JOURNÉES ÉQUATIONS AUX DÉRIVÉES PARTIELLES

ANDRÉ UNTERBERGER

From pseudodifferential analysis to modular form theory

Journées Équations aux dérivées partielles (1999), p. 1-11

<http://www.numdam.org/item?id=JEDP_1999____A15_0>

Journées *Équations aux dérivées partielles*
Saint-Jean-de-Monts, 31 mai–4 juin 1999
GDR 1151 (CNRS)

From pseudodifferential analysis to modular form theory

André Unterberger

Abstract

Taking advantage of methods originating with quantization theory, we try to get some better hold — if not an actual construction — of Maass (non-holomorphic) cusp–forms. We work backwards, from the rather simple results to the mostly devious road used to prove them.

1. Introduction.

We first freshen up the reader's memory on the subject of non-holomorphic modular forms. With $G = SL(2,\mathbb{R})$, the Poincaré upper half-plane can be considered as the homogeneous space $\Pi = G/K$, $K = SO(2)$, if G is to act on Π by the usual formula

$$g.z = \frac{az+b}{cz+d}, \qquad g = \left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right). \tag{1.1}$$

The hyperbolic Laplacian

$$\Delta = -y^2 \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) \tag{1.2}$$

is a G-invariant differential operator on Π. With $\Gamma = SL(2,\mathbb{Z})$, one then defines non-holomorphic modular forms (*e.g.* [1] or [4]) as those Γ-invariant functions on Π which are eigenfunctions of Δ, bounded by some power of $y = \text{Im } z$ in the fundamental domain

$$F = \{z \in \Pi\colon\ -\frac{1}{2} < x < \frac{1}{2}\ ;\ |z| > 1\}. \tag{1.3}$$

Explicit examples of such are constants, and (non-holomorphic) Eisenstein series $E_{\frac{1-\nu}{2}}$ defined as

$$E_{\frac{1-\nu}{2}}(z)\colon = \frac{1}{2} \sum_{\substack{|m|+|n|\neq 0 \\ (n,m)=1}} \left(\frac{|mz-n|^2}{\text{Im } z}\right)^{\frac{\nu-1}{2}} \tag{1.4}$$

when Re $\nu < -1$, then by complex continuation as a meromorphic function of ν in the entire complex plane. The function $E_{\frac{1-\nu}{2}}$ satisfies the equation $\Delta E_{\frac{1-\nu}{2}} = \frac{1-\nu^2}{4} E_{\frac{1-\nu}{2}}$.

The examples that precede correspond to the isolated eigenvalue 0 and (for ν pure imaginary) to the continuous part of the spectrum of some appropriate extension of Δ as an operator on $L^2(\Gamma\backslash\Pi)$, the space of functions on Π which are Γ-invariant, and square-integrable with respect to the invariant measure $d\mu(z) = \frac{dx\,dy}{y^2}$ once restricted to F. The automorphic Laplacian also has a discrete spectrum. Here is an easy proof that we do not yet have a full set of (generalized or not) eigenfunctions: take the Poisson bracket of two distinct Eisenstein series and observe that it is, in some obvious sense, an odd function under the symmetry $z \mapsto -\bar{z}$, while constants and Eisenstein series are even functions under the same symmetry. The Selberg trace formula yields considerably more precise results, to the effect that there are infinitely many linearly independent genuine eigenfunctions of Δ lying in $L^2(\Gamma\backslash\Pi)$ (such functions are hereafter referred to as cusp-forms, or Maass cusp-forms); moreover, there is an $N(\lambda)$ equivalent, a simple constant times λ.

Still, the nature of the eigenvalues of the automorphic Laplacian, and that of its eigenfunctions, is essentially a mystery. There are some elements of answer, based on the consideration of certain complicated Dirichlet series (*cf.* the theory of Kloosterman sums), the poles and residues of which correspond to the eigenvalues of Δ and to the Fourier coefficients (*cf.* *infra*) of associated cusp-forms. Our present results go in the same direction: however, some of our Dirichlet series have rather simple coefficients, and residues of appropriate generalizations of Eisenstein series yield at once the Maass cusp-forms, without any need for resumming Fourier series. Moreover, our methods are unrelated to the two ones practised by all authors after Selberg: the one based on the consideration of the so-called Poincaré series, and that based on the use of the integral kernel of the resolvant. Rather, they rely on facts of structure inspired by pseudodifferential analysis, even though, at the end, it would be possible to dispense with the latter one.

