

GORM SALOMONSEN

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Geometric Heat Kernel Coefficients for APS-Type Boundary Conditions.

Gorm Salomonsen.

Abstract

I present an alternative way of computing the index of a Dirac operator on a manifold with boundary and a special family of pseudodifferential boundary conditions. The local version of this index theorem contains a number of divergence terms in the interior, which are higher order heat kernel invariants. I will present a way of associating boundary terms to those divergence terms, which are rather local of nature.

0. Introduction.

Given a closed realization of a Dirac-type operator D between vector-bundles E_1 and E_2 over a compact manifold K of dimension n with smooth boundary. If the realization of D is given by pseudodifferential boundary conditions it is known [Gr1], [GS] that there exist coefficients $a_{i,k}$ and $a'_{i,k}$ such that

$$\mathrm{tr}(\varphi e^{-t\Delta_i}) \sim \sum_{0 < k < n} a_{i,k} t^{-\frac{k}{2}} + \sum_{k=-\infty}^0 (a_{i,k} \log(t) + a'_{i,k}) t^{-\frac{k}{2}}, \quad \text{for } t \rightarrow 0.$$

Here $\Delta_1 = D^*D$, $\Delta_2 = DD^*$ and $\varphi \in C^\infty(K, \mathrm{End}(E))$. The unprimed coefficients are locally determined, whereas the primed coefficients are globally determined. By locally determined we here mean, that there exist local formulas $a_{i,k}(x)$ on K and $b_{i,k}(z)$ on ∂K , such that

$$a_{i,k} = \int_K a_{i,k}(x) dx + \int_{\partial K} b_{i,k}(z) dz.$$

In the interior of K there is a local contribution to $a'_{i,k}$ also, given by the standard local formulas for closed manifolds. That means, that the non-local contribution to $a'_{i,k}$ arises from the behaviour of $\mathrm{tr}(\varphi e^{-t\Delta_i})(x, x)$ arbitrarily close to the boundary only.

In this paper we will for some special cases construct boundary contributions $b_k(z)$ for $k < 0$ corresponding to the $a'_{i,k}$. These boundary contributions depend

on the choice of self-adjoint boundary conditions inside a class of pseudodifferential boundary conditions, which we will define using some recent results of Gilles Carron. The proof of the self-adjointness is a modified version of a proof for the self-adjointness of a similar operator in [Sa1]. In addition to Carron's condition of non-parabolicity at infinity it makes use of Assumption 2.2, which is non-trivial to establish, but which can still often be established for manifolds with fixed or asymptotically fixed geometry near infinity. It has apparently still not been established in the case of the extension of a manifold with corners considered in [HMM], [Mü].

We will have $E = E_1 = E_2$, $D = D^*$, $n := \dim(K) \in 2\mathbb{N}$, and φ will be the involution corresponding to a superstructure on E . Thus the expansions we will be interested in are the expansions of

$$\mathrm{tr}(\tau e^{-tD^2}).$$

By standard arguments it follows that in this case the $a'_{i,k}$ vanish for $k \neq 0$. This does however not mean, that they vanish locally. It just means, that the boundary contributions cancel with the contributions from the interior.

Each of the pseudo-differential boundary conditions, for which we will be able to define formulae at the boundary, corresponds to a way to extend K to a manifold X with ∂X isometric to ∂K , product structure at the boundary and which are such that the distance from K to X is constant. The boundary terms are such that they split into a local term and a term, which depends only on the derivatives of the various structures on X along the shortest geodesic from $z \in \partial K$ to ∂X . They are thus not local in the sense that they can be computed in terms of the derivatives of the various structures in a point at the boundary, but they are still less than global and nicely given in terms of the geometric construction, which leads to the pseudodifferential boundary conditions. For $k = 0$ the corresponding term in the case, where K has product structure close to the boundary, is the density of the η -invariant of the boundary plus a term involving the kernel, which can not be given in the same way. If D is replaced by the operator $D + \tau m$ we shall see that all b_k are indeed global for k even. The global contributions are in this case multiples of the global contribution for $k = 0$ in the case $m = 0$.

We remark that it does not follow from the construction that the terms $b_k(z)$ are really related to the local behaviour of the heat super-trace. There could be an additional divergence term.

