ldots \\ a_{\alpha-1}, & a_\alpha, & \ldots, & a_{2\alpha-2} \\ a_{\alpha+1}, & a_{\alpha+2}, & \ldots, & a_{2\alpha} \end{vmatrix}, \tag{61}$$

where the determinant differs from that for c_α only in the last row. *The derivative of c_α is obtained by increasing the index of each element in the last row of c_α by a single unit, and then multiplying the whole determinant by 2α.*

This formula assumes an interesting form if we remember that

$$a_k = \frac{1}{k!} \frac{d^k w}{dz^k}.$$

As a result of formula (61) we may regard the invariants d_α as being known in terms of the coefficients $a_1, a_2, \ldots, a_{2\alpha}$, in the form of sums of two determinants.

The direct proof of (57), which has been omitted, may be based upon a calculation very similar to the one just completed.

We proceed now to study the distribution of the poles and zeros of these rational osculants. The osculating linear function is $\frac{T_1}{U_1}$, where

$$T_1 = \begin{vmatrix} a_1, & a_0 z \\ a_2, & a_1 z + a_0 \end{vmatrix}, \qquad U_1 = \begin{vmatrix} a_1, & z \\ a_2, & 1 \end{vmatrix}.$$

Therefore we find the formulae

$$(62) \qquad p = \frac{a_1}{a_2}, \qquad e = \frac{a_0 a_1}{a_0 a_2 - a_1^2},$$

where p is the pole, and e the zero of the osculating linear function.

The expression for the osculating quadratic function depends upon a_0, a_1, a_2, a_3 and a_4. If, in this expression, we replace a_4 by an arbitrary parameter λ, we obtain *a* one-parameter family of quadratic functions each of which will have third order contact with the function $w = f(z)$ at $z = 0$. We shall speak of these quadratic functions as *penosculating quadratics*. The osculating quadratic is that penosculating quadratic for which λ has the value a_4.

The denominator of the general penosculating quadratic is

$$(63) \qquad \begin{vmatrix} a_1 & a_2 & z^2 \\ a_2 & a_3 & z \\ a_3 & \lambda & 1 \end{vmatrix} = (\lambda a_2 - a_3^2) z^2 + (a_2 a_3 - \lambda a_1) z + a_1 a_3 - a_2^2.$$

Since its coefficients are linear functions of λ, there will exist in general two values of λ, call them λ_1 and λ_2, for which the linear factors of (63) will become identical. These values of λ are given by

$$(64) \qquad \left.\begin{matrix}\lambda_1 \\ \lambda_2\end{matrix}\right\} = \frac{1}{a_1^2}\left[-a_2(2a_2^2 - 3a_1 a_3) \pm 2(a_2^2 - a_1 a_3)^{\frac{3}{2}}\right].$$

Those penosculating quadratics which correspond to $\lambda = \lambda_1$, and to $\lambda = \lambda_2$ will be the only ones whose two poles coincide. We shall call them the *singular penosculating quadratics*. Let p_1 and p_2 be the poles of the singular penosculating quadratics. Then p_1 and p_2 will be obtained from the factors of (63) by equating λ to λ_1 and λ_2 respectively. Thus we find that

$$(65) \qquad p_1 = \frac{a_2 + \sqrt{a_2^2 - a_1 a_3}}{a_3}, \qquad p_2 = \frac{a_2 - \sqrt{a_2^2 - a_1 a_3}}{a_3},$$

are the poles of the two singular penosculating quadratics.

Let us assume $a_1 \neq 0$, and that the series under consideration has ben reduced to its first canonical form (See Art. 1). Then we shall have

$$w = A_0 + A_1 z + A_2 z^2 + A_3 z^3 + \ldots,$$

where

$$A_0 = a_0, \qquad A_1 = 1, \qquad A_2 = 0, \qquad A_3 = \frac{a_1 a_3 - a_2^2}{a_1^4}, \qquad \ldots.$$

Thus we find

$$p = \infty, \qquad p_1 + p_2 = 0,$$

showing that the four points 0, p, p_1, p_2 form a harmonic set, that is, the cross-ratio

$$(66) \qquad (0, p, p_1, p_2) = -1.$$

Now the cross-ratio of four points in the z plane is left unchanged by all linear transformations of z; to a function of z and the linear function which osculates it at $z = 0$ will correspond the transformed function and its linear osculating function. Consequently we obtain the following theorem.