Before we leave this introduction, let us point out that the separation of variables method makes it possible to write any non-holomorphic modular form f, corresponding to the eigenvalue $\frac{1-\nu^2}{4}$, as a Fourier series (with respect to x)

$$f(z) = a_0\, y^{\frac{1-\nu}{2}} + a_1\, y^{\frac{1+\nu}{2}} + y^{\frac{1}{2}} \sum_{n\neq 0} b_n\, K_{\frac{\nu}{2}}(2\pi|n|y)\, e^{2i\pi nx} . \tag{1.5}$$

For instance, for the function $f(z) = \zeta^*(1-\nu)\, E_{\frac{1-\nu}{2}}(z)$, with

$$\zeta^*(s) :\; = \pi^{-\frac{s}{2}}\, \Gamma(\frac{s}{2})\, \zeta(s)\,, \tag{1.6}$$

the expansion above holds with $a_0 = \zeta^*(1-\nu)$, $a_1 = \zeta^*(1+\nu)$, and $b_n = 2\,|n|^{-\frac{\nu}{2}}\, \sigma_\nu(n)$, where $\sigma_\nu(n)$ denotes the sum of ν-th powers of all positive divisors of n.

2. A first example.

Theorem 2.1 *Given $n \in \mathbb{Z}^\times$, set, for every $m \geq 1$,*

$$a_n(m) :=\sum_{\substack{r \bmod m \\ r(1-r)\equiv 0}} e^{2i\pi n \frac{r}{m}} . \tag{2.1}$$

This can also be written as

$$a_n(m) = \prod_{j=1}^{k} (1 + e^{2i\pi n \frac{x_j}{m}}) \tag{2.2}$$

where, if $m = p_1^{\alpha_1} \dots p_k^{\alpha_k}$ is the decomposition of m as a product of prime factors, x_j is for each $j = 1, \dots, k$ the solution (mod m) of the Chinese remainder problem $x_j \equiv 1$ mod $p_j^{\alpha_j}$, $x_j \equiv 0$ mod $p_\ell^{\alpha_\ell}$ for $\ell \neq j$. Consider, for Re $s > 1$, *the Dirichlet series*

$$\psi_n(s) = \sum_{m \geq 1} a_n(m)\, m^{-s} . \tag{2.3}$$

It extends as a meromorphic function to the half-plane Re $s > 0$, $s \neq 1$. *The points which are poles of at least one of the functions ψ_n are all the points $\frac{\omega}{2}$, ω a non-trivial zero of the zeta-function, and some points $s_k = \frac{1-i\lambda_k}{2}$, $s_k(1-s_k) = \frac{1+\lambda_k^2}{4}$ in the discrete spectrum of the automorphic Laplacian.*

Which points s_k exactly? We answer this question, in terms which depend on the concept of L-function, in the next section.

3. Hecke's theory.

As all PDE practitioners know, one should never study a linear operator Δ without considering at the same time all operators that commute with Δ (fully or principally only), which one can put one's hands on. We have already seen, in the introduction, that it is useful, in our case, to consider the operator $f \mapsto \check{f}$ with $\check{f}(z) = f(-\bar{z})$: this permits to distinguish *even* eigenvalues from *odd* ones. On the other hand, Δ commutes with all Hecke operators T_N, $N \geq 1$. These are not differential operators, but operators of an arithmetic nature:

$$(T_N f)(z) = N^{-\frac{1}{2}} \sum_{\substack{ad = N, d > 0 \\ b \bmod d}} f(\frac{az+b}{d}) . \tag{3.1}$$

The operators from the system $\{\Delta,\ \{T_N\}_{N\geq 1}\}$ are self-adjoint and commute with one another. It is thus possible, for each even eigenvalue $\frac{1+\lambda_k^2}{4}$ of Δ, to find a (finite) basis $\{\mathcal{M}_{k,\ell}\}_\ell$ of *Hecke-Maass* forms, joint eigenfunctions of the whole system, making up an orthonormal set. One can adjust the first Fourier coefficient b_1 in the expansion (1.5) of each $\mathcal{M}_{k,\ell}$ to a positive real value, which will be part of our assumptions. Sometimes, it is better to substitute for $\mathcal{M}_{k,\ell}$ the function $\mathcal{N}_{k,\ell}$, proportional to the former one, normalized by the condition $b_1 = 1$.

