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Similarity Stabilizes Blow-up

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Abstract

The blow-up of solutions to a quasilinear heat equation is studied using a similarity transformation that turns the equation into a nonlocal equation whose steady solutions are stable. This allows energy methods to be used, instead of the comparison principles used previously. Among the questions discussed are the time and location of blow-up of perturbations of the steady blow-up profile.

1. Introduction.

Solutions of the PDE

$$u_t = \Delta u^2 + u^2 \tag{1.1}$$

with nonnegative compact initial data remain nonnegative and compactly supported, and blow up in a finite time, in both the one-dimensional ([SGKM], [BG]) and multi-dimensional cases ([CDE], [CEF]). Furthermore, when appropriately rescaled the solution tends to an asymptotic profile as the blow-up time approaches.

These results have been obtained using delicate comparison arguments. They will be obtained here using only energy methods, via a different scaling method. This approach also yields some new results, such as estimates for the blow-up time.

The basic ideas of this energy method will be discussed here. Complete proofs, additional results, and generalizations to higher-order equations will appear in [S]. Thanks to Philip Rosenau for introducing me to the problems considered here.

2. Similarity.

Since the maximum of u increases rapidly and blows up in a finite time, let us set u equal to a time-dependent positive growth factor times a function that retains dependence on the spatial variables:

$$u(t,x) = \phi(t)v(\tau(t),x) \tag{2.1}$$

Substituting (2.1) into the PDE (1.1) yields

$$\phi' v + \phi\tau' v_\tau = \phi^2\left(\Delta v^2 + v^2\right).$$

In order to make the factors of ϕ balance, set

$$\tau'(t) = \phi(t) \qquad \text{and} \qquad \phi' = \lambda\phi^2. \tag{2.2}$$

The equation then becomes

$$v_\tau = \Delta v^2 + v^2 - \lambda v. \tag{2.3}$$

The factor λ is usually chosen to be one, which yields $\phi = \frac{1}{T-t}$ and $\tau = \log\frac{1}{T-t}$, where T is the blow-up time in the original time variable t. Note that in the new τ variable, the blow-up time has been mapped to infinity.

Theorem ([BG], [CEF], [SGKM]): There exists a sequence of times τ_j tending to infinity such that $v(\tau_j, x)$ tends to a time-independent solution $w(x)$ of (2.3) with $\lambda = 1$, i.e. $0 = \Delta w^2 + w^2 - w$.

One reason this theorem is difficult is that equation (2.3) is unstable: although the particular solution $v(\tau, x)$ exists for all time by construction since its blow-up time has been mapped to $\tau = \infty$, there exist other solutions that blow up in a finite time.

This difficulty will be eliminated here by making λ depend on the solution v. Specifically, we will choose λ so as to make an appropriate L^p norm of v remain constant. In a recent numerical investigation of this PDE ([LR]), λ was chosen so as to make the L^∞ norm of the solution remain constant. Although this choice is natural in numerical calculations, the natural choice for analysis of the PDE turns out to be $p = 3$. .

The functional $\lambda[v]$ is therefore determined by solving for λ in the equation

$$0 = \frac{1}{3}\frac{d}{\tau}\int v^3 = \int v^2 v_\tau = \int v^2(\Delta v^2 + v^2 - \lambda v),$$

which yields

$$\lambda = \frac{\int v^4 + v^2\Delta v^2}{\int v^3} = \frac{\int (v^2)^2 - |\nabla v^2|^2}{\int v^3} \tag{2.4}$$

Taking the τ derivative of this formula while remembering that the denominator $\int v^3$ is independent of τ yields

$$\frac{d}{\tau}\lambda = \frac{\int v v_\tau^2}{\int v^3} \geq 0 \tag{2.5}$$

3. Bounds and convergence.

By construction,

$$\|v\|_{L^3} = \|v_0\|_{L^3} \tag{3.1}$$

By using Gagliardo-Nirenberg inequalities, one obtains from the formula for λ that

$$\lambda \leq C(\|v\|_{L^3}) = C(\|v_0\|_{L^3}) \tag{3.2}$$

In turn, estimate (3.2) plus the definition (2.4) of λ and equation (2.5) for $\lambda_\tau 0$ imply that

$$\|v^2\|_{H^1} \leq c \tag{3.3}$$

and

$$(v^{3/2})_\tau \in L^2(\mathbf{R} \times \mathbf{R}^d) \tag{3.4}$$

where d is the spatial dimension.

