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**The uniaxial potential function (fonction de potentiel) and its orthogonal force function (fonction de force) contrasted**

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SECTION III. — PHYSIQUE MATHÉMATIQUE.

# *The uniaxial potential function (fonction de potentiel) and its orthogonal force function (fonction de force) contrasted;*

PAR G. GREENHILL.

1. To recall the notation, a potential function (P. F.) V satisfies Laplace's equation, and this for uniaxial symmetry about the axis O$x$ can be written.

(A) $$\frac{d}{dx}\left(y\frac{dV}{dx}\right)+\frac{d}{dx}\left(y\frac{dV}{dy}\right)=0.$$

Then a function S exists, which we shall call the force function (F.F.) such that

(1) $$\frac{dS}{dx}=+y\frac{dV}{dy},\qquad \frac{dS}{dy}=-y\frac{dV}{dx};$$

and meridian sections of the surface V and S constant are orthogonal curves. Then conversely

(2) $$\frac{dV}{dy}=\frac{1}{y}\frac{dS}{dx},\qquad \frac{dV}{dx}=-\frac{1}{y}\frac{dS}{dy},$$

(B) $$\frac{d}{dx}\left(\frac{1}{y}\frac{dS}{dx}\right)+\frac{d}{dy}\left(\frac{1}{y}\frac{dS}{dy}\right)=0,$$

analogous to Laplace's equation (A).

Maxwell introduces a factor $2\pi$, which may be inserted when required for a comparison of results.

To standardise the notation, we prefer to use the ordinary Cartesian $x$ and $y$ coordinates, instead of Maxwell's ($b$, A), or the ($\rho$, $z$) or ($\varpi$, $z$) columnar coordinates; and $\varpi$ today is a Greek letter not in use by the printer. But we shall often require in the sequel to change to the notation of Maxwell.

The name Force-Function has been introduced after consideration, and it is submitted to this Congress for its sanction and approval.

And Professor Tait of Edinburg, in his *Dynamics of a Particle*, Chapter II, speaks scornfully of « the unnatural force-function, which disfigures most of the analytical treatises on our subject of Dynamics », still however in use in Germany. Tait meaning that he intends to substitute the name potential function.

So we may be allowed to rescue the discarded name, and give it this new signification, as representing a function constant along a line of force.

Some writers call it the Stream or Current Function, from its hydrodynamical interpretation; others call it the Stokes Function, because Stokes is supposed to be the first to have employed it. But there is always the difficulty of rival claims to be considered, when a man's name is given to a mathematical function.

In many cases where the P. F. V is a complicated function, or given by an intractable integral, it is, found that S the F. F. assumes a form of greater simplicity; a list of some such cases is given herewith in the sequel.

The calculation of the P. F. can be made according to a definite procedure of integration, when the method of dissection has been selected.

But there does not appear to be any similar straight-forward method of integration for the F. F.; the form of the F. F. S is to be inferred from (1), and thence by the partial integration which lead to a result.

**2.** Thus the P. F. of an electrified lens is given by Professor H. M. Macdonald in the *Proceedings of the London Mathematical Society* (*L. M. S.*), XXVI, in the form

(1) $$V = \pi - C + D,$$

where

(2) $$D = \int_u^\infty \sqrt{\left(\frac{\operatorname{ch} u - \cos v}{\operatorname{ch}\zeta - \operatorname{ch} u}\right)} \frac{\operatorname{sh} f\zeta\, f\, d\zeta}{\operatorname{ch} f\zeta - \cos f(v + 2\beta)},$$

and C is derived from D by putting $\beta = 0$; as this V will be found to satisfy Laplace's equation (A) and over the spherical surface $v = -\beta$, and $v = \alpha - \beta$, $C = D$, $V = \pi$, with $f\alpha = \pi$.

The coordinates $(u, v)$ employed here are the conjugate functions in the stereographic conformal representation.

(3) $$x + iy = a \cot \tfrac{1}{2}(v + iu)$$

and then Laplace's equation (A) becomes

(A′) $$\frac{d}{du}\left(y\frac{dV}{du}\right) + \frac{d}{du}\left(y\frac{dV}{dv}\right) = 0,$$

as well as for any other conformal representation of conjugate functions

(4) $$x + iy = f(v + iu).$$

But the determination of the F. F. terms A and B, corresponding to C or D, was a matter of guess work for the most part, when it was found that

(5) $$B = a\int \sqrt{\left(\frac{\operatorname{ch}\zeta - \operatorname{ch} u}{\operatorname{ch} u - \cos v}\right)} \frac{\sin v}{\operatorname{ch}\zeta - \cos v} \frac{\operatorname{sh} f\zeta . f\, d\zeta}{\operatorname{ch} f\zeta - \cos f(v + 2\beta)}$$
$$- a\int \sqrt{\left(\frac{\operatorname{ch} u - \cos v}{\operatorname{ch}\zeta - \operatorname{ch} u}\right)} \frac{\operatorname{sh}\zeta}{\operatorname{ch}\zeta - \cos v} \frac{\sin f(v + 2\beta)\, f\, d\zeta}{\operatorname{ch} f\zeta - \cos f(v + 2\beta)},$$

and A is derived from B by putting $\beta = 0$.

With these values it can be verified that

(6) $$\frac{a}{y}\frac{dB}{du} + \frac{dD}{dv} = 0, \qquad \frac{a}{y}\frac{dB}{dv} - \frac{dD}{du} = 0, \qquad \frac{y}{a} = \frac{\operatorname{sh} u}{\operatorname{ch} u - \cos v},$$

and the same with A and C, as required (*Proceedings of the Royal Society*, vol. 98, 1920).

3. Begin by the consideration of the electrical potential of a circular disc AB with axis O$x$; this is given by

(1) $$\omega = \sin^{-1}\frac{\text{AB}}{\text{AP}+\text{PB}}$$

making $\omega = \frac{1}{2}\pi$ over the disc, and $\omega = 0$ at infinity.

But the corresponding F. F. is an algebraical function

(2) $$\text{A} = \frac{1}{2}\sqrt{[\text{AB}^2 - (\text{AP}-\text{PB})^2]} = \sqrt{(\text{AP}.\text{PB})}\sin\frac{1}{2}\text{APB} = x\,\text{tang}\,\omega,$$

so that A is the semi-conjugate axis of the confocal hyperbola though P, while sin $\omega$ is the excentricity of the confocal ellipse, foci at A and B; and the meridian sections of constant $\omega$ and A are confocal ellipses and hyperbolas.

The charge induced in the disc, if to earth, by a point charge $q$ at P is given by $-q\dfrac{\omega}{\frac{1}{2}\pi}$.

On the conformal representation of this confocal system

(3) $$y + ix = a\,\text{ch}(\eta + i\xi),$$

(4) $$\text{AP},\ \text{PB} = a(\text{ch}\,\eta \pm \cos\xi),$$

(5) $$\sin\omega = \text{sech}\,\eta, \qquad \text{A} = a\sin\xi.$$

More generally then, any oblate spheroid of which AB is the focal circle, insulated and electrified with a charge E, will have a P. F. V, F. F. S and electrical density $\sigma$, such that

(6) $$\left\{\begin{aligned} &\text{V} = \frac{2\text{E}}{\text{AB}}\sin^{-1}\frac{\text{AB}}{\text{AP}+\text{PB}}, \qquad \text{S} = \text{E}\sqrt{\left[1 - \left(\frac{\text{AP}-\text{PB}}{\text{AB}}\right)^2\right]}, \\ &4\pi\sigma = \frac{\text{E}}{\frac{1}{2}(\text{AP}+\text{PB})\sqrt{(\text{AP}.\text{PB})}}; \end{aligned}\right.$$

and this verifies at infinity, where

(7) $$\text{V} \to \frac{2\text{E}}{\text{AB}}\sin^{-1}\frac{\text{AB}}{2\text{OP}} \to \frac{\text{E}}{\text{OP}}.$$

The analogous expression of

$$(8)\quad \begin{cases} \omega_1 = \mathrm{ch}^{-1}\dfrac{\mathrm{AB}}{\mathrm{AP}-\mathrm{PB}}, \\ \mathrm{A}_1 = \frac{1}{2}\sqrt{[(\mathrm{AP}+\mathrm{PB})^2-\mathrm{AB}^2]} = \sqrt{(\mathrm{AP}.\mathrm{PB})}\cos\frac{1}{2}\mathrm{APB} = x\coth\omega_1 \end{cases}$$

would apply to an infinite electrified plate, from which a circular hole AB had been cut out [*American Mathematical Journal* (*A. M. J.*), oct. 1917, p. 346, *Phil. Trans.*, 1891, The Stokes Function, R. A. Sampson].

**4.** With O $y$ in the axis of revolution and in any system of conformal representation by conjugate functions, (A) and (B) in § **1** can be replaced, as in (A′), § **2**, by

$$(\mathrm{C})\qquad \frac{d}{du}\left(x\frac{d\mathrm{V}}{du}\right) + \frac{d}{dv}\left(x\frac{d\mathrm{V}}{dv}\right) = 0,$$

$$(\mathrm{D})\qquad \frac{d}{du}\left(\frac{1}{x}\frac{d\mathrm{S}}{du}\right) + \frac{d}{dv}\left(\frac{1}{x}\frac{d\mathrm{S}}{du}\right) = 0.$$

At a great distance from the axis O$y$, where the variation of $x$ is insensible, these equations reduce to the ordinary equations of two conjugate functions, V and S.

Then for a rod AB in O$y$, or a confocal prolate spheroid, with electric charge E,

$$(1)\quad \mathrm{V} = \frac{2\mathrm{E}}{\mathrm{AB}}\mathrm{th}^{-1}\frac{\mathrm{AB}}{\mathrm{AP}+\mathrm{PB}},\quad \mathrm{S} = \mathrm{E}(\mathrm{AP}-\mathrm{PB}),\quad 4\pi\sigma = \frac{\mathrm{E}}{\frac{1}{2}(\mathrm{AP}+\mathrm{PB})\sqrt{(\mathrm{AP}.\mathrm{PB})}}.$$

The line P. F. of the rod AB with AP, BP $= r_1, r_2$, is obtained from the integral

$$(2)\quad \int_{-a}^{a}\frac{dy'}{\sqrt{[x^2+(y'-y)^2]}} = \mathrm{sh}^{-1}\frac{y+a}{x} - \mathrm{sh}^{-1}\frac{y-a}{x} = \mathrm{ch}^{-1}\frac{r_1}{x} - \mathrm{ch}^{-1}\frac{r_2}{x}$$

$$= \mathrm{sh}^{-1}\frac{a(r_1+r_2)-y(r_1-r_2)}{x^2} = \mathrm{ch}^{-1}\frac{r_1r_2-y^2+a^2}{x^2}$$

$$= 2\,\mathrm{th}^{-1}\sqrt{\left(\frac{r_1r_2-x^2-y^2+a^2}{r_1r_2+x^2-y^2+a^2}\right)} = 2\,\mathrm{th}^{-1}\frac{2a}{r_1+r_2} = 2\,\mathrm{th}^{-1}\frac{2y}{r_1-r_2},$$

and this is infinite along AB.

The separate terms,

(3) $$V = \mathrm{sh}^{-1}\frac{y+a}{x} = \mathrm{ch}^{-1}\frac{r_1}{x}, \qquad S = r_1$$

give the V and S for a positive semi-infinite rod $Ay$, and negative road $Ay'$.

With the break at O, and $a = 0$

(4) $$V = \mathrm{sh}^{-1}\frac{y}{x} = \mathrm{ch}^{-1}\frac{r}{x} = \mathrm{th}^{-1}\frac{y}{r}, \qquad S = r = OP,$$

and the stream sheets for the corresponding streaming motion are spherical and of uniform thickness, the flow issuing from a pole and disappearing at the other pole, with velocity elsewhere inversely as the distance from the axis.

The single line source O$y$ would have

(5) $$V = \mathrm{th}^{-1}\frac{y}{r} + \log x = \frac{1}{2}\log\frac{r+y}{r-y} + \log x = \log(r+y), \qquad S = r - y,$$

and sections of constant V and S are confocal parabolas, with focus at O (*A. J. M.*, XXXIX, p. 356).

Take the case of a spherical vessel, with a source A and a sink B, at opposite ends of a diameter AB; the F. F. or stream function.

(6) $$S = \frac{AN}{AP} + \frac{NB}{PB} - \frac{AP+PB}{AB},$$

makes $S = 0$ over the surface, $S = 1$ along AB, so that the flow is $2\pi$.

But the expression for the P. F. V. is more complicated

(7) $$V = \frac{1}{AP} - \frac{1}{PB} - \frac{2}{AB} + \mathrm{th}^{-1}\frac{AB}{AP+PB}.$$

**5.** In the electrical distribution $\sigma$ over an equipotential,

(1) $$4\pi\sigma = F = -\frac{dV}{dn} = \frac{1}{y}\frac{dS}{ds},$$

where $dn$, $ds$ denote the element of the normal and arc of the meridian of the equipotential V:

$$(2)\qquad \frac{dS}{ds} = 4\pi\sigma y\, ds, \qquad \frac{1}{2}(S_1 - S_2) = \int 2\pi\sigma y\, ds,$$

and this is the charge or electric induction through the zone on the equipotential.

Thence the charge or capacity is inferred from the half difference of the extreme values of the F. F. S.

For the insulated bowl, of radius $c$, electrified to potential $\pi$,

$$(3)\qquad V = \pi - \omega + \frac{c}{r}\omega',$$

$$(4)\qquad S = A + c\omega - c\omega' \cos\varphi.$$

But for the gravitation potential U of the bowl, of uniform superficial density $\sigma$, g/cm$^2$,

$$(5)\qquad \frac{U}{G\sigma} = c\Omega + r'\Omega',$$

$$(6)\qquad \frac{S}{G\sigma} = -acP + AcQ + c^2\Omega\cos\gamma + c^2\Omega'\cos\varphi$$
$$= -\frac{cW}{G\sigma} + \frac{U}{G\sigma} r\cos\varphi,$$

as in the *American Journal of Mathematics* (*A. J. M.*), t. XXXIII, p. 391, W denoting the P. F. of the plate on AB.

Here $\Omega$ is the solid or conical angle subtended at P by the circle on AB, and $\Omega'$ the solid angle subtended at P′, inverse point of P in the sphere or bowl.

The gravitation P. F. and F. F. has been worked out in the *A. J. M.*, 1910, t. XXXIII, p. 394, for a homogeneous plano convex lens, and thence, by addition or subtraction, for a double convex, or concavo-convex lens.

The converse process on p. 399 of deriving the plano-convex lens is troublesome, in leading to an indeterminate form.

6. The electrical problem of a point charge $q$ at Q in the presence of a conducting sphere to earth and of the electrification induced, is at the root of the method of inversion.

