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Abstract

The lecture is devoted to the problem of absolute continuity of the spectrum of periodic operators. A general approach to this problem was suggested by L. Thomas in 1973 for the case of the Schrödinger operator with periodic electric potential. Further application of his method to concrete operators of mathematical physics met analytic difficulties. In recent years several new problems in this area have been solved. We propose a survey of known results in this area, including very recent, and formulate unsolved problems.

1. Introduction

The problem of the absolute continuity of the periodic operators of mathematical physics has been intensively studied in recent years. We propose a survey of the known results and unsolved problems in this area. This is a revised version of surveys [BSu4,5], which includes the most recent results.

1.1. The Floquet decomposition. We study periodic differential operators in $L_2(\mathbb{R}^d)$, $d \geq 2$. Let $\mathcal{P}(\mathbf{x}, \mathbf{D})$ be a linear formally selfadjoint differential expression, $\mathbf{x} \in \mathbb{R}^d$, $\mathbf{D} = -i\nabla$. Suppose that $\mathcal{P}$ is periodic in $\mathbf{x}$ with a lattice of periods Γ. Differential expression $\mathcal{P}(\mathbf{x}, \mathbf{D})$ generates an operator P in $L_2(\mathbb{R}^d)$. Assume that P is selfadjoint. By Ω we denote the elementary cell of the lattice Γ. The elementary cell of the dual lattice is denoted by $\widetilde{\Omega}$. We use notation $H^s(\mathbb{R}^d)$, $H^s(\Omega)$, $s \in \mathbb{R}$, for the Sobolev classes. Symbol $\|\cdot\|$ stands for the norm of function in $L_2(\Omega)$ or the norm of bounded operator in $L_2(\Omega)$.

As usual, we use *the Floquet decomposition* for periodic operators. Introduce a family of operators $P(\mathbf{k})$ in $L_2(\Omega)$. Operator $P(\mathbf{k})$ is defined by expression $\mathcal{P}(\mathbf{x}, \mathbf{D}+\mathbf{k})$ on a set of functions satisfying periodic boundary conditions. Parameter $\mathbf{k} \in \mathbb{R}^d$ is called *quasimomentum*. Operator $P(\mathbf{k})$ is selfadjoint in $L_2(\Omega)$. Assume that $P(\mathbf{k})$

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has *compact resolvent* which depends on $\mathbf{k}$ continuously (in operator norm). We use the so called *Gelfand transform*

$$U : L_2(\mathbb{R}^d) \to L_2(\Omega \times \widetilde{\Omega}) = \int\limits_{\widetilde{\Omega}} \oplus L_2(\Omega)\, d\mathbf{k}.$$

First U is defined on the Schwartz class $\mathcal{S}(\mathbb{R}^d)$ by the formula

$$(Uf)(\mathbf{x};\mathbf{k}) = (\operatorname{mes}\widetilde{\Omega})^{-1/2} \sum_{\mathbf{n}\in\Gamma} e^{-i\mathbf{k}(\mathbf{x}+\mathbf{n})} f(\mathbf{x}+\mathbf{n}), \quad f \in \mathcal{S}(\mathbb{R}^d).$$

Then U is extended by continuity to a *unitary* operator. Operator P is unitarily equivalent to the direct integral of operators $P(\mathbf{k})$

$$UPU^{-1} = \int\limits_{\widetilde{\Omega}} \oplus P(\mathbf{k})\, d\mathbf{k}.$$

Operator $P(\mathbf{k})$ has discrete spectrum. By $E_j(\mathbf{k})$, $j \in \mathbb{Z}$, we denote eigenvalues of $P(\mathbf{k})$, numbered in increasing order counting multiplicity. Functions $E_j(\cdot)$ are continuous. The spectrum of P has *a band structure.* It consists of intervals of the real axis (bands). Each band is the range of the corresponding *band function* $E_j(\cdot)$. The intervals lying between bands (if they exist) are called *gaps.* Bands can overlap.

1.2. The problem of absence of degenerate bands. General arguments do not imply the absence of spectral bands which degenerate into points. These points represent eigenvalues of infinite multiplicity of P. This possibility can be realized, for example, for a periodic elliptic operator of fourth order; the corresponding example can be found in [K1]. However, for periodic operators of mathematical physics, in particular, for periodic elliptic second order operators, the absence of degenerate bands is the most plausible conjecture. It turns out that *if there are no degenerate bands, then the spectrum of the periodic operator is absolutely continuous.* (Then we say that the operator itself is absolutely continuous.) *The absence of degenerate bands is equivalent to the absence of constant band functions.*

The proof of the absence of degenerate bands is a subtle technical problem. The main objects of study are the following: the Schrödinger operator (with periodic magnetic and electric potentials $\mathbf{A}$, V and periodic metric g)

$$H(g, \mathbf{A}, V) = (\mathbf{D} - \mathbf{A}(\mathbf{x}))^* g(\mathbf{x})(\mathbf{D} - \mathbf{A}(\mathbf{x})) + V(\mathbf{x}); \tag{1.1}$$

the Dirac operator (see (4.4)), the Maxwell operator (see (5.2)), the operator of elasticity theory, the problems for periodic waveguides. It is important for applications to consider the operators with discontinuous coefficients. However, even in smooth situations there are many unsolved problems. Each new case requires invention of new specific technical tricks.