With any cusp-form $\mathcal{M}$, one associates its L-function $L(.,\mathcal{M})$ defined as the Dirichlet series

$$L(s,\mathcal{M}) = \sum_{n\geq 1} b_n\, n^{-s}\,, \tag{3.2}$$

the coefficients of which have been borrowed from the Fourier expansion (1.5) with $f = \mathcal{M}$. It is only absolutely convergent when Re s is rather large (just how large depends on unproven conjectures), but a very easy holomorphic continuation to the whole complex plane is possible: it is useful to contribute some extra Γ-factors, substituting

$$L^*(s,\mathcal{M}) \colon = \pi^{-s}\,\Gamma(\frac{s}{2}+\frac{i\lambda_k}{4})\,\Gamma(\frac{s}{2}-\frac{i\lambda_k}{4})\,L(s,\mathcal{M}) \tag{3.3}$$

for $L(s,\mathcal{M})$ if $\mathcal{M}$ is associated with the eigenvalue above, so as to get a simple functional equation.

Theorem 2.1 can then be made fully precise as follows: the eigenvalues $\frac{1+\lambda_k^2}{4}$ which occur as poles of at least one of the ψ_n's are those for which $L(\frac{1}{2},\mathcal{M}) \neq 0$ for at least one *even* cusp-form $\mathcal{M}$ with such an eigenvalue.

It is easy (we shall not do it, from lack of space, but the next section will give the idea) to generalize theorem 2.1 so that the presence of any given eigenvalue should depend on the value of L-functions at generic points, rather than $\frac{1}{2}$. In this way, we can get *all* even eigenvalues of Δ. Also, a modification makes it possible to get the odd eigenvalues as well: in the latter case, the zeros of the zeta-function do not enter the family of poles of the appropriate Dirichlet series.

4. A generating series of sorts for Maass cusp-forms.

Assuming $|\text{Re}\ \nu| < 1$, set

$$F_{\mu,\nu}(z) = \frac{1}{2} \sum_{\substack{n\in\mathbb{Z},m\in\mathbb{Z}^\times \\ m|n(n-1)}} (\frac{m_1}{m_2})^{\frac{\nu}{2}}\,|m|^{\frac{1-i\mu}{2}} \left(\frac{|mz-n|^2}{\text{Im}\ z}\right)^{\frac{i\mu-1}{2}}\,, \tag{4.1}$$

where the pair m_1, m_2 is the pair of positive integers characterized by the conditions $|m| = m_1 m_2,\ m_1|n-1,\ m_2|n$.

One can show that the preceding series converges when $\mathrm{Im}\ \mu > 1 + |\mathrm{Re}\ \nu|$ but that, except for the poles $\mu = i(1 \pm \nu)$, the function

$$\mu \mapsto F_{\mu,\nu}(z) - F_{\mu,\nu}(-\frac{1}{z})$$

extends as a holomorphic function to the larger half-plane $\mathrm{Im}\ \mu > -1+|\mathrm{Re}\ \nu|$. Now, as a function of z, $F_{\mu,\nu}(z)$ is not modular, but it satisfies the equation $\Delta F_{\mu,\nu} = \frac{1+\mu^2}{4} F_{\mu,\nu}$ and it is already $\mathbb{Z}$-periodic. Thus, if one can prove (one can) that, as a function of μ, it extends as a *meromorphic* function to the larger half-plane above, it is clear that the coefficients of its polar parts at poles contained in this half-plane will have to be modular forms!