The potential over the sphere is reduced to zero if a point charge $-q'$ is introduced at Q', inverse point in the sphere of the point Q, adjusted so that at any point P of the sphere

(1) $$\frac{q}{PQ} - \frac{q'}{PQ'} = 0.$$

Then if PA, PB are the bisectors, internal and external, of the angle OPQ'

(2) $$\frac{q'}{q} = \frac{PQ'}{PQ} = \frac{Q'A}{QA} = \frac{Q'B}{QB} = \frac{O'Q'}{OA} = \frac{OA}{OQ} = \sqrt{\frac{OQ'}{OQ}};$$

APB is a right angle, and OQP, OPQ' are similar triangles, with

$$OQP = OPQ' = \theta, \qquad OQ'P = OPQ = \theta',$$

(3) $$\frac{q'}{q} = \frac{PQ'}{PQ} = \frac{OP}{OQ} = \sqrt{\frac{OQ'}{OQ}},$$

so that the capacity of the point images at Q, Q' may be taken as proportional to the square root of OQ, OQ'.

The resultant force F at P is then along the radius PO, and estimated in the direction to the inside of the sphere

(4) $$F = \frac{q' \cos\theta}{PQ'^2} - \frac{q\cos\theta'}{PQ^2} = \frac{q}{PQ^2}\left(\frac{OQ}{OP}\cos\theta - \cos\theta'\right),$$

(5) $$OQ\cos\theta - OP\cos\theta' = QR - PR = QP' = \frac{OQ^2 - OA^2}{QP},$$

(6) $$F = q\frac{OQ^2 - OA^2}{OA.PQ^3} = q\frac{OI^2}{OA.PQ^3} = q\frac{OQ.QQ'}{OA\ PQ},$$

varying inversely as the cube of PQ.

Put $F = 4\pi\sigma$, and $\sigma$ is the density, electric or gravitation, of a surface film, like a coat of paint, to make a centrobaric shell, attracting external points as if concentrated into the image Q'.

**7.** Suppose a succession of electrified Q points outside the sphere to make a ring; the series of Q' images will make another ring, and the electrical field outside the sphere, if uninsulated, is due to these two rings.

Let the Q ring for instance be a circle on diameter AB, presented directly to the sphere, and charged with C coulombs of electricity.

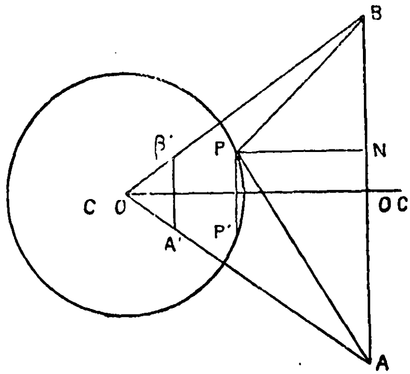
The electrical density $\Delta\sigma$ at a point P on the sphere induced by the charge $q = C\frac{d\theta}{2\pi}$ at Q is given by

(1) $$\Delta\sigma = C\frac{d\theta}{2\pi}\frac{OA^2 - OP^2}{4\pi . OP . PQ^3} = \frac{C\,d\theta}{8\pi^2}\frac{OA^2 - OP^2}{OP . PD}\frac{\cos QPN}{PQ^2},$$

if PN is the perpendicular on the plane AB of the circle, and this is the component attraction on P of the $q$ particle at Q perpendicular to the plane AB, multiplied by $\frac{OA^2 - OP^2}{4\pi . OP . PN}$.

The total $\sigma$ induced at P will be the same multiple of the attraction

Fig. 1.




of the ring perpendicular to its plane, and this attraction is expressible by E, the complete E.I.II.

Because with $\theta = 2\omega$,

(2) $$PQ^2 = PA^2\cos^2\omega + PB^2\sin^2\omega = PA^2\,\Delta^2(\omega, \gamma), \qquad \gamma' = \frac{PB}{PA},$$

and then the component normal attraction is

(3) $$N = \int C\frac{d\theta}{2\pi}\frac{PN}{PQ^3} = \frac{2C}{\pi}\frac{PN}{PA^3}\int_0^{\frac{1}{2}\pi}\frac{d\omega}{\Delta^3\omega},$$

this is reduced by a change to the co-amplitude angle $\omega'$, by

(4) $$\gamma'\tan g\omega\tan g\omega' = 1, \qquad \Delta\omega\,\Delta\omega' = \gamma', \qquad \frac{d\omega}{\Delta\omega} + \frac{d\omega'}{\Delta\omega'} = 0$$

and

$$\int_0^{\frac{1}{2}\pi} \frac{d\omega}{\Delta^3\omega} = \int_0^{\frac{1}{2}\pi} \frac{\Delta\omega'\, d\omega'}{\gamma'^2} = \frac{H}{\gamma'^2},$$

(5) $$N = \frac{2C}{\pi} \frac{PN}{PA^3} \frac{H}{\gamma'^2} = \frac{2C}{\pi} \frac{PN}{PA} \frac{H}{PB^2} = C \frac{PN}{PA.PB^2} \frac{H}{\frac{1}{2}\pi}.$$

Then on the sphere at P, in coulombs/cm²

(6) $$\sigma = C \frac{OA^2 - OP^2}{4\pi.OP} \frac{1}{PA.PB^2} \frac{H}{\frac{1}{2}\pi}.$$

The P. F. at any point R due to this electrified ring on AB is

(7) $$\left\{\begin{aligned} U &= \int_0^{2\pi} \frac{C\,d\theta}{2\pi.RQ} = \frac{C}{\frac{1}{2}\pi.RA} \int_0^{\frac{1}{2}\pi} \frac{d\omega}{\Delta\omega} = \frac{C}{RA} \frac{G}{\frac{1}{2}\pi} \\ &= \frac{C}{\sqrt{(RA.RB)}} \frac{G\sqrt{\gamma'}}{\frac{1}{2}\pi} = \frac{C}{\sqrt{(RA.RB)}} \frac{K\varkappa'}{\frac{1}{2}\pi} \\ &= \frac{C}{\frac{1}{2}(RA + RB)} \frac{K}{\frac{1}{2}\pi}, \\ \gamma' &= \frac{1-\varkappa}{1+\varkappa} = \frac{RA}{RB}, \end{aligned}\right.$$

in a quadric transformation.

If C′ is the charge on the ring A′B′, image of AB in the sphere, the P. F. at R

(8) $$U' = \frac{C'}{RA'} \frac{G}{\frac{1}{2}\pi} = -\frac{C}{RA} \frac{OQ}{OA} \frac{G}{\frac{1}{2}\pi}$$

and then $U + U' = 0$ when R is on the sphere.

The F. F. of the charge C on the ring AB at any point P will be given by $C\,a\,\Omega_1$, where $\Omega_1$ is the solid angle subtended at any point A on the ring by the circle on PP′.

The expansion given in a series of zonal harmonies is worthless and midleading, and gives no indication of the infinities on the ring AB and its image A′B′.

**8.** The typical element of a P. F. may be taken as the reciprocal of R, distance from a fixed point, and when the fixed point is at C in $Ox$, with $OC = c$, and $OP = r$, $xOP = \theta$ and $\cos\theta = \mu$ :

$$(1) \qquad R^2 = c^2 - 2cr\cos\theta + r^2 = c^2 - 2cr\mu + r^2;$$

and with respect to the axis $Ox$ the F. F. is $\cos PCx$.

The expansion of the reciprocal of R is powers of $\frac{r}{c}$ or $\frac{c}{r}$, introduces the coefficients of Legendre, or zonal harmonics as they are also called, denoted by $P_n(\mu)$ such that

$$(2) \qquad \frac{1}{R} = \sum \frac{r^n}{c^{n+1}} P_n(\mu) \quad \left(\frac{r}{c} < 1\right), \qquad \text{or} \qquad \sum \frac{c^n}{r^{n+1}} P_n(\mu) \quad \left(\frac{r}{c} > 1\right),$$

and the discussion of the analytical properties of the zonal harmonic is to be found in every treatise on the subject.

They may be summarised in the relations

$$(3) \qquad \frac{d}{d\mu}(\mu^2 - 1)\frac{dP_n}{d\mu} = n(n+1)P_n,$$

$$(4) \qquad P_n(\mu) = \frac{1}{2^n . n!}\left(\frac{d}{d\mu}\right)^n (\mu^2 - 1)^n,$$

$$(5) \qquad (n+1)P_{n+1} - (2n+1)\mu P_n + nP_{n-1} = 0,$$

$$(6) \qquad (\mu^2 - 1)\frac{dP_n}{d\mu} = (n+1)(P_{n+1} - \mu P_n) = n(\mu P_n - P_{n-1})$$

and so on (*Messenger of Mathematics*, August 1919, R. Hargreaves, *Standard relations of Legendre's Functions*).

Maxwell's interpretation by the coalescence of poles along $Ox$ is explained in his *Electricity and Magnetism*, giving a physical meaning to the résult, such as in the repeated axial differentiation

$$(7) \qquad \left(-\frac{d}{dx}\right)^n \frac{1}{r} = \frac{P_n(\mu)}{r^{n+1}}.$$

But the corresponding typical element of the F. F. will be R, and expanded in a series

$$(8) \qquad R = \sum \left(\frac{r^{n+1}}{c^n}, \text{ or } \frac{c^{n+1}}{r^n}\right) I_n(\mu)$$

and since

(8)' $$R = \int \frac{cr\,d\mu}{R},$$

(9) $$I_n(\mu) = \int P_n(\mu)\,d\mu = \frac{\mu^2-1}{n(n+1)}\frac{dP_n}{d\mu} = \frac{\mu^2-1}{n(n+1)}\frac{d^2I_n}{d\mu^2}$$
$$= \frac{P_{n+1}-\mu P_n}{n} = \frac{\mu P_n - P_{n-1}}{n+1} = \frac{P_{n+1}-P_{n-1}}{2n+1}.$$

But R is not the F. F. of $\frac{1}{R}$, as this is $\cos PCx$ or $\frac{dR}{dx}$.

The F. F. of R. is given already as

(10) $$\text{sh}^{-1}\frac{NP}{CN} = \text{ch}^{-1}\frac{CP}{CN} = \text{th}^{-1}(\sin PCx) = \log\tan\left(\frac{1}{4}\pi + \frac{1}{2}PCx\right) = \ldots$$

Consult R. A. Sampson in the *Phil. Trans.*, 1891, on the Stokes Function, and a discussion of the properties of $I_n(\mu)$.

Similarly, for the F. F. relations of axial differentiation

(11) $$\left(\frac{d}{dx}\right)^n r = \frac{n!\,I_n(\mu)}{r^{n-1}}.$$

Thus for a solid sphere, take

(12) $$V = \frac{1}{r}, \qquad S = \frac{dr}{dx} = \cos\theta,$$

and for the magnetic doublet molecule, or a sphere moving along O$x$ in infinite liquid,

(13) $$V = -\frac{d}{dx}\left(\frac{1}{r}\right) = \frac{x}{r^3}, \qquad S = \frac{d^2r}{dx^2} = -\sin\theta\frac{d\theta}{dx} = \frac{y^2}{r^3}.$$

Hence for a spherical boss on a infinite plane, electrified,

(14) $$V = x - x\frac{a^3}{r^3}, \qquad S = -\frac{1}{2}y^2 - y^2\frac{a^3}{r^3}$$

satisfying the relations $V = 0$ over the surface, and in space

(15) $$\frac{dS}{dx} = y\frac{dV}{dy}; \qquad \frac{dS}{dy} = -y\frac{dV}{dx}.$$

Coaxial doublets at S and T, inverse points in a sphere, centre O and radius OA, with resultant F. F.

$$S_1 - S_2 = PN^2 \frac{OS^3}{SP^3} - PN^2 \frac{OA^3}{TP^3},$$

will have $S_1 - S_2 = 0$ over the sphere, which can be made to screen the motion inside and outside the surface.

A singlet source at S and equal source at T would be screened by a plane bisecting ST at right angles.

But with a source S in the presence of the sphere and outside, the resultant F. F.

$$S_1 = \frac{SM}{SP} + \frac{OA}{OS}\frac{TM}{TP} + \frac{TP - TM}{OA} - \frac{OP - OM}{OA},$$

so that the hydrodynamical image of the source S in the sphere is an image source at T, and a line sink from T to O.

Over the sphere this satisfies the condition

$$S_1 = -\frac{MS}{SP} + \frac{TS - MS}{SP} + \frac{TP + OT}{OA} - 1,$$

and if SP cuts the sphere again in P′, SP.SP′ = OS.ST, so that with OR bisecting PP′ at right angles,

$$S_1 = -2\frac{MS}{SP} + \frac{SP'}{OS} + \frac{SP}{OS} + \frac{OT}{OA} - 1 = -2\frac{MS}{SP} + 2\frac{SR}{OS} - \frac{AT}{OA} = -\frac{AT}{OA},$$

a constant. The P.F. can then be written down, but is of more complicated analytical character, involving a logarithm.

A single source at T inside the sphere would have en image source at S, and a line sink extending from S to infinity, and would require to be associated physically with another equal sink inside the sphere, to draw off the accumulation of the stream.

Thence the special case given already in § 4.

9. The P. F. and F. F. has been calculated in the *Phil. Trans.*, 1919, *Electromagnetic Integrals*, for a number of problems in Electromagnetic and Attraction Theory, and the contrast drawn of the relative simplicity.

Thus for the rim potential P of the circle on the diameter AB $= 2a$, with axis in $Ox$, returning to Maxwell's notation,

$$(1)\qquad P=\int_0^{2\pi}\frac{a\,d\theta}{PQ},\qquad PQ^2=MQ^2+PM^2=A^2+2Aa\cos\theta+a^2+b^2,$$

$$(2)\qquad \omega=\tfrac{1}{2}\theta,\qquad PQ^2=r_3^2\cos^2\omega+r_2^2\sin^2\omega=r_3^2\Delta^2(\omega,\gamma),\qquad \gamma'=\frac{r_2}{r_1},$$

$$(3)\qquad P=\frac{2a}{r_1}\int_0^{\pi}\frac{d\omega}{\Delta\omega}=\frac{4aG}{r_1},\qquad G=\int_0^{\frac{1}{2}\pi}\frac{d\omega}{\Delta(\omega,\gamma)},\qquad \gamma^2=\frac{4Aa}{r_1^2}.$$

But S, the rim F. F. at P proves to be given by the conical angle $\Omega(1-f)$, subtended at any point on the rim of the circle AB by the circle on the diameter PP′ :

$$S=2\pi a\,\Omega(1-f)\qquad (\textit{Phil. Trans.}, \text{p. } 57),$$

the conical angle subtended at P by the rim AB being $\Omega=\Omega(f)$, such that

$$(4)\qquad \Omega(f)+\Omega(1-f)=2\pi-P\frac{b}{a}.$$

The solid angle $\Omega$ subtended at P by the circle on AB is the magnetic P. F. of unit current round the rim AB, or of the equivalent magnetic sheet on AB, magnetized normally; and its F. F. is given by Maxwell's M. in *E. and M.*, § **705**, in consequence of the equation (2), (3); and

$$(5)\qquad \begin{cases} M=\displaystyle\int_0^{2\pi}2\pi Aa\cos\theta\frac{d\theta}{PQ}=\pi\int\left(PQ-\frac{A^2+a^2+b^2}{PQ}\right)d\theta \\ \quad =2\pi\displaystyle\int_0^{\pi}\left[r_1\Delta\omega-\frac{\frac{1}{2}(\gamma_1^2+\gamma_2^2)}{\gamma_1\Delta\omega}\right]d\omega \\ \quad =2\pi\gamma_1[2H-(1+\gamma'^2)G]; \end{cases}$$

G and H, and so M is expressed by the complete E. I, I and II.