1.3. The Thomas approach. For the first time, the absence of degenerate bands for the periodic operator

$$H = -\Delta + V(\mathbf{x}) \tag{1.2}$$

in $L_2(\mathbb{R}^3)$ with $V \in L_2(\Omega)$ was established in the famous paper [T] by L. Thomas. Thomas suggested a general approach which was later used in all the papers devoted to the absolute continuity of periodic operators. Let us explain the Thomas approach taking the operator (1.2) in $\mathbb{R}^d$, $d \geq 2$, as an example, and *assuming for simplicity that* $\Gamma = \mathbb{Z}^d$.

The operator family $H(\mathbf{k})$ in $L_2(\Omega)$ is defined by expression $H(\mathbf{k}) = (\mathbf{D}+\mathbf{k})^2 + V(\mathbf{x})$ on functions in $H^2(\Omega)$ satisfying periodic boundary conditions. Thomas used the analytic extension in the quasimomentum $\mathbf{k}$. The family $H(\mathbf{k})$ is polynomial in $\mathbf{k} \in \mathbb{C}^d$, the resolvent of $H(\mathbf{k})$ is compact. This allows us to apply analytic perturbation theory and the analytic Fredholm alternative. We fix $k_2, \ldots, k_d \in \mathbb{R}$ and put $k_1 = \pi + iy$, $y \in \mathbb{R}$. Then the family $H(\mathbf{k}) = H(y)$ depends on one parameter y. Thomas proved that $H(y)$ is invertible for sufficiently large $|y|$ and

$$\lim_{|y|\to\infty} \|H(y)^{-1}\| = 0. \tag{1.3}$$

The relation (1.3) directly implies the absence of degenerate bands. Indeed, suppose that some band function is constant, i.e. $E_j(\mathbf{k}) = E = \text{const}$. Then, by the analytic Fredholm alternative, the number E is an eigenvalue of the operator $H(y)$ for all $y \in \mathbb{R}$. Let $u = u(y)$ be an eigenfunction with $\|u\| = 1$. Then we have $u = EH(y)^{-1}u$. The norm of the left-hand side is equal to 1, whereas, by (1.3), the norm of the right-hand side tends to 0 as $|y| \to \infty$. We arrive at a contradiction which proves the absence of degenerate bands.

Thus, the problem is reduced to the proof of the relation (1.3). To prove such a relation for particular periodic operators is a difficult technical problem. However, it is easy for the operator (1.2) with $V \in L_\infty$. We start with the "unperturbed" operator $H_0 = -\Delta$ and consider the corresponding operator family $H_0(y)$ in $L_2(\Omega)$. Write down the Fourier series for a periodic function $v(\mathbf{x})$:

$$v(\mathbf{x}) = (2\pi)^{-d/2} \sum_{\mathbf{n}\in\mathbb{Z}^d} \hat{v}_{\mathbf{n}} e^{2\pi i \mathbf{n}\mathbf{x}}.$$

The operator $H_0(y)$ turns into multiplication of the Fourier coefficients $\hat{v}_{\mathbf{n}}$ by the symbol:

$$H_0(y)v(\mathbf{x}) = (2\pi)^{-d/2} \sum_{\mathbf{n}\in\mathbb{Z}^d} h_{\mathbf{n}}(y)\hat{v}_{\mathbf{n}} e^{2\pi i \mathbf{n}\mathbf{x}},$$

$$h_{\mathbf{n}}(y) = (2\pi n_1 + \pi)^2 + (2\pi n_2 + k_2)^2 + \ldots + (2\pi n_d + k_d)^2 - y^2 + 2\pi i y(2n_1 + 1).$$

The real part of the symbol $h_{\mathbf{n}}(y)$ is degenerate for some values of $\mathbf{n}$ and y. However, the imaginary part of the symbol is nondegenerate. The symbol satisfies the estimate $|h_{\mathbf{n}}(y)| \geq c(1+|y|)$. This leads to the estimate for the inverse operator

$$\|H_0(y)^{-1}\| \leq C(1+|y|)^{-1}, \quad y \in \mathbb{R}, \quad C = c^{-1}. \tag{1.4}$$

The next step is to add $V(\mathbf{x})$. When $V \in L_\infty$, this can be done by the elementary estimate

$$\|H(y)u\| \geq \|H_0(y)u\| - \|Vu\| \geq c(1+|y|)\|u\| - \|V\|_\infty\|u\| \geq \frac{c}{2}(1+|y|)\|u\|$$

valid for $|y| \geq y_0 = 2C\|V\|_\infty$. Thus,

$$\|H(y)^{-1}\| \leq 2C(1+|y|)^{-1}, \quad |y| \geq y_0, \tag{1.5}$$

which implies (1.3). When potential gets worse it takes more efforts to prove (1.5), nevertheless V can be taken into account as an additive perturbation.