The results of this paper basically gives a way of extending index theorems of the Atiyah-Patodi-Singer type from product type boundaries to smooth boundaries. What is new is that the boundary conditions are given explicitly. In order to extend the results to other open manifolds than manifolds with cylindrical ends, some "product case" will in each case have to be solved rather explicitly. It would be interesting to find ways to replace the "product case" by a limiting case in the form of a complete Riemannian manifold.

1. Manifolds Non-Parabolic at Infinity.

In this section we will recall the setup introduced by Gilles Carron in [Ca1], [Ca2], [Ca3].

In the following M will be an even dimensional complete Riemannian manifold, and $E \mapsto M$ will be a Dirac bundle, i.e. a Hermitian vector bundle supplied with a connection and a structure of Clifford multiplication, such that the three structures are compatible. Let D be the associated Dirac operator.

In some of the terminology, when we refer to M , we automatically include E .

Definition 1.1. M is said to be non-parabolic at infinity if there exists a compact subset K of M such that for any precompact open subset U of M , there exists a constant $C(U)$ such that for any $\varphi \in C_0^\infty(M, E)$,

$$\|\varphi\|_{L^2(U,E)} \leq C(U) (\|\varphi\|_{L^2(K,E)} + \|D\varphi\|_{L^2(M,E)}). \quad (1.1)$$

Definition 1.2. Let W be the completion of $C_0^\infty(M, E)/\ker(D|_{C_0^\infty(M \setminus K, E)})$ with respect to the norm

$$\|\varphi\|_W^2 = \|\varphi\|_{L^2(K,E)}^2 + \|D\varphi\|_{L^2(M,E)}^2.$$

Then W is a Hilbert space.

We notice that manifolds with cylindrical ends are examples of manifolds non-parabolic at infinity [Ca1], where K can be taken to be any compact set containing all of the non-cylindrical part of M , and that manifolds for which the self-adjoint extension of $D|_{C_0^\infty(M,E)}$ is a Fredholm operator make up other examples.

Definition 1.3. For any manifold X and any Hermitian vector bundle $F \mapsto X$ supplied with a Hermitian connection we define

$$\begin{aligned} W^{k,p}(X, F) &:= \{f \in L^p(X, F) \mid \forall 0 \leq j \leq k : \nabla^j f \in L^p(X, T^*M^{\otimes j} \otimes F)\}, \\ W_0^{k,p}(X, F) &:= \text{Closure of } C_0^\infty(X, F) \text{ in } W^{k,p}(X, F), \\ W^{k,p,loc}(X, F) &:= \{f \in L^{p,loc}(X, F) \mid \forall U \Subset X : f|_U \in W^{k,p}(U, F)\}. \end{aligned}$$

Here $U \Subset X$ means, that U is a precompact open subset of X . If further F is a Dirac bundle with an associated Dirac operator D we define

$$\begin{aligned} H^k(X, F) &:= \{f \in L^2(X, F) \mid \forall 0 \leq j \leq k : D^j f \in L^2(X, F)\}, \\ H_0^k(X, F) &:= \text{Closure of } C_0^\infty(X, F) \text{ in } H^k(X, F), \\ H^{k,loc}(X, F) &:= \{f \in L^{2,loc}(X, F) \mid \forall U \Subset X : f|_U \in H^k(U, F)\}. \end{aligned}$$

□

For the complete Riemannian manifold M all powers of D are essentially self-adjoint on $C_0^\infty(M, E)$ [Ch], so $H_0^k(M, E) = H^k(M, E)$. Further, if the curvature term in the Weizenböck formula is bounded it follows by an application of the Weizenböck formula, $W_0^{1,2}(M, E) = H^1(M, E)$, and in general $W_0^{k,2}(M, E) \hookrightarrow H_0^k(M, E)$, $W^{k,2}(M, E) \hookrightarrow H^k(M, E)$. If $U \Subset M$ we further have $H_0^k(U, E) \hookrightarrow W_0^{k,2}(M, E)$.