This M as defined by Maxwell is negative; so putting

$$M=-2\pi AQ,$$

then $Q\cos\varphi$ is the magnetic potential of the circular sheet on AB magnetized parallel to AB; Q is a vector potential.

The integration of M with respect to $b$ will give L for the induction of a current sheet round a cylinder (*Electromagnetic Integrals*) and this is also the F. F. of a circular area on AB,

$$(6) \qquad L = \int M\,db = \pi a b P + \frac{1}{2} b M - \pi(a^2 - A^2)(2\pi - \Omega)$$
$$= \pi A^2 \frac{b}{a} P + \frac{1}{2} b M - \pi(a^2 - A^2)\Omega(1 - f).$$

Then for the P. F. W of the circular area

$$(7) \qquad W = \int\int \frac{2\pi a\,da\,d\theta}{PQ} = a P + \frac{M}{2\pi} - b\Omega.$$

An integration of L will give N the F. F. of the solid cylinder, at a point in the plane PP′ of its base, and

$$(8) \qquad N = \int L\,db = \pi a\left(\frac{4}{3}A^2 + b^2\right)P$$
$$+ \frac{1}{3}(2A^2 + 2a^2 - b^2)M - \pi b(a^2 - A^2)(2\pi - \Omega)$$
$$= \pi A^2\left(\frac{4}{3}a + \frac{b^2}{a}\right)P$$
$$- \frac{2}{3}\pi A(2A^2 + 2a^2 - b^2)Q - \pi b(a^2 - A^2)\Omega(1 - f).$$

Then $\frac{dN}{da}$ is the F. F. of the cylindrical skin,

$$(9) \qquad \frac{1}{2\pi}\frac{dN}{da} = a W - (A^2 + a^2 + b^2)P - 2aM + 2\pi a b;$$

the $\Omega$ being included in W.

But the P. F. of this solid cylinder, and of its cylindrical skin, introduces an intractable integral, whereas we may say that the F. F. has been expressed in finite terms, that is by tabulated functions.

So too for the F. F. of the disc, when the density varies as some $n$ — th power of the distance from the centre; this can be expressed in finite terms, when the P. F. is intractable.

For the details of these integrations consult the Memoir on *Electromagnetic Integrals* in the *Phil. Trans.*, Dec. 1919.

The skin P. F. of the cylinder is given by

(10) $$\mathrm{I} = \int_0^{2\pi} a\,d\theta \int_0^b \frac{db}{\mathrm{PQ}} = \int_0^{2\pi} a\,d\theta\,\mathrm{th}^{-1}\frac{b}{\mathrm{PQ}}$$

and then with

$$\theta = 2\omega = 2\,\mathrm{am}\,u, \qquad u = e\mathrm{G}, \qquad \mathrm{PQ} = r_1\,\mathrm{dn}\,u,$$

$$\mathrm{sn}\,2f\mathrm{G}' = \frac{b}{r_2}, \qquad \mathrm{cn}\,2f\mathrm{G}' = \frac{a-\mathrm{A}}{r_2}, \qquad \mathrm{dn}\,2f\mathrm{G}' = \frac{a+\mathrm{A}}{r_1},$$

put

$$v = \mathrm{G} + 2f\mathrm{G}'i, \qquad \mathrm{sn}\,v = \frac{r_1}{a+\mathrm{A}}, \qquad \mathrm{cn}\,v = \frac{ib}{a+\mathrm{A}},$$

$$\mathrm{dn}\,v = \frac{a-\mathrm{A}}{a+\mathrm{A}}, \qquad \frac{b}{\mathrm{PQ}} = \frac{i\,\mathrm{cn}\,v}{\mathrm{sn}\,v\,\mathrm{dn}\,u},$$

$$\mathrm{th}^{-1}\frac{b}{\mathrm{PQ}} = i\,\mathrm{tang}^{-1}\frac{\mathrm{cn}\,v}{\mathrm{sn}\,v\,\mathrm{dn}\,u} = \frac{1}{2}i\sin^{-1}\frac{2\,\mathrm{sn}\,v\,\mathrm{cn}\,v\,\mathrm{dn}\,u}{1-\gamma^2\,\mathrm{sn}^2v\,\mathrm{sn}^2u}.$$

$$= \frac{1}{2}i\,\mathrm{am}(u+v) - \frac{1}{2}i\,\mathrm{am}(u-v);$$

(11) $$\mathrm{I} = 2ia\int_0^{\mathrm{G}} [\mathrm{am}(u+v) - \mathrm{am}(u-v)]\,d\,\mathrm{am}\,u$$

an intractable integral.

Expanded in a series

(12) $$\mathrm{th}^{-1}\frac{b}{\mathrm{PQ}} = \sum \frac{1}{2n+1}\left(\frac{b}{\mathrm{PQ}}\right)^{2n+1} = \sum \frac{1}{2n+1}\left(\frac{b^2}{r_1 r_3}\,\frac{\gamma'}{\mathrm{dn}^2 u}\right)^{n+\frac{1}{2}},$$

(13) $$\mathrm{I} = 4a\sum\left(\frac{b^2}{r_1 r_2}\right)^{n+\frac{1}{2}}\frac{\mathrm{P}_n(u)}{2n+1}.$$

(Work out the surface longitudinal pressure due to self attraction.)

**10.** In a change to another system of orthogonal co-ordinates, $u$ and $v$, given by the conformal representation of a conjugate system

(1) $$y + ix = f(v+iu),$$

equation (A) and (B) will change into

(A′) $$\frac{d}{du}\left(y\frac{d\mathrm{V}}{du}\right) + \frac{d}{dv}\left(y\frac{d\mathrm{V}}{dv}\right) = 0,$$

(B′) $$\frac{d}{du}\left(\frac{1}{y}\frac{d\mathrm{S}}{du}\right) + \frac{d}{dv}\left(\frac{1}{y}\frac{d\mathrm{S}}{dv}\right) = 0.$$

Thus for a system of confocal conics, foci at A, B, given by

(2) $y + ix = c\,\mathrm{ch}(v + iu), \qquad y = c\,\mathrm{ch}\,u\cos v, \qquad x = c\,\mathrm{sh}\,u\sin v,$

(3) $$\frac{d^2V}{du^2} + \mathrm{th}\,u\frac{dV}{du} + \frac{d^2V}{dv^2} - \tan v\frac{dV}{dv} = 0,$$

(4) $$\frac{d^2S}{du^2} - \mathrm{th}\,u\frac{dS}{du} + \frac{d^2S}{dv^2} + \tan v\frac{dS}{dv} = 0;$$

or with $\mathrm{ch}\,u = C$, $\cos v = \mu$,

(5) $$\frac{d}{dC}(C^2 - 1)\frac{dV}{dC} + \frac{d}{d\mu}(1 - \mu^2)\frac{dV}{d\mu} = 0,$$

(6) $$(C^2 - 1)\frac{d^2S}{dC^2} + (1 - \mu^2)\frac{d^2S}{d\mu^2} = 0.$$

Where $V = U.W$, $S = Q.R$, and U and Q are functions of $u$ or C only, W and R, of $v$ or $\mu$ only,

(7) $$\frac{d}{dC}(C^2 - 1)\frac{dU}{dC} = hU, \qquad \frac{d}{d\mu}(1 - \mu^2)\frac{dW}{d\mu} = -hW,$$

(8) $$(C^2 - 1)\frac{d^2Q}{dC^2} = kQ, \qquad (1 - \mu^2)\frac{d^2R}{d\mu^2} = -kR.$$

Thus we may take

(9) $$Q = \int U\,dC, \qquad R = \int W\,d\mu$$

or

(10) $$Q = (C^2 - 1)\frac{dU}{dC}, \qquad R = (1 - \mu^2)\frac{dW}{d\mu}.$$

**11.** In the conformal representation of the dipolar co-ordinates, $u$ and $v$, of the stereographic projection of the hemisphere, as in (3) § **2**,

(1) $$y + ix = a\coth\tfrac{1}{2}(u + iv) \quad \text{or} \quad a\cot\tfrac{1}{2}(v + iu),$$

$$y, x = a\frac{\mathrm{sh}\,u, \sin v}{\mathrm{ch}\,u - \cos v}.$$

Then, putting $V = P\sqrt{(\mathrm{ch}\,u - \cos v)}$, Laplace's equation (A′) becomes

(2) $$\frac{d^2P}{du^2} + \coth u\frac{dP}{du} + \frac{d^2P}{dv^2} + \frac{1}{4}P = 0;$$

while, with $S = \frac{R}{\sqrt{(\operatorname{ch} u - \cos v)}}$, (B′) becomes

$$(3)\qquad \frac{d^2R}{du^2} - \coth u \frac{dR}{du} + \frac{d^2R}{dv^2} + \frac{1}{4}R = 0.$$

Then with P, R, varying as $e^{inu}$, and writing C, S for ch $u$, sh $u$,

$$(4)\qquad \frac{d}{dC}(C^2-1)\frac{dP}{dC} = \left(n^2 - \frac{1}{4}\right)P, \qquad (C^2-1)\frac{d^2R}{dC^2} = \left(n^2 - \frac{1}{4}\right)R)$$

so that P is a zonal harmonic or Legendre function of order $i = n - \frac{1}{2}$, while R is an I function, of the form

$$\int P\, dC.$$

These are the toroidal functions considered by W. M. Hicks in *Phil. Trans.*, 1881-4, expressed there in the form

$$(5)\qquad P_n(u) = \frac{1}{\pi}\int_0^{\pi} \frac{d\theta}{(C + S\cos\theta)^{n+\frac{1}{2}}};$$

and then, by differentiation,

$$(6)\qquad S^2\frac{dP_n}{dC} = -\left(n+\frac{1}{2}\right)\frac{1}{\pi}\int \frac{S(S + C\cos\theta)\,d\theta}{(S + C\cos\theta)^{n+\frac{3}{2}}}$$

$$= -\left(n+\frac{1}{2}\right)(CP_n - P_{n+1}) = \left(n^2 - \frac{1}{4}\right)\int P_n\, dC$$

and differentiating again

$$(7)\qquad \left(n - \frac{1}{2}\right)P_n = -P_n - C\frac{dP_n}{dC} + \frac{dP_{n+1}}{dC},$$

$$(8)\qquad \left(n+\frac{1}{2}\right)S^2 P_n = \left(n+\frac{1}{2}\right)C(CP_n - P_{n+1}) - \left(n+\frac{3}{2}\right)(CP_{n+1} - P_{n+2}),$$

$$(9)\qquad (2n+3)P_{n+2} - 4(n+1)CP_{n+1} + (2n+1)P_n = 0,$$

the recurring relation; equivalent to

$$(10)\qquad 4n(CP_n - P_{n+1}) = (2n-1)(P_{n-1} - P_{n+1}),$$

$$(11)\qquad R_n = \int P_n\, dC = -\frac{CP_n - P_{n+1}}{n - \frac{1}{2}} = \frac{P_{n+1} - P_{n-1}}{2n} = \frac{S^2\frac{dP_n}{dC}}{n^2 - \frac{1}{4}};$$

and other relations analogous to the former series for the zonal harmonic of integral order.

**12.** The change is made to the form employed by Macdonald, in *L. M. S.*, XXVI, by the substitution

(1) $$C + S\cos\theta = e^{\zeta} = \frac{PQ^2}{\gamma_1\gamma_2} = \frac{\gamma_1}{\gamma_2}\cos^2\omega + \frac{\gamma_2}{\gamma_1}\sin^2\omega,$$

(2) $$2.\operatorname{ch} u - \operatorname{ch}\zeta = 2C - C - S\cos\theta - \frac{1}{C + S\cos\theta} = \frac{S^2\sin^2\theta}{C + S\cos\theta},$$

(3) $$d\zeta = \frac{-S\sin\theta\, d\theta}{C + S\cos\theta}, \qquad \frac{d\zeta}{\sqrt{(2.\operatorname{ch} u - \operatorname{ch}\zeta)}} = \frac{-d\theta}{\sqrt{(C + S\cos\theta)}},$$

(4) $$P_n(u) = \frac{1}{\pi}\int_0^{\pi} \frac{d\theta}{(C + S\cos\theta)^{n+\frac{1}{2}}} = \frac{1}{\pi}\int_{-u}^{u} \frac{e^{-n\zeta}\, d\zeta}{\sqrt{(2.\operatorname{ch} u - \operatorname{ch}\zeta)}}.$$

The reduction is made to elliptic functions through

(5) $$\begin{cases} e^{\zeta} = (C + S)\cos^2\omega + (C - S)\sin^2\omega = e^u \Delta^2\omega = \dfrac{\operatorname{dn}^2 eG}{\gamma'}, \\ \gamma' = \dfrac{\gamma_2}{\gamma_1} = e^{-u}, \qquad \dfrac{1}{2}\theta = \omega = \operatorname{am} eG; \end{cases}$$

(6) $$e^{\frac{1}{2}(u+\zeta)} = \frac{\operatorname{dn} eG}{\gamma'}, \qquad \operatorname{sh}\frac{1}{2}(u + \zeta) = \frac{\gamma^2 \operatorname{cn}^2 eG}{2\gamma' \operatorname{dn} eG},$$

(7) $$e^{\frac{1}{2}(u-\zeta)} = \frac{1}{\operatorname{dn} eG}, \qquad \operatorname{sh}\frac{1}{2}(u - \zeta) = \frac{\gamma^2 \operatorname{sn}^2 eG}{2\operatorname{dn} eG},$$

(8) $$2\operatorname{ch} u - \operatorname{ch}\zeta = 4\operatorname{sh}\frac{1}{2}(u + \zeta)\operatorname{sh}\frac{1}{2}(u - \zeta) = \frac{\gamma^4 \operatorname{sn}^2 eG \operatorname{cn}^2 eG}{\gamma' \operatorname{dn}^2 eG},$$

(9) $$\frac{d\zeta}{\sqrt{(2.\operatorname{ch} u - \operatorname{ch}\zeta)}} = -2\sqrt{\gamma'}\, de\mathbf{G},$$

(10) $$P_n(u) = \frac{2\sqrt{\gamma'}}{\pi}\int_0^1 \left(\frac{\operatorname{dn}^2 eG}{\gamma'}\right)^n deG$$

beginning with

(11) $$P_0 = \sqrt{\gamma'}\,\frac{G}{\frac{1}{2}\pi}, \qquad P_1 = \frac{H}{\frac{1}{2}\pi\sqrt{\gamma'}},$$

and then proceeding with the sequence equation (9), § 11,

(12) $$P_2 = \frac{4}{3}CP_1 - \frac{1}{3}P_0, \qquad P_3 = \ldots,$$

all the P's $= 1$ at $u = 0$, but at $u = \infty$ they are all infinite.