More detailed expositions (slightly different from each other) of the Thomas approach can be found, for example, in [RSi], [K1], [BSu4].

For the periodic *magnetic* Schrödinger operator $(\mathbf{D} - \mathbf{A}(\mathbf{x}))^2 + V(\mathbf{x})$ the perturbation is of first order and for the corresponding operator depending on y the perturbation contains the term $O(|y|)$. This is why (see [HemHer]) the magnetic operator cannot be treated as an additive perturbation of the "free" operator H_0. This difficulty was overcome in [BSu1] in twodimensional case. The magnetic operator was treated as a "multiplicative" perturbation of H_0 (see Sec. 2 for details). In the case $d \geq 3$ the problem was much more difficult. It was solved by A. Sobolev [So1].

In Section 2 we discuss results for the periodic Schrödinger operator (1.1) in more details. Section 3 is devoted to the Schrödinger operator with delta-like potential supported by a periodic graph. In Section 4 we discuss results for the periodic Dirac operator, in Section 5 – for the periodic Maxwell operator.

2. Absolute continuity of the periodic Schrödinger operator

2.1. Notation. The standard inner product in $\mathbb{C}^d$ is denoted by $\langle \cdot, \cdot \rangle$. Integrals without indication of the integration domain are over $\mathbb{R}^d$. We recall the definition of the classes $L_{p,\infty}(\Omega)$ of the Lorentz scale. For a measurable function f we put

$$\rho_f(t) = \operatorname{mes}\{\mathbf{x} \in \Omega : |f(\mathbf{x})| \geq t\}, \quad t > 0.$$

The class $L_{p,\infty}(\Omega)$, $0 < p < \infty$, consists of all functions f such that

$$\|f\|_{p,\infty} := \sup_{t>0} t(\rho_f(t))^{1/p} < \infty. \tag{2.1}$$

The class $L_{p,\infty}(\Omega)$ is complete with respect to the *quasinorm* (2.1) and it is nonseparable. We single out the separable subspace $L^0_{p,\infty}(\Omega)$ formed by the functions $f \in L_{p,\infty}(\Omega)$ such that $\rho_f(t) = o(t^{-p})$ as $t \to \infty$.

2.2. Definition of the Schrödinger operator. Now we give the precise definition of the operator (1.1). We fix an orthonormal basis $\mathbf{e}_1, \ldots, \mathbf{e}_d$ in $\mathbb{R}^d$. Let $g(\mathbf{x}) = \{g^{jl}(\mathbf{x})\}$, $j, l = 1, \ldots, d$, $\mathbf{A}(\mathbf{x}) = A_1(\mathbf{x})\mathbf{e}_1 + \ldots + A_d(\mathbf{x})\mathbf{e}_d$ and $V(\mathbf{x})$ be Γ-periodic functions in $\mathbb{R}^d$ such that g^{jl}, A_j and V are real-valued and

$$g(\mathbf{x}) > 0, \quad g + g^{-1} \in L_\infty(\Omega), \tag{2.2}$$

$$\mathbf{A} \in L_r(\Omega), \quad r > 2, \quad d = 2; \quad \mathbf{A} \in L^0_{d,\infty}(\Omega), \quad d \geq 3, \tag{2.3}$$

$$V \in L_\rho(\Omega), \quad \rho > 1, \quad d = 2; \quad V \in L^0_{d/2,\infty}(\Omega), \quad d \geq 3. \tag{2.4}$$

Under the conditions (2.2)−(2.4) the operator (1.1) cannot be defined directly by the differential expression. We use the definition via the quadratic form

$$h[u] = h(g, \mathbf{A}, V)[u] = \int (\langle g(\mathbf{D}-\mathbf{A})u, (\mathbf{D}-\mathbf{A})u\rangle + V|u|^2)\,d\mathbf{x}, \quad u \in H^1(\mathbb{R}^d). \tag{2.5}$$

Under conditions (2.2)−(2.4) the form (2.5) is semibounded from below and closed. The selfadjoint operator $H = H(g, \mathbf{A}, V)$ in $L_2(\mathbb{R}^d)$, generated by this form, is, *by definition*, treated as the Schrödinger operator (1.1).

2.3. Conjecture. The following conjecture was formulated in [BSu4].

Conjecture 2.1 *Let g^{jl}, A_j, V be real Γ-periodic functions. Under conditions* (2.2)-(2.4) *the operator $H(g, \mathbf{A}, V)$ generated by the form* (2.5) *is absolutely continuous.*

Conditions of Conjecture 2.1 on $\mathbf{A}$ and V are optimal in the Lorentz scale. In its full generality, Conjecture 2.1 has not yet been justified. The validity of it was proved in a number of particular cases described below.

2.4. Known results. *Below a stands for a constant positive $(d \times d)$-matrix.* We start with the case of a constant metric $g(\mathbf{x}) = a$ and with the "nonmagnetic" case $\mathbf{A}(\mathbf{x}) \equiv 0$.