Let z be any point of the plane in the neighborhood of which the function $f(z)$ is analytic and where $f'(z)$ is not equal to zero. Consider the osculating linear function and the two singular penosculating quadratics of $f(z)$ which belong to the point z, which shall henceforth be called the point of contact. Then, the point of contact and the pole of the osculating linear function are separated harmonically by the poles of the two singular penosculating quadratics.

Of course this gives us the corollary that *the four points mentioned* *are on one and the same circle*, it being understood, as is customary in this theory, that a straight line is to be regarded as a special circle, namely a circle through infinity.

In proving this theorem we made use of the canonical form of the expansion. Of course the theorem may be proved without this device, but not without some calculation. It suffices for this purpose to note that

$$\frac{2}{p} = \frac{1}{p_1} + \frac{1}{p_2}, \tag{67}$$

an equation which is equivalent to (66).

Let us now consider *any* penosculating quadratic, and let us denote its poles by $p_1^{(\lambda)}$ and $p_2^{(\lambda)}$. These poles will be distinct, except when λ is equal to either λ_1 or λ_2, and they will always be the roots of the quadratic equation,

$$(\lambda a_2 - a_3^2)z^2 + (a_2 a_3 - \lambda a_1)z + a_1 a_3 - a_2^2 = 0, \tag{68}$$

which is obtained by equating (63) to zero. According to (65) p_1 and p_2 are the roots of the quadratic

$$a_3 z^2 - 2a_2 z + a_1 = 0. \tag{69}$$

But the harmonic invariant of these two quadratics, namely,

$$(\lambda a_2 - a_3^2)a_1 + (a_1 a_3 - a_2^2)a_3 + (a_2 a_3 - \lambda a_1)a_2$$

is equal to zero for all values of λ. Consequently $p_1^{(\lambda)}$ and $p_2^{(\lambda)}$ are harmonic conjugates with respect to p_1 and p_2. This gives rise to the following theorem.

The poles of every penosculating quadratic, and therefore also the poles of the osculating quadratic, are harmonic conjugates of each other with respect to the poles of the two singular penosculating quadratics.

The same result may also be expressed by saying that the poles of the penosculating quadratics are pairs of an involution.

Equation (68) will furnish, as its roots, the poles of the osculating quadratic if we put $\lambda = a_4$. Therefore we find

$$(70) \qquad q_1 + q_2 = \frac{a_1 a_4 - a_2 a_3}{a_2 a_4 - a_3^2}, \qquad q_1 q_2 = \frac{a_1 a_3 - a_2^2}{a_2 a_4 - a_3^2},$$

where q_1 and q_2 denote the poles of the osculating quadratic.

We now pass to the consideration of penosculating and osculating cubics. The osculating cubic is $\frac{T_3}{U_3}$, and the penosculating cubics are obtained from $\frac{T_3}{U_3}$ by writing an arbitrary parameter λ in place of a_6. Thus the poles of the penosculating cubics are the roots of

$$(71) \qquad \begin{vmatrix} a_1 & a_2 & a_3 & z^3 \\ a_2 & a_3 & a_4 & z^2 \\ a_3 & a_4 & a_6 & z \\ a_4 & a_5 & \lambda & 1 \end{vmatrix} = -Az^3 + Bz^2 - Cz + D = 0,$$

where

$$(72) \qquad \begin{cases} A = \begin{vmatrix} a_2 & a_3 & a_4 \\ a_3 & a_4 & a_5 \\ a_4 & a_5 & \lambda \end{vmatrix}, \qquad B = \begin{vmatrix} a_1 & a_2 & a_3 \\ a_3 & a_4 & a_5 \\ a_4 & a_5 & \lambda \end{vmatrix}, \\ C = \begin{vmatrix} a_1 & a_2 & a_3 \\ a_2 & a_3 & a_4 \\ a_4 & a_5 & \lambda \end{vmatrix}, \qquad D = \begin{vmatrix} a_1 & a_2 & a_3 \\ a_2 & a_3 & a_4 \\ a_3 & a_4 & a_5 \end{vmatrix}. \end{cases}$$