Theorem 4.1 *Assume* $|\mathrm{Re}\ \nu| < 1$. *The function* $\mu \mapsto F_{\mu,\nu}(z)$ *extends as a holomorphic function for* $\mathrm{Im}\ \mu > -1 + |\mathrm{Re}\ \nu|$, $\mu \neq i(1 \pm \nu)$, *except for the following poles: the points* $-i\omega$ *with* $\zeta^*(\omega) = 0$ *contained in this half-plane, and the points* λ_k, $\frac{1+\lambda_k^2}{4}$ *in the even part of the discrete spectrum of* Δ. *Near a point* $-i\omega$, *the function*

$$\mu \mapsto F_{\mu,\nu}(z) - \pi^{\frac{1}{2}} \frac{\Gamma(-\frac{i\mu}{2})}{\Gamma(\frac{1-i\mu}{2})} \frac{\zeta(\frac{1-i\mu-\nu}{2})\,\zeta(\frac{1-i\mu+\nu}{2})}{\zeta(1-i\mu)} E_{\frac{1+i\mu}{2}}(z) \tag{4.2}$$

remains holomorphic. A point λ_k *can only be a simple pole, and the residue there of the function under discussion is*

$$-i\,\pi^{-\frac{\nu}{2}} \frac{\Gamma(-\frac{i\lambda_k}{2})\,\Gamma(\frac{1+i\lambda_k+\nu}{4})}{\Gamma(\frac{1-i\lambda_k-\nu}{4})} \mathcal{M}_{\nu;k}(z)\,, \tag{4.3}$$

where

$$\mathcal{M}_{\nu;k}: = \sum_{\ell} L(\frac{1+\nu}{2}, \mathcal{M}_{k,\ell})\, \mathcal{M}_{k,\ell}\,. \tag{4.4}$$

Thus all even Maass cusp-forms can be obtained as residues of rather simple Eisenstein-like series; again, there are corresponding results in the odd case. It takes considerable work to prove this theorem (or theorem 2.1 as well): one of the main points is an expansion of the product (one would take a Poisson bracket instead in the odd case) of any two Eisenstein series.

5. Products of Eisenstein series.

Theorem 5.1 *Let* ν_1 *et* ν_2 *be complex numbers with* $\mathrm{Re}\ (\nu_1 \pm \nu_2) \neq \pm 1$ *and* $\nu_1, \nu_2 \neq -1, 0, 1$, *finally* $\nu_1 \pm \nu_2 \neq 0$. *Let*

$$\Sigma = \{(\varepsilon_1 = \pm 1,\ \varepsilon_2 = \pm 1) : \varepsilon_1\, \mathrm{Re}\ \nu_1 + \varepsilon_2\, \mathrm{Re}\ \nu_2 < 1\}\,. \tag{5.1}$$

Then

$$\zeta^*(1-\nu_1)\,\zeta^*(1-\nu_2)\,E_{\frac{1-\nu_1}{2}}(z)\,E_{\frac{1-\nu_2}{2}}(z) =$$
$$\sum_{(\varepsilon_1,\varepsilon_2)\in\Sigma} \zeta^*(1-\varepsilon_1\nu_1)\,\zeta^*(1-\varepsilon_2\nu_2)\,E_{1-\frac{\varepsilon_1\nu_1+\varepsilon_2\nu_2}{2}}(z)$$
$$+\sum_{k,\ell} f^{k,\ell}_{\nu_1,\nu_2}\,\|\mathcal{N}_{k,\ell}\|^{-2}\,\mathcal{N}_{k,\ell}(z)$$
$$+\frac{1}{8\pi}\int_{-\infty}^{\infty}\Phi(\nu_1,\nu_2;\lambda)\,E_{\frac{1-i\lambda}{2}}(z)\,d\lambda\,, \quad (5.2)$$

with

$$f^{k,\ell}_{\nu_1,\nu_2} = \frac{1}{2}\,L^*(\frac{1-\nu_1-\nu_2}{2},\mathcal{N}_{k,\ell})\,L^*(\frac{1+\nu_1-\nu_2}{2},\mathcal{N}_{k,\ell}) \quad (5.3)$$

and

$$\Phi(\nu_1,\nu_2;\lambda) =$$
$$\frac{\zeta^*(\frac{1+i\lambda-\nu_1+\nu_2}{2})\,\zeta^*(\frac{1+i\lambda+\nu_1-\nu_2}{2})\,\zeta^*(\frac{1-i\lambda-\nu_1-\nu_2}{2})\,\zeta^*(\frac{1+i\lambda-\nu_1-\nu_2}{2})}{\zeta^*(-i\lambda)}\,. \quad (5.4)$$