Theorem 2.2 *Let $g(\mathbf{x}) = a$ and let $V(\mathbf{x})$ be a real Γ-periodic function satisfying* (2.4). *Then the operator $H(a, 0, V)$ is absolutely continuous.*

Comments. 1) For $d = 3$ and $V \in L_2(\Omega)$ the absolute continuity of the periodic operator $-\Delta + V(\mathbf{x})$ was proved by L. Thomas [T].
2) The result of [T] was extended to arbitrary $d \geq 2$ in [RSi]. For $d = 2, 3$ it was assumed that $V \in L_2(\Omega)$. For $d \geq 4$, $V \in L_s(\Omega)$, $s > d-1$.
3) In [BSu4] the absolute continuity of the operator $H(a, 0, V)$ was justified under condition (2.4) for $d = 2, 3, 4$. However, for $d \geq 5$ more restrictive condition $V \in L^0_{d-2,\infty}(\Omega)$ was imposed.
4) In [Sh2] the absolute continuity of $H(a, 0, V)$ was proved under assumption (2.4) for all $d \geq 3$. Earlier in [Sh1] the same result was obtained for $V \in L_{d/2}(\Omega)$.
5) In [Sh3] the absolute continuity of $H(a, 0, V)$ was proved under condition slightly less restrictive than (2.4), formulated in terms of Fefferman-Phong classes.

Remark 2.3 *Without selfadjointness assumption the Thomas estimate implies the absence of eigenvalues (see [K1]). In particular, the Schrödinger operator $-\Delta + V(\mathbf{x})$ with complex-valued periodic V and the Schrödinger operator $-\Delta\mathbf{1} + \mathcal{V}(\mathbf{x})$ in $L_2(\mathbb{R}^d, \mathbb{C}^k)$ with matrix-valued periodic potential $\mathcal{V}$ have no eigenvalues.*

We see that for a constant metric and in "nonmagnetic" case the assumptions of Theorem 2.2 on potential $V(\mathbf{x})$ coincide with the assumptions of Conjecture 2.1.

Now let the metric be still constant, $g(\mathbf{x}) = a$, but $\mathbf{A}(\mathbf{x}) \not\equiv 0$. For a constant matrix $g = a$, without loss of generality, the magnetic potential $\mathbf{A}$ can be subject to the additional *gauge conditions*

$$\operatorname{div} a\mathbf{A} = 0, \quad \int_\Omega \mathbf{A}(\mathbf{x})\,d\mathbf{x} = 0. \tag{2.6}$$

Theorem 2.4 *Let $g(\mathbf{x}) = a$ and let $V(\mathbf{x})$ and $A_j(\mathbf{x})$ be real Γ-periodic functions such that*

$$V \in L_\rho(\Omega), \quad \rho > 1; \quad \mathbf{A} \in L_r(\Omega), \quad r > 2, \quad d = 2; \tag{2.7}$$

$$V \in L^0_{d/2,\infty}(\Omega), \quad d = 3, 4; \quad V \in L^0_{d-2,\infty}(\Omega), \quad d \geq 5; \tag{2.8}$$

$$\mathbf{A} \in H^s(\Omega), \quad 2s > 3d - 2, \quad d \geq 3. \tag{2.9}$$

Let $\mathbf{A}$ satisfy (2.6). Then the operator $H(a, \mathbf{A}, V)$ is absolutely continuous.

Comments. 1) In [HemHer] the absolute continuity of the magnetic Hamiltonian was proved for small $\mathbf{A} \in C^1$. The difficulties arising without this assumption were analysed.
2) In [BSu1] it was proved that $H(a, \mathbf{A}, V)$ is absolutely continuous for $d = 2$, $\mathbf{A} \in C(\Omega)$, $V \in L_2(\Omega)$. Let us briefly explain the approach of [BSu1] in the case when $a = \mathbf{1}$ and $\mathbf{A} \in C^1$. Consider the Pauli operator $P = (\mathbf{D} - \mathbf{A}(\mathbf{x}))^2 + \partial_1 A_2 - \partial_2 A_1$. By the gauge conditions, there exists a real-valued periodic function $\varphi(\mathbf{x})$ such that $\nabla\varphi = \{A_2, -A_1\}$. The Pauli operator admits a factorization of the form