Now define $v_n(\tau, x) = v(n + \tau, x)$, so that the sequence v_n satisfies estimates (3.1-3.4) uniformly in n. From (3.3-3.4), the Lions-Aubin compactness lemma, and interpolation, we conclude that there exists a function w such that for some subsequence of the v_n,

$$v_n^{3/2} \overset{L^\infty(L^2)}{\longrightarrow} w^{3/2} \tag{3.5}$$

on bounded τ-intervals. Note that since

$$|v_n^3 - w^3| = |v_n^{3/2} - w^{3/2}||v_n^{3/2} + w^{3/2}|,$$

v_n^3 converges to w^3 in $L^\infty(L^1)$, and hence by interpolation v_n^p converges to w^p for $3/2 \leq p \leq 3$. This will now be extended to lower values of p; an extension to higher values follows from the result in the next section.

The convergence of $v^{3/2}$ implies convergence in L^1_{loc}:

$$\left[\int_{|x|\leq k} |v_n - w|\right]^3 \leq c \int_{|x|\leq k} |v_n - w|^3 \leq C \int |v_n^{3/2} - w^{3/2}|^2 \to 0$$

In fact, since v is known to have compact support uniformly in τ, this shows that $v_n \to w$ in $L^\infty(L^1)$

Furthermore, (3.4) plus the definition of v_n show that $(v_n^{3/2})_\tau$ converges to zero, so w is independent of τ.

In order to see what equation the limit w satisfies, first note that since λ is increasing by (2.5) and bounded by (3.1-3.2), $\lambda[v(\tau)]$ converges to some finite λ_∞. Taking the weak limit of equation (2.3) then yields

$$0 = \Delta w^2 + w^2 - \lambda_\infty w \tag{3.6}$$

By the definition of λ, this implies that

$$\lambda[w] = \lambda_\infty \equiv \lim_{\tau\to\infty} \lambda[v(\tau)] \tag{3.7}$$

By a rescaling plus theorems of [SGKM] and [CEF], this means that $w = \lambda_\infty \sum_{j=1}^N Z(x - x_j)$ where Z is the unique compactly-supported radial solution of (3.6) with λ_∞ replaced by 1 and the x_j are such that the regions where $Z(x-x_j) \neq 0$ are disjoint. In one dimension Z has the explicit form $\frac{4}{3}\cos^2(x/4)$.

4. The L^∞ bound.

The results of the previous section ensure that the blow-up time of the original equation (1.1) corresponds to $\tau = \infty$ provided that we define the blow-up time to be the time when the L^3 norm tends to infinity. However, the blow-up time is usually defined by the L^∞ norm tending to infinity. In this section we show that the solution v of (2.3) is uniformly bounded in L^∞, which implies that the two notions of blow-up time are equivalent.

For simplicity, we will only consider here the case when $\lambda(0) > 0$. In general, it is possible to show that λ eventually becomes positive, which allows the argument here to be applied.

Define

$$X_r \equiv \int v^{2^r+1}\,dx \tag{4.1}$$

Applying $\frac{d}{\tau}$, substituting in the equation (2.3), integrating by parts and using Gagliardo-Nirenberg inequalities leads to

$$\frac{d}{\tau}X_r \leq c_1 2^r \left[c_2 c_3^r X_{r-1}^2 - \lambda x_r\right] \tag{4.2}$$