Then, with $eG = V$,

(13) $$\left(n-\tfrac{1}{2}\right)R_n = P_{n+1} - CP_n = \frac{2\sqrt{\gamma'}}{\pi}\int_0^{eG}\left[\frac{\text{dn}^2 v}{\gamma'} - \frac{1}{2}\left(\gamma' + \frac{1}{\gamma'}\right)\right]\left(\frac{\text{dn}^2 v}{\gamma'}\right)^n dv$$

$$= \frac{\gamma^2}{\pi\sqrt{\gamma'}}\int^{V}(\text{cn}^2 v - \text{sn}^2 v)\left(\frac{\text{dn}^2 v}{\gamma'}\right)^n dv$$

starting with

(14) $$R_0 = \frac{G - H - (H - \gamma'^2 G)}{\frac{1}{2}\pi\sqrt{\gamma'}} = \frac{AQ}{\frac{1}{2}\pi\sqrt{(\gamma_1\gamma_2)}},$$

(15) $$2nR_n = P_{n+1} - P_{n-1},$$

starting with $n = 1$,

(16) $$2R_1 = P_2 - P_0 = \frac{4}{3}(CP_1 - P_x) = \frac{(1+\gamma'^2)H - 2\gamma'^2 G}{\frac{3}{2}\pi\gamma'\sqrt{\gamma'}};$$

and thence by the sequence equation (9), § **11**, to the series

(17) $$(2n+3)R_{n+1} - 4nCR_n + (2n-3)R_{n-1} = 0.$$

**15.** In a Quadric Transformation of these Elliptic Functions, to a new modulus

(1) $$\varkappa = \frac{1-\gamma'}{1+\gamma'} = \text{th}\,\frac{1}{2}u,$$

(2)
$$e^{\zeta} = \frac{\text{dn}^2 eG}{\gamma'} = \frac{1+\varkappa\,\text{sn}(1-2e)K}{1-\varkappa\,\text{sn}(1-2e)K},$$

$$\text{th}\,\frac{1}{2}\zeta = \varkappa\,\text{sn}(1-2e)K, \qquad \text{ch}\,\frac{1}{2}\zeta = \frac{\text{dn}\,2eK}{\varkappa'}, \qquad \text{sh}\,\frac{1}{2}\zeta = \frac{\varkappa\,\text{cn}\,2eK}{\varkappa'},$$

$$\gamma'\,\text{tn}^2 eG = \frac{\text{sh}\,\frac{1}{2}(u-\zeta)}{\text{sh}\,\frac{1}{2}(u+\zeta)} = \frac{\text{th}\,\frac{1}{2}u - \text{th}\,\frac{1}{2}\zeta}{\text{th}\,\frac{1}{2}u + \text{th}\,\frac{1}{2}\zeta}$$

$$= \frac{1-\text{sn}(1-2e)K}{1+\text{sn}(1-2e)K} = \frac{\text{dn}\,2eK - \text{cn}\,2eK}{\text{dn}\,2eK + \text{cn}\,2eK},$$

$$\frac{1}{2}d\zeta = -\varkappa\,\text{sn}\,2eK.d(2eK);$$

and

(3) $$\frac{d\zeta}{\sqrt{(2.\mathrm{ch}\,u - \mathrm{ch})\zeta}} = \frac{\frac{1}{2}d\zeta}{\sqrt{\left(\mathrm{ch}^2\frac{1}{2}u - \mathrm{ch}^2\frac{1}{2}\zeta\right)}}$$
$$= \frac{-\varkappa\,\mathrm{sn}\,2e\mathrm{K}.d(2e\mathrm{K})}{\frac{\varkappa}{\varkappa'}\mathrm{sn}\,2e\mathrm{K}} = -2\varkappa' de\,\mathrm{K},$$

(4) $$\int_{-u,\zeta}^{u}\frac{d\zeta}{\sqrt{(2.\mathrm{ch}\,u - \mathrm{ch}\,\zeta)}} = 2\varkappa'\mathrm{K}, \quad 2e\varkappa'\mathrm{K} = 2\mathrm{G}\sqrt{\gamma'}, \quad 2e\mathrm{G}\sqrt{\gamma'}; \quad \varkappa'\mathrm{K} = \mathrm{G}\sqrt{\gamma'}.$$

Then

(5)
$$r^2 = \gamma_1\gamma_2 e^{\zeta}, \qquad \frac{1}{2}(\gamma_1^2 + \gamma_2^2) = \gamma_1\gamma_2\,\mathrm{ch}\,u,$$

$$2ay\cos\theta = \frac{1}{2}(\gamma_1^2 + \gamma_2^2) - r^2 = \gamma_1\gamma_2(\mathrm{ch}\,u - e^{\zeta}),$$

$$\mathrm{ch}\,u - e^{\zeta} = \frac{1}{2}\left(\frac{1}{\gamma'} + \gamma'\right) - \frac{\mathrm{dn}^2 e\mathrm{G}}{\gamma'} = \frac{1+\varkappa^2}{\varkappa'^2} - \frac{\mathrm{dn}\,2e\mathrm{K} + \varkappa\,\mathrm{cn}\,2e\mathrm{K}}{\mathrm{dn}\,2e\mathrm{K} - \varkappa\,\mathrm{cn}\,2e\mathrm{K}}$$
$$= \frac{1 + \varkappa^2 - (\mathrm{dn}\,2e\mathrm{K} + \varkappa\,\mathrm{cn}\,2e\mathrm{K})^2}{\varkappa'^2}$$
$$= 2\,\frac{1 - \mathrm{dn}^2 2e\mathrm{K} - \varkappa\,\mathrm{cn}\,2e\mathrm{K}\,\mathrm{dn}\,2e\mathrm{K}}{\varkappa'^2},$$

$$\frac{\mathrm{M}}{2\sqrt{(\gamma_1\gamma_2)}} = \int_{-u}^{u}\frac{(\mathrm{ch}\,u - e^{\zeta})\,d\zeta}{\sqrt{(2.\mathrm{ch}\,u - \mathrm{ch}\,\zeta}}$$
$$= -2\int_0^1(1 - \mathrm{dn}^2 2e\mathrm{K} - \varkappa\,\mathrm{cn}\,2e\mathrm{K}\,\mathrm{dn}\,2e\mathrm{K})\frac{2\mathrm{K}\,de}{\varkappa'} = -4\frac{\mathrm{K}-\mathrm{E}}{\varkappa'},$$

$$\mathrm{E} = \int_0^1 \mathrm{dn}^2 2e\mathrm{K}.2\mathrm{K}\,de,$$

$$\mathrm{M} = -4\pi(\gamma_1 + \gamma_2)(\mathrm{K} - \mathrm{E}), \qquad \varkappa' = \frac{2\sqrt{(\gamma_1\gamma_2)}}{\gamma_1 + \gamma_2}.$$

**14.** The integral considered by Macdonald, *L. M. S.*, XXVI, p. 165,

(1) $$\mathrm{Q}_n(u) = \int_u^{\infty}\frac{e^{-n\zeta}d\zeta}{\sqrt{(2.\mathrm{ch}\,\zeta - \mathrm{ch}\,u)}}$$

is a zonal harmonic or toroidal function of the second kind, and of order $i = n - \frac{1}{2}$; and by the substitution

$$(2) \qquad e^{\zeta} = \mathrm{C} + \mathrm{S}\,\mathrm{ch}\,\theta', \qquad 2(\mathrm{ch}\,\zeta - \mathrm{ch}\,u) = \frac{\mathrm{S}^2\,\mathrm{sh}^2\,\theta'}{\mathrm{C} + \mathrm{S}\,\mathrm{ch}\,\theta'}$$

it takes the form employed by W. M. Hicks

$$(3) \qquad \mathrm{Q}_n(u) = \int_0^{\infty} \frac{d\theta'}{(\mathrm{C} + \mathrm{S}\,\mathrm{ch}\,\theta')^{n+\frac{1}{2}}}.$$

The reduction to the elliptic function relation is made through the substitution

$$(4) \quad e^{-\zeta} = \gamma' \,\mathrm{sn}^2 f\mathrm{G}', \qquad \gamma' = e^{-u}, \qquad \mathrm{Q}_n(u) = 2\sqrt{\gamma'}\int_0^1 (\gamma'\,\mathrm{sn}^2 f\mathrm{G}')^n\, df\mathrm{G}'.$$

Because

$$(5) \quad \left\{ \begin{aligned} & e^{-\frac{1}{2}(\zeta+u)} = \gamma'\,\mathrm{sn}\,f\mathrm{G}', \qquad e^{-\frac{1}{2}(\zeta-u)} = \mathrm{sn}\,f\mathrm{G}', \\ & 2.\mathrm{ch}\,\zeta - \mathrm{ch}\,u = \frac{\mathrm{cn}^2 f\mathrm{G}'\,\mathrm{dn}^2 f\mathrm{G}'}{\gamma'\,\mathrm{sn}^2 f\mathrm{G}'}, \qquad d\zeta = -\frac{2\,\mathrm{cn}\,f\mathrm{G}'\,\mathrm{dn}\,f\mathrm{G}'}{\mathrm{sn}\,f\mathrm{G}'}\,d(f\mathrm{G}'); \end{aligned} \right.$$

$$(6) \quad \int_{u,\zeta}^{\infty} \frac{d\zeta}{\sqrt{(2.\mathrm{ch}\,\zeta - \mathrm{ch}\,u)}} = 2\,\mathrm{G}\sqrt{\gamma'}, \qquad 2f\mathrm{G}'\sqrt{\gamma'}, \qquad \infty > \zeta > u.$$

Then in the Quadric Transformation

$$(7) \quad \left\{ \begin{aligned} & e^{-\zeta} = \gamma'\,\mathrm{sn}^2 f\mathrm{G}' = \frac{1 - \mathrm{dn}\,f\mathrm{K}'}{1 + \mathrm{dn}\,f\mathrm{K}'}, \qquad \mathrm{th}\,\frac{1}{2}\zeta = \mathrm{dn}\,f\mathrm{K}', \\ & \mathrm{sech}\,\frac{1}{2}\zeta = \varkappa'\,\mathrm{sn}\,f\mathrm{K}', \qquad \frac{1}{2}d\zeta = -\frac{\mathrm{cn}\,f\mathrm{K}'}{\mathrm{sn}\,f\mathrm{K}'}\,df\mathrm{K}', \\ & 2.\mathrm{ch}\,\zeta - \mathrm{ch}\,u = 4\left(\mathrm{ch}^2\frac{1}{2}\zeta - \mathrm{ch}^2\frac{1}{2}u\right) = 4\,\frac{\mathrm{cn}^2 f\mathrm{K}'}{\varkappa'^2\,\mathrm{sn}^2 f\mathrm{K}'}; \end{aligned} \right.$$

$$(8) \qquad \frac{d\zeta}{\sqrt{(2.\mathrm{ch}\,\zeta - \mathrm{ch}\,u)}} = -\varkappa'\,df\mathrm{K}';$$

$$(9) \quad \int_{u,\zeta}^{\infty} \frac{d\zeta}{\sqrt{(2.\mathrm{ch}\,\zeta - \mathrm{ch}\,u)}} = \varkappa'\mathrm{K}', \quad f\varkappa'\mathrm{K}'; \qquad \varkappa'\mathrm{K}' = 2\,\mathrm{G}'\sqrt{\gamma'}; \qquad \frac{\mathrm{K}'}{\mathrm{K}} = 2\,\frac{\mathrm{G}'}{\mathrm{G}}.$$

The corresponding elliptic function expressions of $\mathrm{R}_n(u)$ in (13), § **12**, irrespective of a constant factor, are then

$$(10) \qquad \mathrm{R}_n(u) = \int (\mathrm{P}_n, \mathrm{Q}_n)\, d\mathrm{C} = \int_{-u,u}^{u,\infty} e^{\pm n\zeta}\sqrt{(2.\mathrm{ch}\,u - \mathrm{ch}\,\zeta)\,d\zeta}$$

and so reducible in the same way to

$$(11)\qquad R_n(u) = \int_0^1 \left(\frac{\mathrm{dn}^2 eG}{\gamma'}\right)^n \mathrm{sn}^2 eG\,\mathrm{cn}^2 eG\,.\,G\,de$$

or

$$\int^1 (\gamma' \mathrm{sn}^2 fG')^n \mathrm{cn}^2 fG'\,\mathrm{dn}^2 fG'\,.\,G'\,df.$$

**15.** Change $\zeta$ into $i\omega$ in (1), § **14**, to obtain another form of Q, not derivable however directly from the form in (1), (10), § **14**.

$$(1)\qquad \begin{cases} Q_n(u) = \displaystyle\int_0^\pi \frac{\cos n\omega\, d\omega}{\sqrt{(2C - \cos\omega)}}, \\ \displaystyle\frac{dQ_n}{dC} = -\int \frac{\cos n\omega\, d\omega}{(2C - \cos\omega)^{\frac{3}{2}}} = \left(n + \frac{1}{2}\right)\frac{CQ_n - Q_{n+1}}{S^2}; \end{cases}$$

and then, with

$$(2)\qquad \begin{cases} \frac{1}{2}(\pi - \omega) = \mathrm{am}\, f\mathrm{K}', \qquad -\dfrac{d\omega}{\sqrt{(2C - \cos\omega)}} = \varkappa'\, df\mathrm{K}', \\ \varkappa' = \mathrm{sech}\,\frac{1}{2}u, \qquad \varkappa = \mathrm{th}\,\frac{1}{2}u = \dfrac{1-\gamma'}{1+\gamma'}; \end{cases}$$

$$(3)\qquad Q_n(u) = \int_0^1 \cos n(\pi - 2\,\mathrm{am}\, f\mathrm{K}')\varkappa'\, df\mathrm{K}'.$$