Since the coefficients of (71) are linear functions of λ, the *poles of every penosculating cubic constitute a triple of a cubic involution.*

The product zU_2, where U_2 is the denominator of the osculating quadratic, will be the denominator of one penosculating cubic, namely that one which corresponds to $\lambda = \infty$. *There also exists a penosculating cubic, one of whose poles coincides with the pole of the osculating linear function.*

In order to determine the essential properties of this latter cubic, let us assume that the given expansion for $w = f(z)$ is in the first canonical form, so that

$$(73) \qquad a_1 = 1, \qquad a_2 = 0, \qquad a_3 = A_3, \qquad \ldots$$

we then find

$$(74) \qquad p = \infty, \qquad p_1 + p_2 = 0, \qquad q_1 + q_2 = -\frac{A_4}{A_3^2}, \qquad q_1 q_2 = -\frac{1}{A_3}.$$

In order that $p = \infty$ may be a root of (71), we must have $A = 0$, and this condition furnishes the following equation for λ:

$$\lambda = \frac{A_4}{A_3^2}(2A_3A_5 - A_4^2). \tag{75}$$

Let r_1 and r_2 be the remaining two poles of this special penosculating cubic. Then r_1 and r_2 are the roots of the quadratic

$$B_0 z^2 - C_0 z + D = 0,$$

obtained from (71) by substituting for λ the value given by (75). Consequently we have

$$\frac{1}{r_1} + \frac{1}{r_2} = \frac{C_0}{D}, \qquad \frac{1}{r_1 r_2} = \frac{B_0}{D}. \tag{76}$$

But we find

$$B_0 = -\frac{D}{A_3^3}(A_3A_5 - A_4^2), \qquad C_0 = \frac{A_4 D}{A_3},$$

so that

$$\frac{1}{r_1} + \frac{1}{r_2} = \frac{A_4}{A_3}, \qquad \frac{1}{r_1 r_2} = \frac{A_4^2 - A_3A_5}{A_3^2}. \tag{77}$$

From (74) we find

$$\tau = \frac{2q_1q_2}{q_1 + q_2} = \frac{2A_3}{A_4}, \tag{78}$$

and from (77) we find similarly

$$\frac{2r_1r_2}{r_1 + r_2} = \frac{2A_3}{A_4}, \tag{79}$$

so that

$$\frac{2}{\tau} = \frac{1}{q_1} + \frac{1}{q_2} = \frac{1}{r_1} + \frac{1}{r_2}. \tag{80}$$

According to this equation, τ is the harmonic conjugate of the origin, both with respect to q_1, q_2 and with respect to r_1, r_2. The point τ will appear so frequently in our theorems as to make it desirable to give it a name. We therefore formulate the following definition. *The point τ, which is the harmonic conjugate of the point of contact with respect to the poles of the osculating quadratic, shall be called the quadratic satellite of the point of contact.*

We may now express the contents of equations (80) as follows. *Consider the special penosculating cubic one of whose poles coincide with the pole of the osculating linear function. Its other two poles will be separated harmonically by the point of contact and the quadratic satellite.*

Still another formulation of this theorem is the following. *The involution determined by the two point-pairs* (q_1, q_2) *and* (r_1, r_2) *has as its double points the point of contact and its quadratic satellite.*

We have characterized geometrically, the denominators of two of the penosculating cubics, namely

$$z(z-q_1)(z-q_2) \quad \text{and} \quad (z-p)(z-r_1)(z-r_2),$$

where the latter may be replaced by $(z-r_1)(z-r_2)$ if the expansion of $f(z)$ is assumed to be in its canonical form. The poles of any penosculating cubic will then be the roots of the equation

$$(81) \qquad z(z-q_1)(z-q_2)+\mu(z-r_1)(z-r_2)=0,$$

which is merely another form for the cubic involution (71).