$$P = e^{-\varphi}(D_1 + iD_2)e^{2\varphi}(D_1 - iD_2)e^{-\varphi}.$$

This factorization is very convenient since each factor is either differential operator with constant coefficients or multiplication by a positive function. This allows us to treat the corresponding operator $P(y)$ as *a multiplicative perturbation* of the free operator $H_0(y)$ and to prove an analogue of the estimate (1.4) for $P(y)^{-1}$. After that we can take V into account as an additive perturbation.
3) Conditions on potentials $\mathbf{A}$ and V for $d = 2$ were relaxed to (2.7) in [BSu2].
4) For $d \geq 3$ there is no convenient factorization for the Pauli operator. The problem was much more difficult. The absolute continuity of $H(a, \mathbf{A}, V)$ was established by A. Sobolev [So1] for $\mathbf{A} \in C^{2d+3}(\Omega)$, $V \in L_{d-1}(\Omega)$. Later the condition on $\mathbf{A}$ was relaxed to (2.9) [So2]. A. Sobolev used pseudodifferential operators on torus. Above (for $H = -\Delta + V$ and for the magnetic operator in twodimensional case) we could choose an arbitrary direction (in Γ) of complex quasimomentum. It turned out that for $d \geq 3$ and $\mathbf{A} \not\equiv 0$ we cannot take this direction arbitrarily. Nevertheless, for a given $\mathbf{A}$ there exists an appropriate direction of the quasimomentum such that an analogue of the estimate (1.5) holds.
5) In [KL] considerations of A. Sobolev were included in a more general framework. As in [So2], the condition on $\mathbf{A}$ was relaxed to (2.9). The absence of eigenvalues for the operator $H(a, \mathbf{A}, V)$ with complex periodic potentials was established.
6) The conditions on $V(\mathbf{x})$ were relaxed to (2.8) in [BSu4]. It is not known if the conditions on V can be relaxed to (2.4) for $d \geq 5$.

Thus, in the case of a constant metric the absolute continuity of the operator $H(a, \mathbf{A}, V)$ is established. For $d = 2$ this is done under the natural assumptions (2.7). For $d \geq 3$ this is done for sufficiently smooth magnetic potential $\mathbf{A}(\mathbf{x})$, and the order of smoothness is a linear function of d.

The most difficult thing is to remove the requirement that the metric should be constant. There are only two cases when the problem is solved. The first case is the case of a "scalar" metric, $d \geq 2$. Let $g(\mathbf{x}) = \omega^2(\mathbf{x})a$, where a is a positive constant matrix and $\omega(\mathbf{x})$ is a Γ-periodic real-valued function such that

$$0 < \omega_0 \leq \omega(\mathbf{x}) \leq \omega_1 < \infty, \quad \mathbf{x} \in \mathbb{R}^d. \tag{2.10}$$

The following generalization of Theorems 2.2, 2.4 to the case of a "scalar" metric was established in [BSu4].

Theorem 2.5 *Under assumptions of Theorem* 2.2 *(respectively, Theorem* 2.4*), let* $\omega(\mathbf{x})$ *be a* Γ*-periodic function subject to* (2.10)*. Let* $V_\omega(\mathbf{x}) := \operatorname{div} a \nabla \omega(\mathbf{x})$ *satisfy the same assumptions as* $V(\mathbf{x})$*, i.e.* (2.4) *(respectively,* (2.7), (2.8)*). Then the operator* $H(\omega^2 a, 0, V)$ *(respectively,* $H(\omega^2 a, \mathbf{A}, V)$*) is absolutely continuous.*

The proof of Theorem 2.5 is based on the simple relation

$$H(\omega^2 a, \mathbf{A}, V) = \omega H(a, \mathbf{A}, \omega^{-2} V + \omega^{-1} V_\omega)\omega \tag{2.11}$$

between the operator with the metric $\omega^2 a$ and the operator with the constant metric a but another electric potential.

The second important case is the twodimensional case. The absolute continuity of $H(g, \mathbf{A}, V)$ for $d = 2$ was established by A. Morame [M1].

Theorem 2.6 *Let* $d = 2$, $g^{jl}(\mathbf{x})$, $A_j(\mathbf{x})$ *and* $V(\mathbf{x})$ *be real,* Γ*-periodic functions such that*

$$g(\mathbf{x}) = \{g^{jl}(\mathbf{x})\} > 0, \quad g \in C^\infty(\mathbb{R}^2), \tag{2.12}$$

$$\mathbf{A} \in C^\infty(\mathbb{R}^2), \quad V \in L_\infty(\mathbb{R}^2). \tag{2.13}$$

Then the operator $H(g, \mathbf{A}, V)$ *is absolutely continuous.*

The technique of [M1] is very involved. The approach is based on a study of the Pauli operator with a metric. The method of [M1] does not give the standard estimate of the form (1.5), the main elements of the Thomas scheme are used in [M1] in a slightly different way. In [M1] it was assumed that $\det g = 1$. We can easily remove this requirement using the analogue of relation (2.11).

Comment. P. Kuchment [K2] recently communicated to the author that in the case $d = 2$ *the global isothermal coordinates exist.* Then the proof of Theorem 2.6 can be significantly simplified. By an appropriate change of variables the periodic operator $H(g, \mathbf{A}, V)$ can be reduced to the periodic operator with *a scalar metric* and, generally speaking, another lattice of periods. This allows us to apply Theorem 2.5. At the same time conditions on g, $\mathbf{A}$ and V are much wider than (2.12), (2.13).

When $d \geq 3$ and metric is of general form the problem of the absolute continuity of $H(g, \mathbf{A}, V)$ is open.