As in the *Am. J. Math.*, XXXIX, p. 375, this is convertible in to the form (3), § **14**, by the substitution

$$(4)\qquad \sin\frac{1}{2}\omega = \frac{\sqrt{(C + S\,\mathrm{ch}\,\theta')}}{\mathrm{sh}\,\frac{1}{2}u + \mathrm{ch}\,\frac{1}{2}u\,\mathrm{ch}\,\theta'}, \qquad \cos\frac{1}{2}\omega = \frac{\mathrm{ch}\,\frac{1}{2}u\,\mathrm{sh}\,\theta'}{\mathrm{sh}\,\frac{1}{2}u + \mathrm{ch}\,\frac{1}{2}u\,\mathrm{ch}\,\theta'};$$

$$(5)\qquad \frac{d\omega}{\sqrt{(2C - \cos\omega)}} = \frac{d\theta'}{\sqrt{(C + S\,\mathrm{ch}\,\theta')}}.$$

Here again, substitute

$$(6)\qquad \mathrm{ch}\,\frac{1}{2}\theta' = \sec\xi, \qquad \mathrm{sh}\,\frac{1}{2}\theta' = \mathrm{tang}\,\xi, \qquad d\theta' = 2\sec\xi\, d\xi, \qquad e^{-u} = \gamma';$$

$$(7)\qquad C + S\,\mathrm{ch}\,\theta' = e^u\,\mathrm{ch}^2\frac{1}{2}\theta' - e^{-u}\,\mathrm{sh}^2\frac{1}{2}\theta' = \frac{\sec^2\xi\,\Delta^2\xi}{\gamma'},$$

and

(8) $$\frac{d\theta'}{\sqrt{(C+S\,\mathrm{ch}\,\theta')}} = 2\sqrt{\gamma'}\frac{d\xi}{\Delta\xi} = 2\sqrt{\gamma'}\,d(f\mathrm{G}'), \qquad \xi = \mathrm{am}\, f\mathrm{G}';$$

(9) $$C + S\,\mathrm{ch}\,\theta' = \frac{\mathrm{dn}^2 f\mathrm{G}'}{\gamma'\,\mathrm{cn}^2 f\mathrm{G}'} = \frac{1}{\gamma'\,\mathrm{sn}^2(1-f)\mathrm{G}'},$$

and then, as in (10), § **12**,

(10) $$Q_n(u) = 2\sqrt{\gamma'}\int_0^1 [\gamma'\,\mathrm{sn}^2(1-f)\mathrm{G}']^n \mathrm{G}'\,df = 2\sqrt{\gamma'}\int (\gamma'\,\mathrm{sn}^2 f\mathrm{G}')^n \mathrm{G}'\,df.$$

A quadric transformation connects up this $\omega$ and $\theta'$ through the relation,

(11) $$\mathrm{dn}\, f\mathrm{K}' = \sqrt{\left(1 - \varkappa'^2 \sin^2\frac{1}{2}\omega\right)} = \frac{\mathrm{ch}\frac{1}{2}u + \mathrm{sh}\frac{1}{2}u\,\mathrm{ch}\,\theta'}{\mathrm{sh}\frac{1}{2}u + \mathrm{ch}\frac{1}{2}u\,\mathrm{ch}\,\theta'},$$

(12) $$\frac{1 - \mathrm{dn}\, f\mathrm{K}'}{1 + \mathrm{dn}\, f\mathrm{G}'} = e^{-u}\,\mathrm{th}^2\frac{1}{2}\theta' = \gamma'\,\mathrm{sn}^2 f\mathrm{G}',$$

(13) $$\mathrm{sn}\, f\mathrm{K}' = \frac{(1+\gamma')\,\mathrm{sn}\, f\mathrm{G}'}{1+\gamma'\,\mathrm{sn}^2 f\mathrm{G}'}, \qquad \mathrm{cn}\, f\mathrm{K}' = \frac{\mathrm{cn}\, f\mathrm{G}'\,\mathrm{dn}\, f\mathrm{G}'}{1+\gamma'\,\mathrm{sn}^2 f\mathrm{G}'},$$

the usual formulas of the complementary Quadric Transformation; and

(14) $$\sqrt{\varkappa}\,\mathrm{tn}\, f\varkappa' = \frac{\gamma\,\mathrm{sn}\, f\mathrm{G}'}{\mathrm{cn}\, f\mathrm{G}'\,\mathrm{dn}\, f\mathrm{G}'} = \frac{\mathrm{cn}(1-f)\mathrm{G}'}{\mathrm{cn}\, f\mathrm{G}'}$$

$$= \sqrt{\left(\frac{\mathrm{dn}\, 2f\mathrm{G}' - \mathrm{cn}\, 2f\mathrm{G}'}{\mathrm{dn}\, 2f\mathrm{G}' + \mathrm{cn}\, 2f\mathrm{G}'}\right)} = \sqrt{\frac{1-\mathrm{sn}(1-2f)\mathrm{G}'}{1+\mathrm{sn}(1-2f)\mathrm{G}'}},$$

(15) $$\frac{\mathrm{dn}\, f\mathrm{K}'}{\sqrt{\varkappa}} = \sqrt{\frac{1+\gamma'\,\mathrm{sn}(1-2f)\mathrm{G}'}{1-\gamma'\,\mathrm{sn}(1-2f)\mathrm{G}'}} = \sqrt{\left(\frac{1+\gamma'}{1-\gamma'}\right)}\,\frac{1-\gamma'\,\mathrm{sn}^2 f\mathrm{G}'}{1+\gamma'\,\mathrm{sn}^2 f\mathrm{G}'}.$$

**16.** The reduction can be carried out in the same manner as for the toroidal function

(1) $$\mathrm{P}_n(u) = \int \frac{e^{\pm n\zeta}\,d\zeta}{\sqrt{(2.\mathrm{ch}\,\zeta + \cos\theta)}},$$

solution of the D. E., with $\cos\theta = \mu$,

(2) $$\frac{d}{d\mu}(1-\mu^2)\frac{d\mathrm{P}}{d\mu} + \left(n^2 - \frac{1}{4}\right)\mathrm{P} = 0.$$

through the substitution

$$(3)\qquad e^{\frac{1}{2}\zeta} = \cot\frac{1}{2}\varphi = \frac{\sin\varphi}{1-\cos\varphi}, \qquad \frac{1}{2}d\zeta = -\frac{\partial\varphi}{\sin\varphi},$$

$$(4)\qquad \left\{\begin{aligned} 2.\mathrm{ch}\,\zeta + \cos\theta &= e^{\zeta} + 2\cos\theta + e^{-\zeta} = \frac{1+\cos\varphi}{1-\cos\varphi} + 2\cos\theta + \frac{1-\cos\varphi}{1+\cos\varphi} \\ &= 4\,\frac{1-\sin^2\frac{1}{2}\theta\sin^2\varphi}{\sin^2\varphi} = 4\,\frac{\Delta^2(\varphi,\varkappa)}{\sin^2\varphi}, \\ \varkappa &= \sin\frac{1}{2}\theta, \end{aligned}\right.$$

$$(5)\qquad \frac{du}{\sqrt{(2.\mathrm{ch}\,\zeta + \cos\theta)}} = -\frac{d\varphi}{\Delta\varphi} = -d(e\mathrm{K}), \qquad \varphi = \mathrm{am}\,e\mathrm{K},$$

$$(6)\qquad \int_{u,\ \infty}^{\infty,\ -u} \frac{d\zeta}{\sqrt{(2.\mathrm{ch}\,\zeta + \cos\theta)}} = e\mathrm{K}, \qquad \int_{0,\ -\infty}^{\infty,\ 0} = \mathrm{K}.$$

$$(7)\qquad \sqrt{(2\mathrm{c.h}\,\zeta + \cos\theta)} = 2\,\frac{\mathrm{dn}\,e\mathrm{K}}{\mathrm{sn}\,e\mathrm{K}} = \frac{2\varkappa'}{\mathrm{cn}\,(1-e)\mathrm{K}}.$$

In the quadric transformation

$$(8)\qquad \left\{\begin{aligned} \frac{\mathrm{dn}\,e\mathrm{K}}{\sqrt{\varkappa'}} &= \frac{\mathrm{dn}\,\frac{1}{2}e\mathrm{G} + \mathrm{dn}\left(1-\frac{1}{2}e\right)\mathrm{G}}{2\sqrt{\gamma'}}, \\ \frac{\mathrm{dn}\,(1-e)\mathrm{K}}{\sqrt{\varkappa'}} &= \frac{\mathrm{dn}\,\frac{1}{2}(1-e)\mathrm{G} + \mathrm{dn}\,\frac{1}{2}(1+e)\mathrm{G}}{2\sqrt{\gamma'}}, \end{aligned}\right.$$

$$(9)\qquad \mathrm{th}^2\frac{1}{2}\zeta = \frac{\mathrm{ch}\,\frac{1}{2}\zeta - 1}{\mathrm{ch}\,\frac{1}{2}\zeta + 1} = \frac{1-\mathrm{sn}\,2e\mathrm{K}}{1+\mathrm{sn}\,2e\mathrm{K}} = \frac{\mathrm{cn}^2\,\frac{1}{2}(1+e)\mathrm{K}}{\mathrm{cn}^2\,\frac{1}{2}(1-e)\mathrm{K}},$$

$$(10)\qquad \left\{\begin{aligned} \frac{1+\varkappa\,\mathrm{sn}\,e\mathrm{K}}{1-\varkappa\,\mathrm{sn}\,e\mathrm{K}} &= \frac{1+\varkappa}{1-\varkappa}\left[\frac{1-\varkappa\,\mathrm{sn}^2\,\frac{1}{2}(1-e)\mathrm{K}}{1+\varkappa\,\mathrm{sn}^2\,\frac{1}{2}(1-e)\mathrm{K}}\right] = \frac{\mathrm{dn}^2\,\frac{1}{2}(1-e)\mathrm{G}}{\gamma'}, \\ \mathrm{th}^2\,\frac{1}{2}\zeta &= \gamma'\,\mathrm{tn}^2\,\frac{1}{2}(1-e)\mathrm{G}, \qquad \gamma' = \mathrm{tang}^2\,\frac{1}{4}(\pi-\theta), \end{aligned}\right.$$

$$(11)\qquad \mathrm{sn}\,e\mathrm{K} = \mathrm{sech}\,\frac{1}{2}\zeta, \qquad \mathrm{cn}\,e\mathrm{K} = \mathrm{th}\,\frac{1}{2}\zeta, \qquad \mathrm{dn}\,e\mathrm{K} = \sqrt{\frac{\mathrm{ch}\,\zeta + \cos\theta}{\mathrm{ch}\,\zeta + 1}},$$

and

$$(12)\quad \begin{cases} \operatorname{sn}(1-e)K = \sqrt{\dfrac{\operatorname{ch}\zeta - 1}{\operatorname{ch}\zeta + \cos\theta}}, \qquad \operatorname{cn}(1-e)K = \sqrt{\dfrac{1+\cos\theta}{\operatorname{ch}\zeta + \cos\theta}}, \\ \operatorname{dn}(1-e)K = \cos\frac{1}{2}\theta\sqrt{\dfrac{\operatorname{ch} + 1}{\operatorname{ch}\zeta + \cos\theta}}. \end{cases}$$

**17.** Then there are the definitions of the toroidal function in the form

$$(1)\quad \begin{cases} P_n(\cos\theta) = \dfrac{1}{\pi}\displaystyle\int_{-\theta}^{\theta} \dfrac{\cos n\varphi\, d\varphi}{\sqrt{(2.\cos\varphi - \cos\theta)}}, \\ R_n(\cos\theta) = \dfrac{1}{\pi}\displaystyle\int \sqrt{(2.\cos\varphi - \cos\theta)}\cos n\varphi\, d\varphi, \end{cases}$$

leading to Mehler's definition of the zonal harmonic of order $m$ when $n$ is replaced by $m + \frac{1}{2}$.

Here in a reduction to the elliptic function we substitute, as in pendulum motion,

$$(2)\qquad \sin\frac{1}{2}\varphi = \sin\frac{1}{2}\theta\sin\chi, \qquad \cos\frac{1}{2}\theta = \Delta\chi, \qquad \varkappa = \sin\frac{1}{2}\theta,$$

$$(3)\qquad 2.\cos\varphi - \cos\theta = 4.\sin^2\frac{1}{2}\theta - \sin^2\frac{1}{2}\varphi = 4\sin^2\frac{1}{2}\theta\,\Delta^2\chi,$$

$$(4)\qquad \frac{1}{2}d\varphi = \frac{\sin\frac{1}{2}\theta\cos\chi\, d\chi}{\Delta\chi}, \qquad \frac{d\varphi}{\sqrt{(2.\cos\varphi - \cos\theta)}} = \frac{d\chi}{\Delta\chi},$$

and then $\cos m\varphi$ must be expanded in power of $\cos^2\frac{1}{2}\varphi$ to obtain a resemblance with previous results.

When $n$ is half an odd integer,

$$n = m + \frac{1}{2}, \qquad \cos n\varphi = \cos m\varphi\cos\frac{1}{2}\varphi - \sin m\varphi\sin\frac{1}{2}\varphi,$$

and the integration can be effected algebraically, to the ordinary form of the zonal harmonic of Legendre.

The solution of the D. E. for the Mehler function,

$$(5)\qquad \frac{d}{dC}(C^2 - 1)\frac{dM}{dC} = -\left(m^2 + \frac{1}{4}\right)M,$$

is obtained from Macdonald's form of $P_n$ and $Q_n$ by a change of $n^2$ into $-m^2$, and will then be given by the form

$$(6)\qquad M(u) = \int_{-u,\,u}^{u,\,\infty} \frac{\cos m\xi\, d\zeta}{\sqrt{(2.\mathrm{ch}\, u - \mathrm{ch}\,\zeta)}},$$

reducible as before to definite elliptic integrals (J.-W. Hobson, *Phil. Trans.*, 1896. — W. Burnside, *Messenger of Mathematics*, t. XIV, p. 122. — W.-D. Niven, *L. M. S.*, t. XXVIII, p. 218; *Am. J. of Math.*, t. XXXIX, p. 371).

The verification of the D. E. (5) is troublesome because of the infinity arising at the limits, and it requires the same treatment as Macdonald's C and D in § 2.