Let s_1, s_2, s_3 be the roots of this cubic for a given value of μ, so that s_1, s_2, s_3 form a triple of the cubic involution. We find

$$(82) \quad \Sigma s_i = q_1+q_2-\mu, \qquad \Sigma s_i s_j = q_1 q_2 - \mu(r_1+r_2), \qquad s_1 s_2 s_3 = -\mu r_1 r_2,$$

and therefore

$$(83) \qquad \begin{cases} \Sigma s_i s_j - (r_1+r_2)\Sigma s_i = q_1 q_2 - (q_1+q_2)(r_1+r_2), \\ s_1 s_2 s_3 - r_1 r_2 \Sigma s_i = -(q_1+q_2) r_1 r_2. \end{cases}$$

The equations (83), being independent of μ, represent those properties of the triples of our involution, which are common to all of them.

Let us determine those triples of the involution which have two coincident points. They correspond to those penosculating cubics which have a multiple pole, and which we shall call the *singular* penosculating cubics. For such a triple we may put

$$s_1 = s_2 = d, \qquad s_3 = e,$$
$$\Sigma s_i = 2d+e, \qquad \Sigma s_i s_j = d^2 + 2de, \qquad s_1 s_2 s_3 = d^2 e.$$

If we substitute these values in (83), we find

$$(84)\qquad \begin{cases} d^2+2de-(r_1+r_2)(2d+e)=q_1q_2-(q_1+q_2)(r_1+r_2), \\ d^2e-r_1r_2(2d+e)=-(q_1+q_2)r_1r_2, \end{cases}$$

whence

$$(85)\qquad \begin{cases} e=\dfrac{r_1r_2[2d-(q_1+q_2)]}{d^2-r_1r_2}, \\ d=\dfrac{(r_1+r_2)e^2+[q_1q_2-r_1r_2-(q_1+q_2)(r_1+r_2)]e+(q_1+q_2)r_1r_2}{2(e-r_1)(e-r_2)}. \end{cases}$$

If the value of e from (85) is substituted in the first equation of (84), we obtain *the biquadratic equation*

$$(86)\qquad d^4-2(r_1+r_2)d^3+[3r_1r_2-q_1q_2+(q_1+q_2)(r_1+r_2)]d^2 -2r_1r_2(q_1+q_2)d+r_1r_2q_1q_2=0,$$

for the double points d_k *of the cubic involution.* To each of these four double points there corresponds, by means of (85) a point e_k at its *companion*, and we obtain in general four singular penosculating cubics each of which has one of the points d_k as a double pole and whose remaining pole, in general distinct from d_k, will be e_k.

Of course equation (86) might also have been obtained by equating to zero the Jacobian of the pencil of binary forms (71).

We may write (86) as follows

$$(87)\qquad \frac{1}{d^4}-\frac{4}{\tau}\frac{1}{d^3}+\left(\frac{3}{q_1q_2}-\frac{1}{r_1r_2}+\frac{4}{\tau^2}\right)\frac{1}{d^2}-\frac{4}{q_1q_2\tau}\frac{1}{d}+\frac{1}{q_1q_2r_1r_2}=0.$$

Consequently we find

$$(88)\qquad \begin{cases} \displaystyle\sum_{i=1}^{4}\frac{1}{d_i}=\frac{4}{\tau}, \qquad \sum_{i,j=1}^{4}\frac{1}{d_id_j}=\frac{4}{\tau^2}+\frac{3}{q_1q_2}-\frac{1}{r_1r_2}, \\ \displaystyle\sum_{i,j,k=1}^{4}\frac{1}{d_id_jd_k}=\frac{4}{q_1q_2\tau}, \qquad d_1d_2d_3d_4=q_1q_2r_1r_2. \end{cases}$$