If we assume that X_{r-1} is bounded by some B_{r-1} then we find from (4.2) that X_r is bounded by

$$B_r \equiv \frac{c_2}{\lambda(0)} c_3^r B_{r-1}^2 \tag{4.3}$$

provided that this bound holds at time zero, since if we estimate X_{r-1} in (4.2) by its constant bound B_{r-1} and $-\lambda$ by $-\lambda(0)$ then the solution of the resulting autonomous ODE cannot cross the point at which its derivative vanishes. If we set

$$B_r = \frac{\lambda(0)}{c_2}(z_r)^{2^r}$$

then the recursion formula (4.3) becomes

$$Z_r = Z_{r-1} c_3^{r/2^r}, \tag{4.4}$$

whose solution is

$$Z_r = \beta c_3^{\sum j/2^j} \longrightarrow Z_\infty < \infty$$

where β can be chosen large enough so that all the bounds $X_r \leq B_r$ are indeed satisfies at time zero. Hence

$$\|v\|_{L^\infty} = \lim X_r^{1/(2^r+1)} \leq \lim Z_r = Z_\infty < \infty$$

5. Estimates for the blow-up time and number of bumps.

By combining the two parts of (2.2) we obtain

$$\lambda(\tau(t))\tau'(t) = \frac{\phi'(t)}{\phi(t)},$$

which can be integrated to yield

$$\int_0^{\tau(t)} \lambda(\tau_1)\,d\tau_1 = \log\phi(t) \tag{5.1}$$

provided that we choose $\phi(0)$ to equal 1. After using (2.2) once more, equation (5.1) can be re-written in the form

$$e^{-\int_0^{\tau(t)} \lambda(\tau_1)\,d\tau_1}\tau'(t) = 1,$$

which integrates to yield

$$T = \int_0^\infty e^{-\int_0^\tau \lambda(\tau_1)\, d\tau_1}\, d\tau,$$

where T is the blow-up time in the original time variable t.

If $\lambda(0) > 0$ then by using the fact that λ is increasing we find the upper and lower bounds

$$\frac{1}{\lambda_\infty} = \int_0^\infty e^{-\int_0^\tau \lambda_\infty} \le T \le \int_0^\infty e^{-\int_0^\tau \lambda(0)} = \frac{1}{\lambda(0)} \tag{5.2}$$

for the blow-up time T.

Even if $\lambda(0) < 0$ when λ is defined by (2.4), an upper bound can be obtained by defining λ to be $\frac{\int v^2}{\int v}$, which implies that $\frac{d}{\tau} \int v = 0$. Since $(\int v)^2 \le \int v^2 \int_{supp\ v} 1$, this implies that

$$\lambda \ge \frac{\int v}{L} = \frac{\int v(0)}{L},$$

where L is the maximal volume of the support of v. Since estimate (5.2) can be expressed more generally as

$$\frac{1}{\lambda_{max}} \le T \le \frac{1}{\lambda_{min}}, \tag{5.3}$$

this yields $T \le \frac{L}{\int v_0}$.

One way to obtain a lower bound for T is to combine estimate (3.2) with (5.2). A more precise bound is obtained from the following lemma, in which λ is defined via (2.4):

Lemma: $\sup \frac{\lambda[v]}{\left(\int v^3\right)^{1/3}} = \frac{\lambda[Z]}{\left(\int Z^3\right)^{1/3}}$

Proof: Since the functional in the lemma is homogeneous of order zero, we may fix the value of $\int v^3$. Use any v as the initial value v_0 of the solution of (2.3). Since λ is increasing and its limit λ_∞ equals $\lambda[w]$, where w is a limit as $\tau \to \infty$, $\lambda(w) \ge \lambda(v)$. Hence the maximum occurs for some possible value of $w = \lambda[w] \sum_{j=1}^N Z(x - x_j)$. Since

$$\int v^3 = \int w^3 = N\lambda[w]^3 \int Z^3, \tag{5.4}$$

the maximum occurs for $N = 1$. Since $\lambda[Z] = 1$, we therefore find that

$$T \ge \frac{1}{\lambda_{max}} = \frac{(\int Z^3)^{1/3}}{(\int v^3)^{1/3}} \tag{5.5}$$

When the set where v_0 takes values near its maximum is large then the alternative estimate $T \ge 1/\max v_0$ obtained via a comparison argument may be better. When that set is small and so is the volume of the support of v_0 then (5.5) is likely to be better.