**18.** In Sonine's integral, *Math. Ann.*, t. XVI, quoted by Macdonald in *L. M. S.*, t. XXVI, p. 160,

$$(1)\qquad J = \int_0^\infty J_n(\varkappa r)\, J_n(\varkappa r')\, d\varkappa = \frac{1}{\frac{1}{2}\pi (rr')^n} \int_0^{r'} \frac{x^{2n}\, dx}{\sqrt{(r^2 - x^2 . r'^2 - x^2)}}.$$

Here we substitute

$$(2)\qquad \varkappa\sqrt{(rr')} = 2e^{\frac{1}{2}\zeta}, \qquad \varkappa x = 2, \qquad x = \sqrt{(rr')}\, e^{-\frac{1}{2}\zeta}, \qquad \frac{r}{r'} = e^u;$$

and introducing the Fourier function in place of Bessel's,

$$(3)\qquad \sqrt{(rr')}\, J = \int_{-\infty}^{\infty} J_n\left[2e^{\frac{1}{2}(\zeta+u)}\right] J_n\left[2e^{\frac{1}{2}(\zeta-u)}\right] e^{\frac{1}{2}\zeta}\, d\zeta$$

$$= \int F_n(e^{\zeta+u})\, F_n(e^{\zeta-u})\, e^{(n+\frac{1}{2})\zeta}\, d\zeta$$

$$= \frac{1}{\pi}\int_u^\infty \frac{e^{-n\zeta}\, d\zeta}{\sqrt{(2.\mathrm{ch}\,\zeta - \mathrm{ch}\, u)}} = Q_n(u),$$

$$(4)\qquad \sqrt{(rr')} = \frac{2a}{\sqrt{(2.\mathrm{ch}\, u - \cos v)}}, \qquad \varkappa a = e^{\frac{1}{2}\zeta}\sqrt{(2.\mathrm{ch}\, u - \cos v)},$$

$$(5)\qquad r^2 - x^2 = rr'(e^u - e^{-\zeta}), \qquad r'^2 - x^2 = rr'(e^{-u} - e^{-\zeta}),$$

$$(6)\qquad \frac{dx}{\sqrt{(r^2 - x^2 . r'^2 - x^2)}} = \frac{-d\zeta}{2\sqrt{(rr')}\sqrt{(2.\mathrm{ch}\,\zeta - \mathrm{ch}\, u)}}.$$

**19.** For the solid angle $\Omega$, and its dissections, refer to the *Trans-*

*American Math. Society*, oct. 1907, p. 54, for the definitions and results. Begin, as in *Phil. Trans.*, 1919, § 13, *Electromagnetic Integrals*, with the form

(1) $$\Phi = \Omega(\mathrm{PY}) = \int_0^{2\pi} \frac{\mathrm{PQ}^2}{\mathrm{PY}^2} \frac{b\,d\theta}{\mathrm{PQ}},$$

obtained for the perimeter $\Phi$ of the sphero-conic, reciprocal to the cone on the circular base AB, vertex at P, of which the solid angle

$$\Omega = 2\pi - \Phi.$$

Here PY is the perpendicular on the tangent of the circle at P; and in Maxwell's notation with $y = \mathrm{A}$,

$$\mathrm{PA} = r_1, \qquad \mathrm{PB} = r_2, \qquad \theta = 2\omega,$$

(2) $$\mathrm{PQ}^2 = r_1 r_2 e^\zeta, \qquad \mathrm{PA}^2 = r_1 r_2 e^u, \qquad \mathrm{PB}^2 = r_1 r_2 e^{-u},$$

(3) $$\mathrm{PY}^2 = \mathrm{PQ}^2 - \mathrm{QY}^2 = r_1 r_2 e^\zeta - \mathrm{A}^2 \sin^2\theta,$$

(4) $$4\mathrm{A}a \sin^2 \frac{1}{2}\theta = \mathrm{PA}^2 - \mathrm{PQ}^2 = r_1 r_2 (e^u - e^\zeta),$$

(5) $$4\mathrm{A}a \cos^2 \frac{1}{2}\theta = \mathrm{PQ}^2 - \mathrm{PB}^2 = r_1 r_2 (e^\zeta - e^{-u}),$$

(6) $$4a^2 . \mathrm{QY}^2 = 4\mathrm{A}^2 a^2 \sin^2\theta = 2 r_1^2 r_2^2 (\mathrm{ch}\, u + \mathrm{ch}\, \zeta) e^\zeta$$
$$= 2 r_1 r_2 (\mathrm{ch}\, u - \mathrm{ch}\, \zeta) \mathrm{PQ}^2,$$

(7) $$\frac{\mathrm{QY}^2}{\mathrm{PQ}^2} = \frac{r_1 r_2}{2a^2}(\mathrm{ch}\, u - \mathrm{ch}\, \zeta) = \frac{\mathrm{ch}\, u - \mathrm{ch}\, \zeta}{\mathrm{ch}\, u - \cos v}, \qquad \frac{\mathrm{PY}^2}{\mathrm{PQ}^2} = \frac{\mathrm{ch}\, \zeta - \cos v}{\mathrm{ch}\, u - \cos v}.$$

With another dissection of $\Omega$ into elements by QNQ′ slices,

(8) $$\Omega(\mathrm{PN}) = \int_0^{2\pi} \frac{\mathrm{QN}^2}{\mathrm{PN}^2} \frac{b\theta}{\mathrm{PQ}}.$$

(9) $$\mathrm{PN}^2 = \mathrm{PQ}^2 - \mathrm{QN}^2 = r_1 r_2 e^\zeta - a^2 \sin^2\theta$$
$$= \mathrm{PQ}^2 \left[ 1 - \frac{\frac{1}{2} r_1 r_2}{\mathrm{A}^2} (\mathrm{ch}\, u - \mathrm{ch}\, \zeta) \right]$$
$$= \mathrm{PQ}^3 \left[ 1 - \frac{(\mathrm{ch}\, u - \cos v)(\mathrm{ch}\, u - \mathrm{ch}\, \zeta)}{\mathrm{sh}^2 u} \right]$$
$$= \frac{\mathrm{PQ}^2}{\mathrm{sh}^2 u} (\mathrm{ch}\, u - \cos v)(\mathrm{ch}\, \zeta - \cos v').$$

(10) $$\cos v' = \frac{1 - \mathrm{ch}\, u \cos v}{\mathrm{ch}\, u - \cos v}, \qquad \sin v' = \frac{\mathrm{sh}\, u \sin u}{\mathrm{ch}\, u - \cos v}, \qquad \mathrm{tang}\, \frac{1}{2} v\, \mathrm{tang}\, \frac{1}{2} v' = \mathrm{th}\, \frac{1}{2} u,$$

interpreted geometrically in paragraph **25**, and then with

$$\frac{1}{2}(\pi - v) = \operatorname{am} f\mathrm{K}', \qquad \frac{1}{2}(\pi - v') = \operatorname{am}(1-f)\mathrm{K}'.$$

(11) $$\Omega(\mathrm{PN}) = \int_0^u \sqrt{\left(\frac{\operatorname{ch} u - \operatorname{ch}\zeta}{\operatorname{ch} u - \cos v}\right)} \frac{2\sin v\, d\zeta}{\operatorname{ch}\zeta - \cos v'}.$$

To connect up $\Omega(\mathrm{PN})$ and $\Omega(\mathrm{PY})$, the Addition Theorem of the *E. I.*, III is required; so take

(12) $$\mathrm{I} = \sin^{-1} \frac{\sin\theta}{\sqrt{(\operatorname{ch}\zeta - \cos v . \operatorname{ch}\zeta - \cos v')}}$$

$$= \cos^{-1} \frac{\sin u}{\sqrt{(\operatorname{ch} u - \cos v)}} \sqrt{\left(\frac{\operatorname{ch} u - \operatorname{ch}\zeta}{\operatorname{ch}\zeta - \cos v . \operatorname{ch}\zeta - \cos v'}\right)}$$

then

(13) $$\frac{d\mathrm{I}}{d\zeta} = \sqrt{\left(\frac{\operatorname{ch} u - \cos v}{\operatorname{ch} u - \operatorname{ch}\zeta}\right)} \frac{\frac{1}{2}\sin v}{\operatorname{ch}\zeta - \cos v} + \sqrt{\left(\frac{\operatorname{ch} u - \operatorname{ch}\zeta}{\operatorname{ch} u - \cos v}\right)} \frac{\frac{1}{2}\sin v}{\operatorname{ch}\zeta - \cos v'}$$

and integrating between $0 < \zeta < u$ and multiplying by 4

(14) $$2\pi = \Omega(\mathrm{PY}) + \Omega(\mathrm{PN}).$$

(15) $$\mathrm{PQ}^2 = r^2 = \mathrm{A}^2 + 2\mathrm{A}a\cos\theta + a^2 + b^2 = r_1 r_2 e^{\zeta},$$

(16) $$-2\mathrm{A}a\sin\theta\, d\theta = r_1 r_2 e^{\zeta} d\zeta = \mathrm{PQ}^2 d\zeta.$$

(17) $$2\mathrm{A}a\sin\theta = \sqrt{(r_1 r_2)}\sqrt{(2 . \operatorname{ch} u - \operatorname{ch}\zeta)}\,\mathrm{PQ},$$

(18) $$\frac{d\theta}{\mathrm{PQ}} = -\frac{1}{\sqrt{(r_1 r_2)}} \frac{d\zeta}{\sqrt{(2 . \operatorname{ch} u - \operatorname{ch}\zeta)}},$$

(19) $$\frac{b\,d\theta}{\mathrm{PQ}} = -\frac{\sin v}{\sqrt{(\operatorname{ch} u . \cos v)}} \frac{d\zeta}{\sqrt{(2 . \operatorname{ch} u - \operatorname{ch}\zeta)}},$$

(20) $$\Omega(\mathrm{PY}) = \int_0^u \sqrt{\left(\frac{\operatorname{ch} u - \cos v}{\operatorname{ch} u - \operatorname{ch}\zeta}\right)} \frac{2\sin v\, d\zeta}{\operatorname{ch}\zeta - \cos v},$$

and so this $\Omega$ is seen to satisfy Laplace's equation, in the procedure of § **5**, $f = 1$.

The substitution

(21) $$\operatorname{ch}\zeta = s, \qquad \frac{d\zeta}{\sqrt{(2 . \operatorname{ch} u - \operatorname{ch}\zeta)}} = \frac{\sqrt{2}\, ds}{\sqrt{\mathrm{S}}},$$

$$\mathrm{S} = 4 . s_1 - s . s - s_2 . s - s_3, \qquad s_1 = \operatorname{ch} u, \qquad s_2 = 1, \qquad s_3 = -1,$$

$$\sigma = \cos v, \qquad -\Sigma = 4\sin^2 v(\operatorname{ch} u - \cos v), \qquad \varkappa^2 = \frac{s_1 - s_2}{s_1 - s_3} = \operatorname{th}^2 \frac{1}{2} u,$$

brings back the form

(22) $$\Phi = 4\int_{s_2}^{s_1} \frac{\frac{1}{2}\sqrt{-\Sigma}}{s-\sigma}\frac{ds}{\sqrt{S}} = 2\pi f + 4\mathrm{K}\,\mathrm{zn}\,f\mathrm{K}';$$

(23) $$\left\{\begin{aligned} &\frac{s-s_2}{s_1-s_2} = \frac{\mathrm{sh}^2\frac{1}{2}\zeta}{\mathrm{sh}^2\frac{1}{2}u} = \mathrm{cn}^2 2e\mathrm{K}, \qquad \frac{s-s_3}{s_1-s_3} = \frac{\mathrm{ch}^2\frac{1}{2}\zeta}{\mathrm{ch}^2\frac{1}{2}u} = \mathrm{dn}^2 2e\mathrm{K},\\ &\mathrm{th}\frac{1}{2}\zeta = \varkappa\,\mathrm{sn}(1-2e)\mathrm{K};\end{aligned}\right.$$

(24) $$\frac{s_2-\sigma}{s_2-s_3} = \sin^2\frac{1}{2}v = \mathrm{cn}^2 f'\mathrm{K}, \qquad \frac{1}{2}(\pi - v) = \mathrm{am}\,f\mathrm{K}'.$$

This I is the $\mathrm{I}_9$ of the *Trans. A. M. S.*, 1907, where

(25) $$\mathrm{I}_9 = \sin^{-1}\frac{b\sin\theta.\mathrm{PQ}}{\mathrm{PY.PN}} = \cos^{-1}\frac{\mathrm{R}}{\mathrm{PY.PN}},$$

(26) $$\mathrm{R} = \mathrm{A}a\cos^2\theta + (\mathrm{A}^2 + a^2 + b^2)\cos\theta + \mathrm{A}a = \frac{\mathrm{A}}{a}\mathrm{ND.NE}.$$

and then, by a straightforward differentiation,

(27) $$\begin{aligned}\frac{d\mathrm{I}_9}{d\theta} &= -\frac{\dfrac{\mathrm{R}}{\mathrm{PY.PN}}\left(\dfrac{1}{\mathrm{R}}\dfrac{d\mathrm{R}}{d\theta} - \dfrac{1}{2}\dfrac{d.\mathrm{PY}^2}{\mathrm{PY}^2.d\theta} - \dfrac{1}{2}\dfrac{d.\mathrm{PN}^2}{\mathrm{PN}^2.d\theta}\right)}{\dfrac{b\sin\theta.\mathrm{PQ}}{\mathrm{PY.PN}}}\\ &= \left(\frac{\mathrm{PQ}^2}{\mathrm{PY}^2} + \frac{\mathrm{QN}^2}{\mathrm{PN}^2}\right)\frac{b}{\mathrm{PQ}}\\ &= \frac{d\Omega(\mathrm{PY})}{d\theta} + \frac{d\Omega(\mathrm{PN})}{d\theta},\end{aligned}$$

after reduction.