The first of these equations is capable of a simple interpretation. Let us put

$$(89)\qquad \frac{1}{d_i}+\frac{1}{d_j}=\frac{2}{d_{ij}}, \qquad \frac{1}{d_k}+\frac{1}{d_l}=\frac{2}{d_{kl}},$$

where i, j, k, l represent the indices 1, 2, 3, 4, arranged in any order. Then we shall have, according to (88),

$$\frac{1}{d_{ij}} + \frac{1}{d_{kl}} = \frac{2}{\tau}. \tag{90}$$

Consequently *the four double poles of the four singular penosculating cubics have the following property. Select any two of them, d_i and d_j, and determine the harmonic conjugate, d_{ij}, of the point of contact with respect to them. Proceed in the same way with the other two double poles, d_k and d_l, giving rise to the point d_{kl}. The points d_{ij} and d_{kl} will be harmonic conjugates of each other with respect to the point of contact and its quadratic satellite.*

The four points d_i may be arranged in two pairs in six different ways, thus giving rise to six pairs (d_{ij}, d_{kl}). According to the theorem just proved, each of these pairs determines, with the point of contact, a circle passing through the point τ. We obtain in this way six circles of the pencil determined by the point of contact and its quadratic satellite.

A similar relation may be obtained from (86). Let us put

$$f_{ij} = \frac{1}{2}(d_i + d_j), \qquad f_{kl} = \frac{1}{2}(d_k + d_l), \qquad r = \frac{1}{2}(r_1 + r_2). \tag{91}$$

Then we find from (86),

$$\frac{1}{2}(f_{ij} + f_{kl}) = r. \tag{92}$$

But in deducing equation (86) we used the canonical expansion, so that the pole of the osculating linear function p was at infinity. Consequently equations (91) and (92) indicate that (f_{ij}, p) and (d_i, d_j) are harmonic pairs, that (f_{kl}, p) and (d_k, d_l) are harmonic, etc. We obtain therefore the following second property of the points d_i.

Select any two, d_i and d_j, of the four points d_1, d_2, d_3, d_4, and determine the harmonic conjugate, f_{ij}, of the pole p of the osculating linear function with respect to them. Proceed in the same way with the remaining two points, d_k and d_l, giving rise to the point f_{kl}. The resulting two points, f_{ij} and f_{kl}, will be harmonic *conjugates of each other with respect to the pole of the osculating linear function p, and the point r, which is itself the harmonic conjugate of p with respect to r_1 and r_2.*

We obtain in this way six circles of the pencil determined by p and r.

Let us put further

$$(93)\qquad q=\frac{1}{2}(q_1+q_2),\qquad d_{ijkl}=\frac{1}{2}(d_{ij}+d_{kl}),\qquad f_{ijkl}=\frac{2f_{ij}f_{kl}}{f_{ij}+f_{kl}},$$

so that q is the harmonic conjugate of p with respect to q_1 and q_2, d_{ijkl} is the harmonic conjugate of p with respect to d_{ij} and d_{kl}, and f_{ijkl} is the harmonic conjugate of the point of contact with respect to f_{ij} and f_{kl}. We shall have

$$(94)\qquad q_1q_2=q\tau,\qquad r_1r_2=r\tau.$$

If we make use of the coefficient of d^2 and the constant term of equation (86), we find

$$(95)\qquad \begin{aligned} f_{ij}d_{ij}+f_{kl}d_{kl}+f_{ij}f_{kl}&=3r\tau-q\tau+4qr,\\ d_{ij}d_{kl}f_{ij}f_{kl}&=qr\tau^2.\end{aligned}$$

On account of (93), the last equation may be written

$$(96)\qquad d_{ijkl}f_{ijkl}=q\tau=q_1q_2,$$

so that d_{ijkl} and f_{ijkl} are a pair of the involution

$$(97)\qquad zz'=q_1q_2,$$

of which $z=0, z'=\infty$, and $z=q_1$, $z'=q_2$, constitute two other pairs.

Thus the involution of which the point of contact and the pole of the osculating linear function form one pair, and for wich the poles of tke osculating quadratic form a second pair, also contains the pair d_{ijkl}, f_{ijkl}.