Equation (5.4) can also be used to obtain a one-sided estimate for the number N of copies of Z in the limit. Note first that since both $\int v_0^3$ and λ_∞ are independent of the subsequence, (5.4) shows that N is the same for all limits w. Next, (5.4) also yields

$$N = \frac{\int v_0^3}{\lambda_\infty^3 \int Z^3} \le \frac{\int v_0^3}{\int Z^3} T^3$$

Applying either of the above upper bounds for T therefore yields an upper bound for N. Since N is an integer, in order to show that it equals one it suffices to obtain any bound smaller than 2. For example, there exist initial data that obtain their maximal value at two points separated by more than the minimal distance 4π between the maxima of two functions $Z(x-x_j)$ in dimension one, but for which our estimate shows that $N < 1.99$.

On the other hand, there can be no nontrivial lower bound for N since initial data leading to a two-bump limit can be transformed into data leading to just one bump by multiplication by a function that is one near one of the bumps and less than but arbitrarily close to one near the other, since this will make the first bump blow up slightly before the second. In other words, an arbitrarily small change in the initial data can reduce the number of bumps to just one.

6. Perturbation results.

Since equation (2.3) is stable in the sense that some L^p norm, depending on the choice of the functional λ, is preserved, we can consider small perturbations of the limit solution Z. Only the one-dimensional case will be considered here. Suppose that

$$v = Z(x - \varepsilon X(\tau)) + \varepsilon w(x - \varepsilon X(\tau)) + O(\varepsilon^2), \tag{6.1}$$

where $X(\tau)$ is chosen so as to minimize $\int [v - Z(x - \varepsilon X)]^2$. Then

$$0 = \frac{d}{X} \int [v - Z(x - \varepsilon X)]^2 = 2\varepsilon \int (v - Z) Z_x = 2\varepsilon \int v Z_x,$$

so

$$\int v Z_x = 0. \tag{6.2}$$

Taking the τ derivative of this equation yields

$$0 = \frac{d}{\tau} \int v Z_x(x - \varepsilon X(\tau)) = \int \left[(v^2)_{xx} + v^2 - \lambda v\right] Z_x - \varepsilon \int v Z_{xx} X' \tag{6.3}$$

First use (6.2) in (6.3) to eliminate the term involving λ. Solve what remains for X', substitute in (6.1), integrate by parts, and use (6.2) and the facts that $\int Z^p Z_x = 0$ and $Z_{xx} = \frac{\chi_{Z>0}}{6} - \frac{Z}{4}$ to obtain

$$X' = -\frac{\frac{3}{2} \int Z Z_x w}{\int Z_x^2} + O(\varepsilon) \tag{6.4}$$

In order to determine X we therefore need to know $\int Z Z_x w$. The terms of order ε in the equation for v can be written as

$$w_\tau = 2(1 + \partial_x^2)(Zw) - w + (1 - \lambda) Z + X' Z_x$$

Upon multiplying this equation by $Z Z_x$, integrating by parts and using the equation satisfied by Z, noting that $\int Z^2 Z_x = 0$ and calculating the integrals involving only Z and its derivatives, we obtain

$$\frac{d}{\tau} \int Z Z_x w = X' \int Z Z_x^2 = -\int Z Z_x w,$$

which shows that

$$\int ZZ_x w = e^{-\tau} \int ZZ_x w_0 \tag{6.5}$$

Substituting (6.5) into (6.4), integrating from $\tau = 0$ to $\tau = \infty$, and using (6.2) to calculate $X(0)$ up to $O(\varepsilon)$ shows that the perturbation displaces the center of the blow-up by an amount

$$\varepsilon \left[X(0) + (X(\infty) - X(0))\right] + O(\varepsilon) = -\varepsilon \left[\frac{9}{2\pi} \int Z_x w_0 + \frac{27}{4\pi} \int ZZ_x w_0\right] + O(\varepsilon^2)$$

A similar but somewhat more complicated procedure determines the $O(\varepsilon)$ change to the blow-up time.

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