The proof of theorem 5.1 is very technical. It is based on the use of the Radon transform and on a non-trivial extension of the so-called Rankin-Selberg *unfolding* method: this is a trick for recovering the coefficient Φ of the continuous part from the Roelcke-Selberg expansion — *i.e.*, the spectral decomposition — of a reasonably general function in $L^2(\Gamma\backslash\Pi)$. Hyperfunctions are made use of in our version, since the problems of complex continuation which arise here demand that one should add holomorphic functions with disjoint domains!

Also, it is necessary for the application to theorems 2.1 and 4.1 to make a detailed study (which follows from a somewhat deeper examination, under the scrutiny of Hecke's theory, of the discrete terms in the expansion above) of the complex continuation of the Dirichlet series in two variables

$$\zeta_n(s,t) = \sum_{\substack{m_1m_2\neq 0\\(m_1,m_2)=1}} |m_1|^{-s}\,|m_2|^{-t}\,e^{2i\pi n\frac{\overline{m}_2}{m_1}}\,, \quad (\overline{m}_2 m_2 \equiv 1 \bmod m_1). \quad (5.5)$$

6. The Radon transform, pseudodifferential analysis and modular forms.

It is possible, as we found out while preparing this lecture, to describe the Radon transform which has just been alluded to (in fact, a slightly more interesting object!), usually a topic in Harmonic Analysis, in terms more suitable to a PDE environment: namely, as a link between the space of Cauchy data associated to the Lax-Phillips scattering theory for the automorphic wave equation on one hand, and a concept of automorphic Weyl symbols on the other hand.

Our first theorem works in a non-arithmetic setting: the hyperbolic Laplacian, as well as the square-root of $\Delta - \frac{1}{4}$, is considered on the half-plane, not in the fundamental domain. The three-dimensional domain C is the forward light-cone

$$C = \{\eta \colon \eta_0 > 0,\ \eta_0^2 - \eta_1^2 - \eta_2^2 > 0\}, \tag{6.1}$$

identified as in (6.4) below with the set of positive-definite 2×2-matrices and, if h is an even function on $\mathbb{R}^2$ (a distribution would be fine too), Qh is the function on C defined by

$$h(\xi_1, \xi_2) = (Qh)\left(\frac{\xi_1^2 + \xi_2^2}{2}, \frac{\xi_1^2 - \xi_2^2}{2}, \xi_1\xi_2\right). \tag{6.2}$$

The operator $\Box$ is the standard wave operator

$$\Box = \frac{\partial^2}{\partial\eta_0^2} - \frac{\partial^2}{\partial\eta_1^2} - \frac{\partial^2}{\partial\eta_2^2}. \tag{6.3}$$

Theorem 6.1 *Recall from [3], p.11, that under the map*

$$(t, z) \mapsto \begin{pmatrix} \eta_0+\eta_1 & \eta_2 \\ \eta_2 & \eta_0-\eta_1 \end{pmatrix} = e^t \begin{pmatrix} \frac{|z|^2}{y} & \frac{x}{y} \\ \frac{x}{y} & \frac{1}{y} \end{pmatrix} \tag{6.4}$$

from $\mathbb{R} \times \Pi$ to C and under the gauge transformation $u \mapsto W = e^{-\frac{t}{2}} u$, the equation $\Box W = 0$ inside C is equivalent to the wave equation

$$\frac{\partial^2 u}{\partial t^2} + (\Delta - \frac{1}{4})\, u = 0, \tag{6.5}$$

in which Δ denotes the hyperbolic Laplacian on Π.