Let us assume that d_1 and d_2 are distinct. Then

$$(z-d_1)^2(z-e_1)\quad\text{and}\quad(z-d_2)^2(z-e_2),$$

will be two distinct forms of the cubic involution (71), and this involution may be rewritten in the form

$$(98)\qquad (z-d_1)^2(z-e_1)=\nu(z-d_2)^2(z-e_2).$$

Let s_1, s_2, s_3 be the roots of (98) for a given value of ν. We shall have

$$(s_k-d_1)^2(s_k-e_1)=\nu(s_k-d_2)^2(s_k-e_2),$$
$$(k=1,2,3),$$

and therefore, by division,

$$\left(\frac{s_k-d_1}{s_k-d_2}:\frac{s_i-d_1}{s_i-d_2}\right)^2\left(\frac{s_k-e_1}{s_k-e_2}:\frac{s_i-e_1}{s_i-e_2}\right)=1,$$

or

$$(99)\qquad (d_1,d_2,s_k,s_i)^2(e_1,e_2,s_k,s_i)=1\qquad (i,k=1,2,3).$$

Thus the relations between the poles, s_1, s_2, s_3, of any penosculating cubic and the poles of the four singular penosculating cubic are given by the double-ratio equations (99). *In particular these relations will hold for the poles of the osculating cubic.*

A similar relation may be obtained between s_1, s_2, s_3 and the points p, q_1, q_2, r_1, r_2, by starting from the form (81) of the cubic involution, if we remember that in our canonical form $z=\infty$ takes the place of $z=p$.

Some of the features of the theory developed so far may be extended at once to the osculating rational functions of higher order. There will always exist one penosculating rational function of degree α, one of whose poles is the point of contact, and whose $\alpha-1$ other poles are the same as the poles of the osculating rational function of degree $\alpha-1$. There will exist another penosculating rational function of degree α, one of whose poles coincides with the pole of the osculating linear function. The poles of all of the other penosculating functions of degree α will be determined by an involution of degree α, of which the two special integral rational functions of degree α just mentioned are the base forms. Among these penosculating functions there will be $2(\alpha-1)$ which are singular, that is, which have at least two coincident poles.

Our theoreme have all been concerned with properties of the *poles* of the various osculating functions. *But the same relations are also true of their zeros.*

To prove this statement, it suffices to note that the rational function $\frac{T_\alpha}{U_\alpha}$ has contact of order 2α with $f(z)$ at $z = 0$, so that the expansion of

$$U_\alpha f(z) - T_\alpha,$$

begins with the term of order at least $2\alpha + 1$. Since $a_0 \neq 0$, we find that the expansion of

$$T_\alpha \frac{1}{f(z)} - U_\alpha,$$

will then also begin with a term of order at least $2\alpha + 1$.

Therefore, *the rational osculants of* $\frac{1}{f(z)}$ *are the reciprocals of the corresponding rational osculants of* $f(z)$. Consequently *the same relations which we have found between the poles of the rational osculants of* $f(z)$ *must also hold between their zeros.* The same statement also applies to the point-sets for which the various osculants assume a given value k, even if k is different from zero or infinity.

We have already pointed out that the rational osculating functions of order $1, 2, 3, \ldots, \alpha, \ldots$, which we have studied are the functions which normally occupy the positions in the principal diagonal of the Padé table. We have also studied certain classes of penosculating functions which have not been considered by Padé, and it appears clearly from our exposition that these penosculants are needed for the geometric interpretation. We have not so far considered any of the rational functions of Padé's table which are not on the principal diagonal. But our original objection to these functions, namely the non-invariant character of their definition, may now be disposed of, at least if we use some canonical form for our power series so that the point at infinity becomes the pole of the osculating linear function or some other invariantly defined point. Consequently we may now regard all of the functions of the Padé table as legitimate objects of our theory, opening up a large field for further investigation.

The theoreme wich have been established on the question of the convergence of a sequence of Padé fractions are of course of special interest. We merely mention that they provide, among other things, criteria as to whether a given power-series does or does not represent a meromorphic function, and that, in the former case, the poles and zeros of the rational osculants approach in definite fashion the poles and zeros of the function $f(z)$ as limits.

We close this section with a discussion of the convergence of the canonical expansion which we have used so frequently; we have postponed this discussion, because we shall make use in it of some of the results which have been obtained since the canonical expansion was first introduced.