3. The Schrödinger operator with delta-like potential supported by a periodic graph

3.1. The most recent result [BSuSht] concerns the absolute continuity of the Schrödinger operator $-\Delta + \delta_\Sigma(\mathbf{x})$ in $L_2(\mathbb{R}^2)$ with delta-like potential supported by a periodic system of curves Σ. We study more general operator

$$H_\Sigma = (\mathbf{D} - \mathbf{A}(\mathbf{x}))^* \omega^2(\mathbf{x}) a (\mathbf{D} - \mathbf{A}(\mathbf{x})) + V(\mathbf{x}) + \sigma(\mathbf{x})\delta_\Sigma(\mathbf{x}). \tag{3.1}$$

Assume that A_j, V and ω are real Γ-periodic functions satisfying conditions (2.7) and (2.10) and also the condition

$$V_\omega = \operatorname{div} a \nabla \omega \in L_\rho(\Omega), \quad \rho > 1. \tag{3.2}$$

The "graph" Σ is periodic: $\mathbf{x} \in \Sigma \Rightarrow \mathbf{x} + \mathbf{n} \in \Sigma$, $\mathbf{n} \in \Gamma$, and

$$\Sigma_\Omega = \Sigma \cap \Omega = \bigcup_{j=1}^{N} \gamma_j, \quad \gamma_j \in C^2, \tag{3.3}$$

where γ_j is a simple smooth (of C^2 class) curve. There are no restrictions on the value of angles at points of intersection of the curves γ_j; graph Σ can be disconnected. Finally, $\sigma(\mathbf{x})$ is a Γ-periodic real-valued function on Σ such that

$$\sigma \in L_p(\Sigma_\Omega), \quad p > 1. \tag{3.4}$$

Consider the quadratic form in $L_2(\mathbb{R}^2)$

$$h_\Sigma[u] = \int (\omega^2(\mathbf{x}) \langle a(\mathbf{D} - \mathbf{A})u, (\mathbf{D} - \mathbf{A})u \rangle + V(\mathbf{x})|u|^2)\, d\mathbf{x} + \int_\Sigma \sigma(\mathbf{x})|u|^2\, ds(\mathbf{x}), \tag{3.5}$$

$u \in H^1(\mathbb{R}^2)$. Under the above assumptions the form (3.5) is semibounded from below and closed. It generates a selfadjoint operator H_Σ which formally corresponds to the expression (3.1).

Theorem 3.1 *Let A_j, V, ω be real-valued and Γ-periodic functions in $\mathbb{R}^2$ satisfying conditions* (2.7), (2.10), (3.2). *Let Σ be a Γ-periodic system of curves satisfying* (3.3). *Let σ be a Γ-periodic real-valued function on Σ satisfying* (3.4). *Then the operator H_Σ generated by the form* (3.5) *is absolutely continuous.*

A similar result was recently obtained by the same authors in the case $d = 3$ when Σ is a periodic system of surfaces.

3.2. Theorem 3.1 implies the absolute continuity of another selfadjoint operator which acts in $L_2(\Sigma)$ – the "Dirichlet-to-Neumann" operator Λ. This result has application to the physics of photonic crystals (see [K3]). For simplicity, first we assume that Σ is connected and divides $\mathbb{R}^2$ into similar bounded domains Ω_j, $j \in \mathbb{N}$. For a given function $\varphi \in L_2(\Sigma)$ we look for solution of the Dirichlet problem in each domain Ω_j: $\Delta u_j = 0$ (in Ω_j), $u_j|_{\partial\Omega_j} = \varphi$. Let $\frac{\partial u_j}{\partial \nu_j}(\mathbf{x})$, $\mathbf{x} \in \partial\Omega_j$, be the derivative of u_j with respect to the external normal to $\partial\Omega_j$. We put $\psi(\mathbf{x}) = \sum_j \frac{\partial u_j}{\partial \nu_j}(\mathbf{x})$; $\psi(\mathbf{x})$

is defined in smooth points of Σ. The operator $\Lambda : \varphi \mapsto \psi$ is called Dirichlet-to-Neumann operator. The precise definition of the operator Λ in the case of arbitrary periodic graph Σ satisfying (3.3) can be given in terms of quadratic form. In [K3] P. Kuchment conjectured that Λ has absolutely continuous spectrum. Now this conjecture is justified in [BSuSht]: absolute continuity of the operator $-\Delta + \delta_\Sigma(\mathbf{x})$ implies absolute continuity of Λ.

4. The periodic Dirac operator

Let $\alpha_1, \cdots, \alpha_{d+1}$ be the Dirac matrices, i. e. the Hermitian $(M \times M)$-matrices, $M = 2^{d-1}$, which satisfy the relations $\alpha_j\alpha_l + \alpha_l\alpha_j = 2\delta_{jl}\mathbf{1}$, $j, l = 1, \ldots, d+1$, where $\mathbf{1}$ is the unit matrix. Let $V(\mathbf{x})$, $V_0(\mathbf{x})$, $A_j(\mathbf{x})$ be real-valued Γ-periodic functions. We assume that

$$\mathbf{A} \in L_r(\Omega), \quad V_0 \in L_r(\Omega), \quad V \in L_r(\Omega), \quad r > 2, \quad d = 2, \tag{4.1}$$

$$V_0, \quad V \in C(\mathbb{R}^d), \quad d \geq 3. \tag{4.2}$$

The magnetic potential is required to satisfy the gauge conditions

$$\operatorname{div} \mathbf{A} = 0, \quad \int_\Omega \mathbf{A}(\mathbf{x})\, d\mathbf{x} = 0. \tag{4.3}$$