Consider on one hand the (classical) Cauchy problem

$$\begin{cases} \frac{\partial^2 u}{\partial t^2} + (\Delta - \frac{1}{4})\, u = 0 \\ u(0, z) = f_0(z) \\ \frac{\partial u}{\partial t}(0, z) = f_1(z)\ , \end{cases} \tag{6.6}$$

on the other hand the characteristic problem

$$\begin{cases} \check{W} = 0 \ \text{ in } \ \mathbb{R}^3 \backslash C \\ \Box \check{W} = Qh. \frac{d\eta_1\, d\eta_2}{\eta_0} \end{cases} \tag{6.7}$$

in which $\check{W}$ denotes the function W extended by 0 in $\mathbb{R}^3 \backslash C$, and Qh is the function on ∂C associated by (6.2) to some even function h on $\mathbb{R}^2$: the right-hand side of (6.7) is the measure supported by ∂C with density Qh with respect to the (Lorentz-invariant) canonical one.

Using appropriate Hilbert spaces of data, as indicated by the formula

$$\|h\|^2_{L^2_{\text{even}}(\mathbb{R}^2)} = 2\pi^2 \left[\|(\Delta - \frac{1}{4})^{\frac{1}{2}} f_0\|^2_{L^2(\Pi)} + \|f_1\|^2_{L^2(\Pi)} \right], \tag{6.8}$$

one can set up a one-to-one correspondence between the two problems.

Moreover, through the map $(f_0, f_1) \mapsto h$ *so defined, the operator* $\begin{pmatrix} 0 & I \\ -\Delta + \frac{1}{4} & 0 \end{pmatrix}$ *the study of which plays the major part in [3] is taken to the Euler operator* $i\pi\mathcal{E} := \frac{1}{2}(\xi_1 \frac{\partial}{\partial \xi_1} + \xi_2 \frac{\partial}{\partial \xi_2} + 1)$ *on* $\mathbb{R}^2$.

We now move to an arithmetic (*i.e.*, Γ-invariant) environment. For the space $L^2(\Pi)$ we substitute of course $L^2(\Gamma\backslash\Pi)$, substituting at the same time $|\Delta - \frac{1}{4}|^{\frac{1}{2}}$ for $(\Delta - \frac{1}{4})^{\frac{1}{2}}$ to allow for the isolated eigenvalue corresponding to constant eigenfunctions. However, much more care is called for when one wishes to define a space $L^2_{\text{even}}(\Gamma\backslash\mathbb{R}^2)$ in a natural way.

For, now, $G = SL(2, \mathbb{R})$ acts on $\mathbb{R}^2$ in a linear way, and there is no fundamental domain for the action of Γ (the orbit of any (ξ_1, ξ_2) with $\frac{\xi_1}{\xi_2} \notin \mathbb{Q}$ is dense in $\mathbb{R}^2$). Thus, there are no non-constant reasonable Γ-invariant functions on $\mathbb{R}^2$. However, there is a natural concept of Γ-invariant, or automorphic for short, distribution: the Dirac comb on $\mathbb{R}^2$, or the Dirac distribution at the origin, are the obvious examples. To define a useful Hilbert space of even tempered automorphic distributions is another matter, which can be dealt with in different, but equivalent, ways. The simplest one is based on the use of the following two families of coherent states in $L^2(\mathbb{R})$:

$$u_z(t) = 2^{\frac{1}{4}} \left(\text{Im}\, \frac{1}{\bar{z}}\right)^{\frac{1}{4}} \exp \frac{i\pi\, t^2}{\bar{z}},$$
$$u_z^1(t) = 2^{\frac{5}{4}}\, \pi^{\frac{1}{2}} \left(\text{Im}\, \frac{1}{\bar{z}}\right)^{\frac{3}{4}} t \exp \frac{i\pi\, t^2}{\bar{z}}, \tag{6.9}$$

parametrized by $z \in \Pi$: using the metaplectic representation, observe that they constitute total spaces in the two subspaces of $L^2(\mathbb{R})$ consisting of even (*resp.* odd) functions. If $\mathfrak{S}$ is an even tempered automorphic distribution, and $\text{Op}(\mathfrak{S})$ is the operator with Weyl symbol $\mathfrak{S}$, then the functions $(u_z|\text{Op}(\mathfrak{S})u_z)$ and $(u_z^1|\text{Op}(\mathfrak{S})u_z^1)$ are Γ-invariant as functions of z, and the Hilbert space $L^2_{\text{even}}(\Gamma\backslash\mathbb{R}^2)$ can be defined as the one associated with the norm such that