Let us suppose that the original series

$$w = a_0 + a_1 z + a_2 z^2 + \ldots,$$

is convergent for

$$|z| < R.$$

We obtained the canonical expansion by putting

$$z_1 = \frac{a_1 z}{1 - \frac{a_2}{a_1} z} = \frac{a_1 p z}{p - z},$$

where p is the pole of the osculating linear function. From our method of deriving the canonical series it is evident that it will be convergent if we have simultaneously

$$|z| < R, \qquad |z_1| < |a_1|\,|p|,$$

or

$$|z| < R, \qquad |z| < |p - z|,$$

whence

$$|z| < R, \qquad |z| < \frac{1}{2}|p|.$$

Thus, *the canonical expansion will surely be convergent in a circle of center $z = 0$ and of radius r, where r is the smaller of the two positive numbers*, R and $\frac{1}{2}|p|$.

V. — The osculating logarithm.

We saw in Art. 3 that the absolute differential invariant

$$\theta = \frac{\sqrt{\{w, z\}}}{w'},$$

has a constant value $\theta = k$ for any function of the form

$$w = \frac{1}{k\sqrt{2}} \log \frac{z-a}{z-b} + l. \tag{100}$$

Since such a function involves four arbitrary constante, a, b, k, and l, we may expect to find a logarithmic function which has third order contact with a given analytic function

$$w = f(z) = a_0 + a_1 z + a_2 z^2 + \ldots,$$

at the origin. The resulting logarithmic function shall be called the *osculating logarithm*.

We see at once that for the determination of k we have

$$k = \theta = \frac{\sqrt{\{w, z\}}}{w'}, \tag{101}$$

where, of course, the value of θ for $z = 0$ is meant, and where a definite determination should be chosen for the square root. However, a change in this determination has no actual significance since it will be balanced by an interchange in the values of a and b, the two singular points of the osculating logarithm.

With this value for k, we find from (100),

$$\frac{dw}{dz} = \frac{1}{\theta\sqrt{2}}\left[\frac{1}{z-a} - \frac{1}{z-b}\right],$$

$$\frac{d_2 w}{dz^2} = \frac{1}{\theta\sqrt{2}}\left[-\frac{1}{(z-a)^2} + \frac{1}{(z-b)^2}\right].$$

In order that (100) may represent the osculating logarithm, these value of w, $\frac{dw}{dz}$, $\frac{d^2 w}{dz^2}$ must be equal to a_0, a_1, $2a_2$ respectively for $z = 0$.

We obtain the following conditions

$$a_0 = \frac{\theta\sqrt{2}}{1}\log\frac{a}{b} + l,$$

$$a_1 = \frac{1}{\theta\sqrt{2}}\left(\frac{1}{b} - \frac{1}{a}\right), \qquad 2\,a_2 = \frac{1}{\theta\sqrt{2}}\left(\frac{1}{b^2} - \frac{1}{a^2}\right).$$

From the last two equations we conclude by division

$$(102) \qquad \frac{1}{a} + \frac{1}{b} = 2\frac{a_2}{a_1} = \frac{2}{p},$$

that is; *the singular points of the osculating logarithm are harmonic conjugates of each other with respect to the point of contact and the pole of the osculating linear function.*

Since we have also found

$$-\frac{1}{a} + \frac{1}{b} = a_1\theta\sqrt{2},$$

we obtain the formulae

$$(103) \qquad \frac{2}{a} = \frac{2\,a_2}{a_1} - a_1\theta\sqrt{2}, \qquad \frac{2}{b} = \frac{2\,a_2}{a_1} + a_1\theta\sqrt{2},$$

whence

$$(104) \quad a = \frac{a_1\left[a_2 + \sqrt{12(a_1a_3 - a_2^2)}\right]}{8a_2^2 - 6a_1a_3}, \qquad b = \frac{a_1\left[a_2 - \sqrt{12(a_1a_3 - a_2^2)}\right]}{8a_2^2 - 6a_1a_3},$$

and finally

$$(105) \qquad l = a_0 + \frac{1}{\theta\sqrt{2}}\log\frac{2\,a_2 - a_1^2\theta\sqrt{2}}{2\,a_2 + a_1^2\theta\sqrt{2}},$$

that branch of the logarithm being selected in (105) such that the expansion of (100) will actually have a_0 as its constant term.