Under the assumptions (2.9), (4.1), (4.2) **the Dirac operator**

$$\mathcal{D}(\mathbf{A}, V_0, V) := \sum_{j=1}^{d} (D_j - A_j(\mathbf{x}))\alpha_j + V_0(\mathbf{x})\alpha_{d+1} + V(\mathbf{x})\mathbf{1}, \quad d \geq 2, \tag{4.4}$$

is selfadjoint in $L_2(\mathbb{R}^d)$ on the domain $H^1(\mathbb{R}^d)$. Here $V(\mathbf{x})$ is an electric potential; $\mathbf{A}(\mathbf{x})$ is a magnetic potential; usually $V_0 = m = \text{Const}$, where m is the mass of a particle, but it is more convenient to consider more general case.

The first result on the absolute continuity of the periodic Dirac operator was obtained by L. Danilov [D1] in the case $d \geq 3$, $\mathbf{A} \equiv 0$, $V_0 = m$. The main result of [D1] concerns the case $V \in C(\mathbb{R}^d)$. In [D2] the assumptions on $V(\mathbf{x})$ were relaxed, but still $\mathbf{A} \equiv 0$. The case $\mathbf{A} \not\equiv 0$ is much more difficult.

In [BSu3] the following theorem was proved.

Theorem 4.1 *Let A_j, V, V_0 be real-valued Γ-periodic functions satisfying* (2.9), (4.1) – (4.3). *Then the Dirac operator* (4.4) *is absolutely continuous.*

The approach of [BSu3] is based on relation between the square of the Dirac operator and the magnetic Schrödinger operator.

Independently L. Danilov [D3] obtained the same result in the case $d = 2$. Recently L. Danilov [D4] communicated to the author that the condition (2.9) in Theorem 4.1 can be replaced by the assumption

$$\mathbf{A} \in C^1(\mathbb{R}^d) \cap H^q_{\text{loc}}(\mathbb{R}^d), \quad 2q > d - 2, \quad d \geq 3,$$

or by the assumption that $d \geq 3$, $\mathbf{A} \in C^1$ and the Fourier series for $\mathbf{A}$ is absolutely convergent.

5. The periodic Maxwell operator

Let $\varepsilon(\mathbf{x})$, $\mu(\mathbf{x})$ be positively definite (3×3)-matrices with real-valued entries which are Γ-periodic in $\mathbf{x} \in \mathbb{R}^3$. Assume that

$$\varepsilon_0 \mathbf{1} \leq \varepsilon(\mathbf{x}) \leq \varepsilon_1 \mathbf{1}, \quad \mu_0 \mathbf{1} \leq \mu(\mathbf{x}) \leq \mu_1 \mathbf{1}, \quad \varepsilon_0,\ \varepsilon_1,\ \mu_0,\ \mu_1 > 0. \tag{5.1}$$

The selfadjoint Maxwell operator M acts in the Hilbert space

$$\mathcal{H} := \{(u,v) \in L_2(\mathbb{R}^3, \mathbb{C}^3; \varepsilon) \oplus L_2(\mathbb{R}^3, \mathbb{C}^3; \mu) : \ \mathrm{div}\varepsilon u = 0, \ \ \mathrm{div}\mu v = 0\}.$$

The operator M is given by the formula

$$M = \begin{pmatrix} 0 & i\varepsilon^{-1}\mathrm{rot} \\ -i\mu^{-1}\mathrm{rot} & 0 \end{pmatrix}, \quad \mathrm{Dom}M = \{(u,v) \in \mathcal{H} : \ \mathrm{rot}u \in L_2, \mathrm{rot}v \in L_2\}. \tag{5.2}$$

Conjecture 5.1 *Let $\varepsilon(\mathbf{x})$, $\mu(\mathbf{x})$ be Γ-periodic (3×3)-matrices with real-valued entries satisfying (5.1). Then the Maxwell operator M is absolutely continuous.*

Conjecture 5.1 was justified only in "isotropic" case when $\varepsilon(\mathbf{x})$ and $\mu(\mathbf{x})$ are positive functions (not matrices).

Theorem 5.2 *Let $\varepsilon(\mathbf{x})$ and $\mu(\mathbf{x})$ be positive periodic functions such that $\varepsilon,\ \mu \in C^2$. Then the Maxwell operator M is absolutely continuous.*

Comments. 1) In the case $\varepsilon,\ \mu \in C^\infty$ the absolute continuity of M was proved in [M2]. A. Morame found the useful relation (see (5.10), (5.11)) for the square of M. 2) The proof of [M2] can be simplified. At the same time conditions on ε and μ can be relaxed to $\varepsilon,\ \mu \in C^2$. Below we give a simple proof of Theorem 5.2.