$$\begin{aligned} \|\mathfrak{S}\|^2_{L^2_{\text{even}}(\Gamma\backslash\mathbb{R}^2)} : &= \frac{1}{2} \| z \mapsto (u_z|\text{Op}(\mathfrak{S})u_z) \|^2_{L^2(\Gamma\backslash\Pi)} \\ &+ \frac{1}{8} \| \, |\Delta - \frac{1}{4}|^{-\frac{1}{2}} (z \mapsto (u_z^1|\text{Op}(\mathfrak{S})u_z^1)) \|^2_{L^2(\Gamma\backslash\Pi)} . \end{aligned} \tag{6.10}$$

In the Γ-invariant case, theorem 6.1 extends as follows.

Theorem 6.2 *The map* $(f_0,\, f_1) \mapsto h$ *defined in theorem* 6.1 *extends to the* Γ-*invariant case. If one decomposes it as* $(f_0,\, f_1) \mapsto \mathfrak{S} \mapsto h$, *with* $h = 2^{\frac{1}{2}}\, \pi^{-i\pi\mathcal{E}}\, \Gamma(i\pi\mathcal{E})\, \mathfrak{S}$, *i.e. (using for distributions the same notation as for functions)*

$$h(\xi) = 2^{\frac{1}{2}} \int_0^\infty \mathfrak{S}(t^{\frac{1}{2}}\,\xi)\, e^{-2\pi t}\, t^{-\frac{1}{2}}\, dt, \qquad \xi \in \mathbb{R}^2\,, \tag{6.11}$$

one then has

$$\frac{1}{2}\,\|\mathfrak{S}\|^2_{L^2_{\mathrm{even}}(\Gamma\backslash\mathbb{R}^2)} = \|\,|\Delta - \frac{1}{4}|^{\frac{1}{2}}\, f_0\|^2_{L^2(\Gamma\backslash\Pi)} + \|f_1\|^2_{L^2(\Gamma\backslash\Pi)}\,, \tag{6.12}$$

where the right-hand side is just the square of the norm introduced by Lax and Phillips on the space of Cauchy data.

7. The composition of Weyl symbols.

Besides substituting for the matrix-operator $\begin{pmatrix} 0 & I \\ -\Delta + \frac{1}{4} & 0 \end{pmatrix}$ the simple first-order operator $i\pi\,\mathcal{E}$, theorem 6.2 has another advantage. It endows the Lax-Phillips space of Cauchy data with the algebraic structure transferred from the Weyl composition of symbols. It is of course interesting to make this transfer explicit. This calls for a preliminary study of how the composition # of (even, for simplicity) Weyl symbols behaves under the decomposition of symbols into homogeneous terms.

Using a Fourier transformation, set for any such symbol $h = \int_{-\infty}^{\infty} h_\lambda\, d\lambda$, where the function h_λ is homogeneous of degree $-1 - i\lambda$: this latter function can be recovered from the function $h^\flat_\lambda$ on the real line, defined as $h^\flat_\lambda(s) = h_\lambda(s, 1)$. Then there should exist an integral kernel $K_{\lambda_1,\lambda_2;\lambda}(s_1, s_2; s)$, depending on the three real parameters indicated, that should, under reasonable conditions (say, when dealing with Hilbert-Schmidt operators) make the formula

$$(h_1 \# h_2)^\flat_\lambda(s) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} d\lambda_1\, d\lambda_2 \int_{\mathbb{R}^2} K_{\lambda_1,\lambda_2;\lambda}(s_1, s_2; s)\; (h_1)^\flat_{\lambda_1}(s_1)\, (h_2)^\flat_{\lambda_2}(s_2)\, ds_1\, ds_2 \tag{7.1}$$