Here with $s$ as variable, and $\cos v' = \sigma'$,

(28) $$\left\{\begin{aligned}\Omega(\mathrm{PN}) &= 4\int_{s_2}^{s_1}\frac{s_1-s}{s-\sigma'}\sqrt{\left(\frac{s_2-\sigma'.\sigma'-s_3}{s_1-\sigma'}\right)}\frac{ds}{\sqrt{S}}\\ &= 4\int\frac{\frac{1}{2}\sqrt{-\Sigma'}}{s-\sigma'}\frac{ds}{\sqrt{S}} - 4\sqrt{\left(\frac{s_2-\sigma'.\sigma'-s_3}{s_1-s_3.s_1-\sigma'}\right)}\int\frac{\sqrt{(s_1-s_3)}\,ds}{\sqrt{S}}\\ &= 2\pi f' + 4\mathrm{K}\,\mathrm{zn}\,f'\mathrm{K}' - 4\mathrm{K}\varkappa'^2\frac{\mathrm{sn}\,f'\,\mathrm{cn}\,f'}{\mathrm{dn}\,f'}\\ &= 2\pi(1-f) - 4\mathrm{K}\,\mathrm{zn}\,f\mathrm{K}',\\ &\qquad f + f' = 1.\end{aligned}\right.$$

This is $\Omega(1-f)$, so that

$$(29)\qquad \Omega(1-f)+\Omega(f)=2\pi-4\mathrm{K}\,\mathrm{zn}\,f\mathrm{K}'-4\mathrm{K}\,\mathrm{zn}\,f'\mathrm{K}'$$
$$=2\pi-4\mathrm{K}\,\kappa'^2\,\mathrm{sn}\,f\mathrm{K}'\,\mathrm{sn}\,f'\mathrm{K}'=2\pi-\mathrm{P}\frac{b}{a}.$$

**20.** Sometimes $r$ is a convenient variable to use. Then for any point Q on the circle AB at angular distance $\mathrm{AOQ}=\theta=2\omega$ from A, and with $\mathrm{AP}=r_1$, $\mathrm{BP}=r_2$,

$$(1)\quad \begin{cases} r_1^2-r_2^2=4ay, \qquad r_1^2+r_2^2=2(x^2+y^2+a^2), \\ \mathrm{PQ}^2=r^2=x^2+y^2+2ay\cos\theta+a^2=r_1^2\cos^2\omega+r_2^2\sin^2\omega=r_1^2\,\Delta^2(\omega,\gamma), \\ \gamma'=\dfrac{r_2}{r_1}, \qquad r_1^2-r^2=(r_1^2-r_2^2)\sin^2\omega, \qquad r^2-r_2^2=(r_1^2-r_2^2)\cos^2\omega, \\ d\theta=2\,d\omega=\dfrac{dr^2}{\sqrt{(r_1^2-r^2.\,r^2-r_2^2)}}, \end{cases}$$

$$(2)\qquad \frac{d\theta}{\mathrm{PQ}}=\frac{d\theta}{r}=\frac{2\,dr}{\sqrt{(r_1^2-r^2.\,r^2-r_2^2)}}=\frac{2\,d\omega}{r_1\,\Delta\omega}=\frac{2\,d\mathrm{G}}{r_1}.$$

Applied to the integral for Maxwell's M,

$$(3)\qquad -\mathrm{M}=\int_0^{2\pi}\frac{2\pi a y\cos\theta\,d\theta}{\mathrm{PQ}}=\pi\int(x^2+y^2+a^2-r^2)\frac{d\theta}{\mathrm{PQ}}$$
$$=\frac{1}{2}\pi(r_1^2+r_2^2)\int\frac{d\theta}{\mathrm{PQ}}-\pi\int\mathrm{PQ}\,d\theta$$
$$=\pi(r_1^2+r_2^2)\int_{r_2}^{r_1}\frac{dr}{\sqrt{\mathrm{R}}}-2\pi\int\frac{r^2\,dr}{\sqrt{\mathrm{R}}}\qquad(\mathrm{R}=r_1^2-r^2.\,r^2-r_2^2)$$
$$=\pi\frac{r_1^2+r_2^2}{r_1}\int_0^{\frac{1}{2}\pi}\frac{d\omega}{\Delta\omega}-2\pi r_1\int\Delta\omega\,d\omega,$$

$$(4)\qquad \frac{\mathrm{M}}{\pi r_1}=2\mathrm{H}-(1+\gamma'^2)\mathrm{G},$$

as before.

With Minchin's dissection of the circle AB into sectors radiating from M (*Phil. Mag.*, Feb. 1894) and taking $r$ as the variable,

(5) $$\Omega(\mathrm{MQ}) = 4\int_{r_2}^{r_1} \frac{\frac{1}{2}(a^2 - \mathrm{A}^2)}{r^2 - b^2}\frac{b\,dr}{\sqrt{\mathrm{R}}}$$

and with a new variable $t$,

(6) $$\left\{\begin{aligned} \frac{r^2}{r_1 r_2} &= e^{\zeta} = \frac{t - t_3}{\sqrt{(t_1 - t_3 \,.\, t_2 - t_3)}}, \\ \frac{r_1}{r_2} &= e^{u} = \frac{t_1 - t_3}{\sqrt{(\qquad)}}, \\ \frac{r_2}{r_1} &= e^{-u} = \frac{t_2 - t_3}{\sqrt{(\qquad)}}, \end{aligned}\right.$$

(7) $$\left\{\begin{aligned} \frac{r_1^2 - r^2}{r_1 r_2} &= \frac{t_1 - t}{\sqrt{(\qquad)}} = e^{u} - e^{\zeta}, \\ \frac{r^2 - r_2^2}{r_1 r_2} &= \frac{t - t_2}{\sqrt{(\qquad)}} = e^{\zeta} - e^{-u}, \end{aligned}\right.$$

(8) $$\left\{\begin{aligned} \frac{b^2}{r_1 r_2} &= \frac{\tau - t_3}{\sqrt{(\qquad)}} = e^{-\delta}, \\ \frac{r^2 - b^2}{r_1 r_2} &= \frac{t - \tau}{\sqrt{(\qquad)}} = e^{\zeta} - e^{-\delta}, \\ \frac{r_1^2 - b^2}{r_1 r_2} &= \frac{t_1 - \tau}{\sqrt{(\qquad)}} = e^{u} - e^{-\delta}, \\ \frac{r_2^2 - b^2}{r_1 r_2} &= \frac{t_2 - \tau}{\sqrt{(\qquad)}} = e^{-u} - e^{-\delta}, \end{aligned}\right.$$

$$\infty > t_1 > t > t_2 > \tau > t_3 > -\infty, \qquad \delta > u > \zeta > -u,$$

(9) $$\Omega(\mathrm{MQ}) = 4\int_{t_3}^{t_2} \frac{\frac{1}{2}\sqrt{(-\mathrm{T}_b)}}{\tau - t}\frac{dt}{\sqrt{\mathrm{T}}} = \int_{-u}^{u}\sqrt{\left(\frac{\operatorname{ch}\delta - \operatorname{ch}u}{\operatorname{ch}u - \operatorname{ch}\zeta}\right)}\frac{2e^{-\delta}\,d\zeta}{e^{\zeta} - e^{-\delta}}.$$

Here, to correspond with the preceding,

(10) $$e^{\zeta} = \frac{\operatorname{dn}^2 e\mathrm{G}}{\gamma'}, \qquad \gamma' = e^{-u}, \qquad \omega = \frac{1}{2}\theta = \operatorname{am} e\mathrm{G},$$

(11) $$e^{u} - e^{\zeta} = \frac{\gamma^2 \operatorname{sn}^2 e\mathrm{G}}{\gamma'}, \qquad e^{\zeta} - e^{-u} = \frac{\gamma^2 \operatorname{cn}^2 e\mathrm{G}}{\gamma'},$$

(12) $$e^{u} - e^{-\delta} = \frac{\operatorname{dn}^2 f\mathrm{G}'}{\gamma'}, \qquad e^{-u} - e^{-\delta} = \gamma' \operatorname{cn}^2 f\mathrm{G}', \qquad e^{-\delta} = \gamma' \operatorname{sn}^2 f\mathrm{G}'.$$

In the second Quadric Transformation, between $\gamma'$ and $\varkappa'$,

$$(13)\quad \left\{\begin{aligned} &\operatorname{th}\frac{1}{2}\delta = \frac{1-e^{-\delta}}{1+e^{-\delta}} = \frac{1-\gamma'\operatorname{sn}^2 fG'}{1+\gamma'\operatorname{sn}^2 fG'} = \operatorname{dn} fK', \\ &\operatorname{sn} fK' = \frac{\operatorname{sech}\frac{1}{2}\delta}{\varkappa'} = \frac{\operatorname{ch}\frac{1}{2}u}{\operatorname{ch}\frac{1}{2}\delta} = \cos\frac{1}{2}v, \end{aligned}\right.$$

and so on. Beginning at $f = 0$ on the plate, $f = 1$ on By beyond,

$$(14)\quad \operatorname{sn} 2fG' = \frac{b}{r_2}, \qquad \operatorname{cn} 2fG' = \frac{a-A}{r_2}, \qquad \operatorname{dn} 2fG' = \frac{a+A}{r_1},$$

$$(15)\quad \frac{a^2-A^2}{r_1 r_2} = \operatorname{cn} 2fG' \operatorname{dn} 2fG',$$

which is negative if $f > \frac{1}{2}$,

$$(16)\quad \frac{b^2}{r_1 r_2} = \frac{\tau - t_3}{\sqrt{(\quad)}} = \gamma' \operatorname{sn}^2 2fG', \qquad ABP = \operatorname{am} 2fG',$$

$$(17)\quad \frac{r_2^2-b^2}{r_1 r_2} = \frac{(a-A)^2}{r_1 r_2} = \frac{t_2-\tau}{\sqrt{(\quad)}} = \gamma' \operatorname{cn}^2 2fG',$$

$$(18)\quad \left\{\begin{aligned} &\frac{r_1^2-b^2}{r_1 r_2} = \frac{(a+A)^2}{r_1 r_2} = \frac{t_1-\tau}{\sqrt{(\quad)}} = \frac{\operatorname{dn}^2 2fG'}{\gamma'}, \\ &\operatorname{dn} 2fG' = \cos BAP. \end{aligned}\right.$$

**21.** These substitutions between the variables $r$, $t$, $\zeta$ are recapitulated in the relations

$$(1)\quad \left\{\begin{aligned} &\frac{r_1^2 - r^2 . r^2 - r_2^2}{r_1 r_2 r^2} = \frac{t_1 - t . t - t_2}{(t-t_3)\sqrt{(t_1-t_3 . t_2-t_3)}} = 2 . \operatorname{ch} u - \operatorname{ch}\zeta, \\ &2\frac{dr}{r} = d\zeta = \frac{dt}{t-t_3}, \end{aligned}\right.$$

$$(2)\quad \frac{\sqrt{(r_1 r_2)}\,dr}{\sqrt{R}} = \frac{\dfrac{\frac{1}{2}dt}{t-t_3}}{\sqrt{\dfrac{t_1-t.t-t_2}{(t-t_3)\sqrt{(t_1-t_3.t_2-t_3)}}}} = \frac{\sqrt[4]{(t_1-t_3.t_2-t_3)}\,dt}{\sqrt{T}}$$

$$= \frac{d\zeta}{\sqrt{(2.\operatorname{ch} u - \operatorname{ch}\zeta)}} = 2\sqrt{\gamma'}\,d(eG) = \varkappa'\,d(2eK).$$

and

$$(3)\quad \begin{cases} \dfrac{r^2 - b^2}{r_1 r_2} = \dfrac{t - \tau}{\sqrt{(t_1 - t_3 . t_2 - t_3)}} = e^{\gamma} - e^{-\delta}, \\[2ex] \dfrac{b}{\sqrt{(r_1 r_2)}} = \dfrac{\sqrt{(\tau - t_3)}}{\sqrt{(t_1 - t_3 . t_2 - t_3)}} = \dfrac{\sin \epsilon}{\sqrt{(2 . \operatorname{ch} u - \cos \epsilon)}}, \\[2ex] \dfrac{r_2^2 - b^2, \; r_1^2 - b^2}{r_1 r_2} = \dfrac{t_1 - \tau . \; t_2 - \tau}{\sqrt{(t_1 - t_3 . t_2 - t_3)}}, \\[2ex] \dfrac{a^2 - A^2}{r_1 r_2} = \sqrt{\left(\dfrac{t_1 - \tau . t_2 - \tau}{t_1 - t_3 . t_2 - t_3}\right)} = e^{-\frac{1}{2}\delta} \sqrt{(2 \operatorname{ch} \delta - \operatorname{ch} u)}, \end{cases}$$

$$(4)\quad \begin{cases} r^2 = r_1^2 \cos^2 \tfrac{1}{2}\theta + r_2^2 \sin^2 \tfrac{1}{2}\theta, \\[1ex] \theta = 2 \sin^{-1} \sqrt{\dfrac{r_1^2 - r^2}{r_1^2 - r_2^2}} = 2 \cos^{-1} \sqrt{\dfrac{r^2 - r_2^2}{r_1^2 - r_2^2}}, \end{cases}$$

$$(5)\quad d\theta = \frac{-r\,dr}{\sqrt{R}}, \qquad \frac{d\theta}{PQ} = \frac{-dr}{\sqrt{R}}.$$

Then from *Trans. A. M. S.*, p. 504,

$$(6)\quad \Omega(MQ) = \int_0^{2\pi} \frac{Aa\cos\theta + a^2 - b^2}{MQ^2} \frac{b\,d\theta}{PQ}$$

$$= \int_{r_2}^{r_1} \frac{r^2 - A^2 + a^2 - b^2}{r^2 - b^2} \frac{2b\,dr}{\sqrt{R}}$$

$$= \int \frac{2b\,dr}{\sqrt{R}} + \int \frac{a^2 - A^2}{r^2 - b^2} \frac{2b\,dr}{\sqrt{R}}.$$

And from (9), p. 505, *Trans. A. M. S.*, for the incomplete integrals,

$$(7)\quad \Omega(MQ) - \Omega(PY) = \sin^{-1} \frac{QY . MP}{PY . MQ} = \cos^{-1} \frac{MY . PQ}{PY . MQ} = I,$$

PY being the perpendicular from P on the tangent at Q; and I is the angle between the planes PQY, PQO.

As this angle oscillates and does not revolve, in the complete integrals

$$(8)\quad \Omega(MQ) - \Omega(PY) = \Phi = 2\pi - \Omega(PN) = 2\pi - \Omega.$$

**22.** Referring to *Electromagnetic Integrals*, p. 49 and for fig. 2 and 3, repeated here

$$\text{am} fK' = \psi = \text{DEP} = \text{DPB}, \qquad \text{and then} \qquad \text{DEP} = \text{am}(1-f)K',$$

and p. 43, on fig. 3.

$$\text{am } 2fG' = \chi = \text{ABP}.$$

Then to show am $fG' = \xi$ on the figure, and to draw the coaxial circle *es* touching EP, the point of contact *s* will be where B*s* will bisect the angle ABP; and if the tangent at the lowest point *e* of this circle cuts the circle on ED in *h*,

$$\xi = \text{am} fG' = \text{AB}h.$$

The angles on fig. 2 for modulus $\gamma$ or $\varkappa$ are shown by

$$\text{ABQ} = \frac{1}{2}\theta = \omega = \text{am}\, e\text{G}, \qquad \text{AB}q = \text{am}(1-e)\text{G}$$

and

$$\text{AEQ} = \varphi = \text{am}\, 2e\text{K} \qquad (\text{p. } 49).$$

The geometrical interpretation of the integral for $P_n(u)$ is given on p. 56

$$(1) \qquad P_n(u) = \frac{1}{\pi}\int_0^{\pi}\left(\frac{\text{EA}.\text{EB}}{\text{EQ}^2}\right)^{n+\frac{1}{2}} d\theta$$

and a similar interpretation can be constructed for $Q_n(u)$ in § 14.