Proof of Theorem 5.2. As usual, it is convenient to use an extension of the Maxwell operator. The extended selfadjoint operator $\mathcal{M}$ acts in the Hilbert space

$$\mathfrak{H} = L_2(\mathbb{R}^3, \mathbb{C}^3; \varepsilon) \oplus L_2(\mathbb{R}^3) \oplus L_2(\mathbb{R}^3, \mathbb{C}^3; \mu) \oplus L_2(\mathbb{R}^3)$$

on the domain $\mathrm{Dom}\mathcal{M} = \{(u, \alpha, v, \beta) \in \mathfrak{H} : \ \mathrm{div}\varepsilon u \in L_2, \ \ \mathrm{div}\mu v \in L_2, \ \ \mathrm{rot}u \in L_2, \ \ \mathrm{rot}v \in L_2\}$. The operator $\mathcal{M}$ is given by the formula

$$\mathcal{M}(u, \alpha, v, \beta) = \{-i\nabla\beta + i\varepsilon^{-1}\mathrm{rot}v, \ -i\mathrm{div}\mu v, \ -i\nabla\alpha - i\mu^{-1}\mathrm{rot}u, \ -i\mathrm{div}\varepsilon u\}.$$

The subspace $\mathcal{H}$ reduces $\mathcal{M}$. The part of $\mathcal{M}$ in $\mathcal{H}$ is M. It is sufficient to prove that the operator $\mathcal{M}$ is absolutely continuous. Since $\mathcal{M}$ is elliptic, it is sufficient [K1] to show that there are no eigenvalues. Assume that E is an eigenvalue of $\mathcal{M}$:

$$-i\nabla\beta + i\varepsilon^{-1}\mathrm{rot}v = Eu, \tag{5.3}$$

$$-i\mathrm{div}\mu v = E\alpha, \tag{5.4}$$

$$-i\nabla\alpha - i\mu^{-1}\mathrm{rot}u = Ev, \tag{5.5}$$

$$-i\mathrm{div}\varepsilon u = E\beta. \tag{5.6}$$

Applying divε to (5.3) and taking (5.6) into account, we have: $-\mathrm{div}\varepsilon\nabla\beta = E^2\beta$. Since the operator $-\mathrm{div}\varepsilon\nabla$ is absolutely continuous, then $\beta \equiv 0$. Similarly, by (5.4), (5.5), $\alpha \equiv 0$. Applying rot to (5.3) and (5.5), we obtain

$$\mathrm{rot}\,\varepsilon^{-1}\,\mathrm{rot}v = E^2\mu v, \quad \mathrm{div}\mu v = 0, \tag{5.7}$$

$$\mathrm{rot}\,\mu^{-1}\,\mathrm{rot}u = E^2\varepsilon u, \quad \mathrm{div}\varepsilon u = 0, \tag{5.8}$$

$$\mathrm{rot}v = -iE\varepsilon u, \quad \mathrm{rot}u = iE\mu v. \tag{5.9}$$

We use the following relations found in [M2]:

$$\mathrm{rot}\mu^{-1}\mathrm{rot}u = -\mu^{-1}\varepsilon^{-1/2}\Delta(\varepsilon^{1/2}u) + (\varepsilon\nabla(\varepsilon\mu)^{-1}) \times \mathrm{rot}u + V(\mathbf{x})u, \quad \mathrm{div}\varepsilon u = 0, \tag{5.10}$$

$$\mathrm{rot}\varepsilon^{-1}\mathrm{rot}v = -\varepsilon^{-1}\mu^{-1/2}\Delta(\mu^{1/2}v) + (\mu\nabla(\varepsilon\mu)^{-1}) \times \mathrm{rot}v + \widetilde{V}(\mathbf{x})v, \quad \mathrm{div}\mu v = 0, \tag{5.11}$$

where $V(\mathbf{x})$ and $\widetilde{V}(\mathbf{x})$ are (3×3)-matrices expressed in terms of ε and μ and their derivatives up to second order. Introduce notation $\Phi = \varepsilon^{1/2}u$, $\Psi = \mu^{1/2}v$. From (5.7)–(5.11) we have

$$\left.\begin{aligned} -\Delta\Phi - 2iE(\nabla(\varepsilon\mu)^{1/2}) \times \Psi + \mu V(\mathbf{x})\Phi - E^2\varepsilon\mu\Phi = 0 \\ -\Delta\Psi + 2iE(\nabla(\varepsilon\mu)^{1/2}) \times \Phi + \varepsilon\widetilde{V}(\mathbf{x})\Psi - E^2\varepsilon\mu\Psi = 0 \end{aligned}\right\} \tag{5.12}$$

The system (5.12) can be rewritten as the Schrödinger equation for a pair $U = \{\Phi, \Psi\}$: $-\Delta U + \mathcal{V}(\mathbf{x})U = 0$, with an appropriate periodic matrix-valued potential $\mathcal{V}(\mathbf{x})$. Mention that $\mathcal{V}(\mathbf{x})$ is a nonsymmetric matrix. We arrive at a contradiction since the Schrödinger operator has no eigenvalues (see Remark 2.3). •

In the above argument (in contrast to [M2]) we did not use the Floquet decomposition and the Thomas scheme explicitly.

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