valid in some weak sense. Indeed, one finds that it works with

$$K_{\lambda_1,\lambda_2;\lambda}(s_1, s_2; s) = 2^{-\frac{3}{2}}\,(2\pi)^{\frac{i(-\lambda+\lambda_1+\lambda_2)-2}{2}} \times \sum_{j=0}^{1} i^j\, \frac{\Gamma(\frac{1+i(\lambda+\lambda_1-\lambda_2)+2j}{4})\Gamma(\frac{1+i(\lambda-\lambda_1+\lambda_2)+2j}{4})\Gamma(\frac{1+i(-\lambda-\lambda_1-\lambda_2)+2j}{4})}{\Gamma(\frac{1+i(-\lambda-\lambda_1+\lambda_2)+2j}{4})\Gamma(\frac{1+i(-\lambda+\lambda_1-\lambda_2)+2j}{4})\Gamma(\frac{1+i(\lambda+\lambda_1+\lambda_2)+2j}{4})} \times \chi^j_{i\lambda_1,i\lambda_2;i\lambda}(s_1, s_2; s) \tag{7.2}$$

if we set

$$\chi^1_{\nu_1,\nu_2;\nu}(s_1, s_2; s) = |s_1 - s_2|^{\frac{1}{2}(-1+\nu+\nu_1+\nu_2)}|s_1 - s|^{\frac{1}{2}(-1-\nu+\nu_1-\nu_2)}|s_2 - s|^{\frac{1}{2}(-1-\nu-\nu_1+\nu_2)} \times \mathrm{sign}\left(\frac{s_1 - s_2}{(s - s_1)(s_2 - s)}\right) \tag{7.3}$$

and $\chi^0_{\nu_1,\nu_2;\nu}(s_1, s_2; s)$ is just the same function with the sign on the right-hand side removed.

What this formula says is that the composition of Weyl symbols can be essentially reduced to the study of the two bilinear operations — involving functions of one variable only — associated with the integral kernels $\chi^j_{\nu_1,\nu_2;\nu}(s_1, s_2; s)$, $j = 0$ or 1; actually, one term corresponds to the commutator and the other one to the anticommutator of the two operators involved.

Finally, as another lengthy computation shows, transferring the bilinear operations just referred to to an operation on Lax-Phillips Cauchy data reduces to computing the spectral decomposition of the product or Poisson bracket of any two eigenfunctions of Δ: this explains our interest in these matters, especially in the Γ-invariant case.

8. Conclusion.

Our interest in the Roelcke-Selberg expansion of products (*cf.* theorem 5.1) or Poisson brackets of Eisenstein series might have originated from our desire to make the composition of automorphic Weyl symbols in $\mathbb{R}^2$ fully explicit. Actually, it stemmed from a similar, not identical, question, connected to another species of symbolic calculus. In any case, the Radon transform — in one version or another — led to an extension of the Rankin-Selberg unfolding method or, more generally, to a new way to recover the coefficients of the spectral decompositions of arbitrary functions in $L^2(\Gamma\backslash\Pi)$. Theorem 5.1 and related ones brought benefits in the direction of some new information about Maass cusp forms: these were not unexpected in view of earlier work done in the holomorphic case [5], in connection with the so-called Rankin-Cohen products [2]. Finally, it has become clear to us that considering automorphic distributions on $\mathbb{R}^2$ as Weyl symbols brings together a considerable amount of structure: we hope that the applications of this point of view are far from exhausted.

References

[1] D.Bump, *Automorphic Forms and Representations*, Cambridge Series in Adv.Math. **55**, Cambridge, 1996.

[2] H.Cohen, *Sums involving the values at negative integers of L-functions of quadratic characters*, Math.Ann. **217** (1975), 271–295.

[3] P.D.Lax, R.S.Phillips, *Scattering Theory for Automorphic Functions*, Ann. Math. Studies **87**, Princeton Univ. Press, 1976.

[4] A.Terras, *Harmonic analysis on symmetric spaces and applications* 1, Springer-Verlag, New York-Berlin-Heidelberg-Tokyo, 1985.

[5] A.Unterberger, J.Unterberger, *Algebras of Symbols and Modular Forms*, J. d'Analyse Math. **68** (1996), 121–143.

MATHÉMATIQUES (UPRESA 6056), UNIVERSITÉ DE REIMS, MOULIN DE LA HOUSSE, B.P.1039, F-51687 REIMS CEDEX 2
andre.unterberger@univ-reims.fr