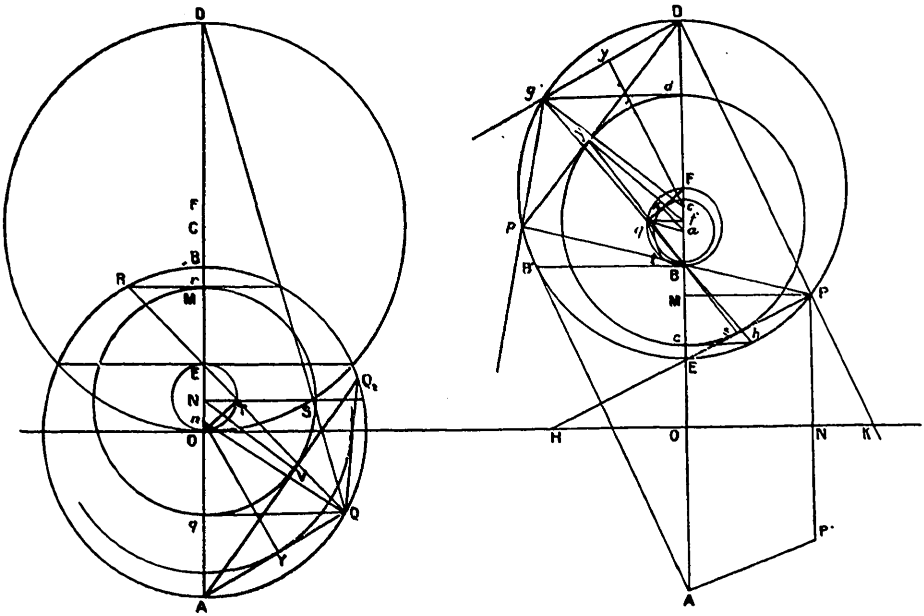
In a dynamical interpretation, where S in fig. 2 oscillates from the level of E, or Q follows it, round the circle on AB with velocity due to the level of F, the time $t = e$T, T denoting the beat of S or period of Q. And on fig. 3, if P describes the circle on DE, with upward gravity, and velocity due to the level of O, proportional to AP or BP, $t = f\text{T}'$, T′ denoting the periodic time of P.

The variable $\theta'$ in § 14 may be interpreted dynamically by putting $\frac{1}{2}\theta' = nt$ in the motion of a particle S round the circle on OD with velocity just sufficient to reach D in a gravity field, the particle making $\frac{n}{\pi}$ beats in small oscillation at O. Then with

$$\text{ODS} = \frac{1}{2}\theta, \quad \cos\frac{1}{2}\theta \,\text{ch}\, \frac{1}{2}\theta' = 1.$$

**23.** With $dg$, $eh$ the segments of the circle on DE on the tangents at the highest and lowest point of the inner coaxial circle on $de$ with centre $f$, $gBh$ is a straight line, and $DBg = \mathrm{am}\, f G'$.

Fig. 2. Fig. 3.




The perpendicular $Fq$ on $gh$ bisects $gh$, so that $f$ and $q$ are at the same level, and

$$(1) \qquad \frac{Ff}{FB} = \sin^2 FBq = \mathrm{sn}^2 f G', \qquad \frac{FB}{FE} = \gamma', \qquad \frac{Ff}{FE} = \gamma' \,\mathrm{sn}^2 f G',$$

$$(2) \qquad \frac{1 - \gamma' \,\mathrm{sn}^2 f G'}{1 + \gamma' \,\mathrm{sn}^2 f G'} = \frac{FE - Ff}{FE + Ff} = \frac{Ef}{fD} = \frac{pr}{rD} = \frac{pB}{BD} = \mathrm{dn}\, f K',$$

with

$$DEp = DPB = \mathrm{am}\, f K',$$

a Quadric Transformation; and

$$(3) \qquad \frac{\mathrm{dn}^2 f G'}{1 + \gamma'} = \frac{\mathrm{dn}\, f K' + \kappa}{1 + \mathrm{dn}\, f K'}.$$

The curvilinear angle DBP made by BD and the circle BPA is $v = \text{APB}$, in the alternate segment; and then

$$(4)\quad \begin{cases} \psi = \text{am}\, fK' = \text{DE}p = \text{DPB}, \qquad \frac{1}{2}\pi - \psi = \text{EPB} = \frac{1}{2}\text{APB} = \frac{1}{2}v, \\ v = \text{APB} = 2\text{EPB} = 2\left(\frac{1}{2}\pi - \text{BPD}\right) = \pi - 2\psi = \pi - 2\,\text{am}\, fK', \end{cases}$$

Equation (10), § 19, is interpreted geometrically with

$$(5)\qquad v' = \text{curvilinear angle DB}p = \text{angle B}p\text{A},$$

in the alternate segment,

$$= 2\text{B}p\text{E} = \pi - 2\text{B}p\text{D} = \pi - 2\text{PED} = \pi - 2\,\text{am}(1-f)K',$$

$$(6)\qquad \frac{1}{2}v' = \text{EDP}, \qquad \frac{1}{2}v = \text{ED}p;$$

$$(7)\qquad \text{tang}\,\frac{1}{2}v\,\text{tang}\,\frac{1}{2}v' = \frac{\text{E}p.\text{EP}}{\text{D}f.\text{DP}} = \frac{\text{EY}}{\text{DZ}} = \frac{\text{EB}}{\text{DB}} = \kappa = \text{th}\,\frac{1}{2}u,$$

where EY, DZ are the perpendiculars on P$p$, and it is the equivalent of the elliptic function relation

$$(8)\qquad \text{tang}\,\frac{1}{2}v\,\text{tang}\,\frac{1}{2}v' = \frac{1}{\text{tn}\, fK'\,\text{tn}(1-f)K'} = \text{th}\,\frac{1}{2}u = \kappa.$$

$$(9)\qquad \kappa\,\text{tn}\, fK'\,\text{tn}(1-f)K' = 1.$$

Proceeding with the geometrical interpretation of the formulas of Quadric Transformation,

$$(10)\qquad \text{D}f = \text{DF} + \text{F}f = \text{DF}(1 + \gamma'\,\text{sn}^2 fG'),$$

$$(11)\qquad \frac{\text{D}r}{\text{DB}} = \frac{pr}{p\text{B}} = \frac{dg}{g\text{B}} = \text{sn}\, fG',$$

by the properties of coaxial circles; and if $r$ is the point of contact with D$p$.

$$(12)\quad \begin{cases} \text{DB} = \text{DF} + \text{FB} = \text{DF}(1 + \gamma'), \qquad \frac{\text{D}r}{\text{DB}} = \frac{dg}{g\text{B}} = \text{sn}\, fG', \\ \text{D}r = \text{DF}(1+\gamma')\,\text{sn}\, fG'; \end{cases}$$

$$(13)\qquad \text{sn}\, fK' = \sin \text{DE}p = \sin \text{DF}r = \frac{\text{D}r}{\text{DF}} = \frac{(1+\gamma')\,\text{sn}\, fG'}{1 + \gamma'\,\text{sn}^2 fG'},$$

$$(14)\quad fr^2 = fd^2 = f\text{B}.f\text{A} = \text{FB}\,\text{cn}^2 fG'.\text{FA}\,\text{dn}^2 fG' = \text{FD}^2\,\text{cn}^2 fG'\,\text{dn}^2 fG',$$

$$(15)\qquad \text{cn}\, fK' = \cos \text{DE}p = \cos \text{D}fr = \frac{fr}{\text{D}f} = \frac{\text{cn}\, fG'\,\text{dn}\, fG'}{1 + \gamma'\,\text{sn}^2 fG'}.$$

Again, with Ft bisecting Pp at right angles,

(16) $$\frac{\text{dn}^2 f K'}{\varkappa} = \frac{\text{dn} f K'}{\text{dn}(1-f) K'} = \frac{pB}{BP} = \frac{pt + tB}{pt - tB},$$

(17) $$\begin{cases} pt = \frac{1}{2} Pp = \frac{1}{2} ED \,\text{dn}\, 2f G' = FD \,\text{dn}\, 2f G', \\ tB = FB \,\text{cn}\, 2f G' = FD . \gamma' \,\text{cn}\, 2f G'; \end{cases}$$

(18) $$\frac{\text{dn}^2 f K'}{\varkappa} = \frac{\text{dn}\, 2f G' + \gamma' \,\text{cn}\, 2f G'}{\text{dn}\, 2f G' - \gamma' \,\text{cn}\, 2f G'} = \frac{1 + \gamma' \,\text{sn}(1 - 2f) G'}{1 - \gamma' \,\text{sn}(1 - 2f) G'}.$$

If B″B is horizontal, AB″ the tangent from A,

(19) $$\gamma' = \cos \frac{B''B}{B''A} = \cos AB''B$$

AB″B is the modular angle for γ, B″E bisects the angle BB″A.

And so all the formulas of the complementary Quadric Transformation can receive a geometrical interpretation on fig. 3, with the notation, as above

$$\psi = DEp = DFr = \text{am} f K', \quad \chi = DBp = \text{am}\, 2f G', \quad DBr = \frac{1}{2} \text{am} f G',$$
$$\xi = DBg = \text{am} f G'.$$

**24.** The Duplication formulas for $fK'$ and $\frac{1}{2} fK'$ are shown on the circle EPD, where

$$\psi = DPp = \text{am} f K', \quad \varphi = DEg = \text{am} \frac{1}{2} f K',$$

so that $pg$, $gD$ are tangents to an inner concyclic circle; here is the kernel of the geometrical interpretation.

The centre of this circle is at $c$ on $gc$, the bisector of the angle

$$Dgp = \pi - DEp = \pi - \text{am} f K,$$

so that

$$Dgc = \frac{1}{2}\pi - \frac{1}{2} \text{am} f K', \quad gcy = \frac{1}{2} \text{am} f K' = \frac{1}{2}\psi,$$

of the circle touches $Dg$ in $y$; and then

$$Dcy = DEg = \text{am} \frac{1}{2} f K' = \varphi, \quad \text{and} \quad Dy = Dc \sin\varphi;$$

(1) $$\frac{gy}{yD} = \frac{gB}{BD} = \text{dn} \frac{1}{2} f K' = \Delta\varphi.$$

and

(2) $$gy = Dc \sin\varphi\,\Delta\varphi, \qquad cy = Dc\cos\varphi;$$

(3) $$gc^2 = Dc^2(\sin^2\varphi\,\Delta^2\varphi + \cos^2\varphi) = Dc^2.D, \qquad D = 1 - \varkappa'^2\sin^4\varphi;$$

(4) $$\frac{\cos\frac{1}{2}\psi}{\cos\varphi} = \frac{cD}{cg} = \frac{1}{\sqrt{D}}, \qquad \cos\frac{1}{2}\operatorname{am} f K' = \frac{\operatorname{cn}\frac{1}{2} f K'}{\sqrt{D}};$$

(5) $$\frac{\sin\frac{1}{2}\psi}{\sin\varphi} = \frac{gy}{gc}\,\frac{cD}{cg} = \frac{\Delta\varphi}{\sqrt{D}}, \qquad \sin\frac{1}{2}\operatorname{am} f K' = \frac{\operatorname{sn}\frac{1}{2} f K' \operatorname{dn}\frac{1}{2} f K'}{\sqrt{D}};$$

(6) $$\frac{\operatorname{tang}\frac{1}{2}\psi}{\operatorname{tang}\varphi} = \frac{gy}{gD} = \Delta\varphi, \qquad \operatorname{tang}\frac{1}{2}\operatorname{am} f K' = \operatorname{tn}\frac{1}{2} f K' \operatorname{dn}\frac{1}{2} f K';$$

(7) $$\operatorname{sn} f K' = 2\sin\frac{1}{2}\operatorname{am} f K' \cos\frac{1}{2}\operatorname{am} f K' = \frac{2\operatorname{sn}\frac{1}{2} f K' \operatorname{cn}\frac{1}{2} f K' \operatorname{dn}\frac{1}{2} f K'}{D}.$$

The Duplication formulas of $fG'$ and $2fG'$ are shown on the circle $FqtB$, where $FBq = \operatorname{am} fG'$, $FBt = \operatorname{am} 2fG'$, so that $Fq$, $qt$ are tangents to an interior concyclic circle; here again is the kernel of the geometry.

The centre of this circle is at $a$ on $qa$, the bisector of the angle

$$Fqt = \pi - \operatorname{am} 2fG',$$

so that

$$Fqa = \frac{1}{2}\pi - \frac{1}{2}\operatorname{am} 2fG', \qquad qaz = \frac{1}{2}\operatorname{am} 2fG' = \frac{1}{2}\chi,$$

if the circle touches $Fq$ in $z$, and then

$$Faz = FBq = \operatorname{am} fG' = \xi.$$

Then

(8) $$Fz = Fa\sin\xi, \qquad qz = Fa\sin\xi\,\Delta\xi, \qquad az = Fa\cos\xi;$$

(9) $$aq^2 = Fa^2(\sin^2\xi\,\Delta^2\xi + \cos^2\xi) = Fa^2.D, \qquad D = 1 - \gamma'^2\sin^4\xi;$$

(10) $$\frac{\cos\frac{1}{2}\chi}{\cos\xi} = \frac{aF}{aq} = \frac{1}{\sqrt{D}}, \qquad \cos\frac{1}{2}\operatorname{am} 2fG' = \frac{\operatorname{cn} fG'}{\sqrt{D}};$$

(11) $$\frac{\sin\frac{1}{2}\chi}{\sin\xi} = \frac{qz}{qa}\,\frac{aF}{Fz} = \frac{\Delta\xi}{\sqrt{D}}, \qquad \sin\frac{1}{2}\operatorname{am} 2fG' = \frac{\operatorname{sn} fG' \operatorname{dn} fG'}{\sqrt{D}};$$

and

$$(12)\qquad \frac{\tan\frac{1}{2}\chi}{\tan\xi} = \frac{qz}{Fz} = \Delta\xi, \qquad \tan\frac{1}{2}\operatorname{am} 2fG' = \operatorname{tn} fG' \operatorname{dn} fG';$$

$$(13)\qquad \operatorname{sn} 2fG' = 2\sin\frac{1}{2}\operatorname{am} 2fG' \cos\frac{1}{2}\operatorname{am} 2fG' = \frac{2\operatorname{sn} fG' \operatorname{cn} fG' \operatorname{dn} fG'}{D}.$$

Similar geometrical interpretation can be given in fig. 2 of the Duplication formulas for $eG$ and $2eG$, where $ABQ = \operatorname{am} eG$, $ABQ_2 = \operatorname{am} 2eG$, and N is the centre of the circle on the diameter $qr$, and $AQ_2$ the tangent from A; also of the Quadric Transformation between

$$\varphi = AEQ = \operatorname{am} 2eK,$$

and

$$\omega = ABQ = \operatorname{am} eG, \qquad \omega' = ABR = \operatorname{am}(1-e)G.$$

Fig. 2 and 3 may be supposed turned about AB into planes at right angles, so as to represent circles linked like the electric and magnetic circuit.

Then the vector product of the electric and magnetic flux is Poynting's vector of flow of energy.