If the function $w = f(z)$ is not a linear function, we may use the second canonical form, for which

$$a_2 = 0, \qquad a_3 = -a_1.$$

We then find

$$p_1 = +1, \qquad p_2 = -1, \qquad p = \infty,$$

$$a = \frac{1}{2}i\sqrt{3}, \qquad b = -\frac{1}{2}i\sqrt{3},$$

so that

$$(106) \qquad (p_1, p_2, a, b) = -\frac{1}{2} - \frac{1}{2} i\sqrt{3}.$$

Moreover, the circles determined by $(0, a, b)$ and $(0, p_1, p_2)$ are (for this canonical form) the imaginary and real axes of the z plane (circles through $p = \infty$). Since angles are preserved by linear transformations we have proved the following theorem.

The cross-ratio determined by the poles of the two singular penosculating quadratics and the singularities of the osculating logarithm is equal to a complex cube root of unity. The circle determined by one of these points pairs with the point of contact is orthogonal to the corresponding circle for the second pair, and these circles meet again in the pole of the osculating linear function.

Let us consider for a moment the conformal representation of the z — plane upon the w — plane established by means of the osculating logarithm. If z moves on a circle of the pencil through a and b, the argument of $\frac{z-a}{z-b}$ will remain constant; if z moves on a circle of the orthogonal family, the modulus of $\frac{z-a}{z-b}$ will remain constant. To the circles of the first family there correspond in the w — plane the straight lines of a family of parallels, and to the circles of the second family will correspond the straight lines of the orthogonal family. Through the point of contact $(z = 0)$ there will pass a circle of each family. If we use the same canonical form as above, the circle through 0, a, and b, is the axis of imaginaries. All circles of the second family have the property that a and b are inverse points with respect to each of them. Since in our canonical form

$$a = -b = \frac{1}{2} i\sqrt{3},$$

the circle of the second family which passes through $z = 0$ is the axis of reals. Both of these lines, regarded as circles, pass through $p = \infty$. We obtain the following result.

Consider the pencil of circles through the singular points of the osculating logarithm, and the orthogonal family of circles. The two circles of the two families which pass through the point of contact meet again in the pole of osculating linear-function.

We may also express this result as follows.

Consider an analytic function $w = f(z)$ *and the osculating logarithm of* $f(z)$ *at* $z = 0$. *Determine the isothermal system of straight lines in the* w *— plane which corresponds, by means of the osculating logarithm, to the isothermal system of circles which is determined by the singular points of the osculating logarithm. To this system of straight lines in the* w *— plane there corresponds, by means of* $w = f(z)$, *an isothermal system of curves in the* z *— plane. The two curves of this system which meet at* $z = 0$ *will have osculating circles which meet again in the pole of the linear osculating function.*

The truth of this statement follows from the preceding theorem, if we remember further that the osculating logarithm has third order contact with $f(z)$ at $z = 0$. Consequently the curves of the two isothermal nets in the z — plane, which correspond to the above mentioned system of lines in the w — plane, by means of the two relations

$$w = f(z) \qquad \text{and} \qquad w = \frac{1}{\theta\sqrt{2}} \log \frac{z-a}{z-b} + l$$

will have the same osculating circles at $z = 0$.

It only remains to put into evidence the fact which is already clear from our discussion, that the quantity k which appears in (100) is actually an invariant for all transformations of the form

$$\overline{w} = w, \qquad \overline{z} = \frac{\alpha z + \beta}{\gamma z + \delta}.$$

For this purpose, let z_1 and z_2 be any two values of z, and w_1 and w_2 the corresponding values of w. Then we find

$$w_2 - w_1 = \frac{1}{k\sqrt{2}} \log \left[\frac{z_2 - a}{z_2 - b} : \frac{z_1 - a}{z_1 - b} \right],$$

whence

$$k\sqrt{2} = \frac{1}{w_2 - w_1} \log(a, b, z_2, z_1),$$

where (a, b, z_2, z_1) denotes the cross-ratio of the four points mentioned. This formula accomplishes our purpose since the right member obviously remains invariant under all transformations of the group under consideration.