In general W is not even a space of sections of M , but we have:

Theorem 1.4. (Carron) *If M is non-parabolic at infinity*

- a) The inclusion $C_0^\infty(M, E) \mapsto H^{1,loc}(M, E)$ extends by continuity to a map $W \mapsto H^{1,loc}(M, E)$.
- b) The linear map $D : W \mapsto L^2(M, E)$ is a Fredholm operator.
- c) If $f \in W$ has support on $\overline{M \setminus K}$ and satisfies $Df = 0$, then $f = 0$.

□

There are a number of indices, which can be considered. First define the extended index

$$\text{Ind}_e(D) := \text{Index}(D : W \mapsto L^2(M, E)).$$

The kernel of $D^* : L^2(M, E) \mapsto W$ is the L^2 -kernel of D , so another description of Ind_e is

$$\text{Ind}_e(D) = \dim(\ker(D|_W)) - \dim(\ker(D|_{H^1(M, E)})).$$

Thus $\text{Ind}_e(D)$ is the dimension of the space of asymptotic behaviours of elements of $\ker(D|_W)$ up to elements of L^2 . In particular $\text{Ind}_e(D) \geq 0$.

Let $\tau \in C^\infty(M, E)$ be a parallel unitary and self-adjoint section anticommuting with the operator of Clifford multiplication by any vector field. Then the ± 1 eigenspaces of τ gives a superstructure $E = E_+ \oplus E_-$ on E . At least one such involution is always given by the volume element in the Clifford algebra. With respect to this superstructure W and $L^2(M, E)$ split into orthogonal direct sums $W = W_+ \oplus W_-$ and $L^2(M, E) = L^2(M, E_+) \oplus L^2(M, E_-)$. Let $D^\pm : W_\pm \mapsto L^2(M, E_\mp)$ be the restriction of D . For f any section of E we write $f = f_+ + f_-$ for the components of f in E_+ and E_- . Under these assumptions we may define

$$\text{Ind}_e(D^+) := \dim(\ker(D|_{W_+})) - \dim(\ker(D|_{L^2(M, E_-)})).$$

2. A Condition Ensuring Self-Adjointness.

In this section we will consider the question, when there is a generalization of the Atiyah-Patodi-Singer boundary conditions on K , such that the operator D_K on K has the desired index, $\text{Ind}_e(D^+)$. It turns out that the existence of such boundary conditions is rather independent of K , but that we can only prove the self-adjointness properties under some conditions on the spectrum of $D_M := D|_{H^1(M, E)}$. For smooth boundaries these results are to a great extent covered by the results of Carron and existing theory of pseudodifferential boundary value problems. Non-smooth boundaries are however of considerable interest for the author, who is working with index theory for manifolds with corners.

The domain, which we will consider as the domain generalizing the domain of the Dirac operator on K with Atiyah-Patodi-Singer boundary conditions, is given by

$$\mathcal{D}(D_K) := \{f|_K \mid f \in W, f_- \in L^2(M, E) \text{ and } Df \in L^2(K, E)\}. \quad (2.1)$$

This domain is the restriction to K of the space

$$\tilde{\mathcal{D}}(D_K) := \{f \in W \mid f_- \in L^2(M, E) \text{ and } Df \in L^2(K, E)\}. \quad (2.2)$$

Domains, which are just as fundamental, and which correspond to augmenting D_K with respect to the scattering matrix instead of with respect to the superstructure in the case of a manifold with cylindrical ends are given by

$$\mathcal{D}(D_{K,S}) := \{f|_K \mid f \in H^1(M, E) \text{ and } Df \in L^2(K, E)\} \oplus \ker(D|_W)|_K. \quad (2.3)$$

$$\tilde{\mathcal{D}}(D_{K,S}) := \{f \in H^1(M, E) \mid Df \in L^2(K, E)\} \oplus \ker(D|_W). \quad (2.4)$$

Lemma 2.1. *If (1.1) holds for M and K , and K is the closure of an open subset of M , then D_K and $D_{K,S}$ are symmetric and closed operators.*

The involution τ anticommutes with D_K and $D_{S,K}$.

Proof: The operator D interchanges W_+ and W_- and if $f \in W_+ \cap \tilde{\mathcal{D}}(D_K)$, $g \in W_- \cap \tilde{\mathcal{D}}(D_K)$,

$$\langle D_K f|_K, g|_K \rangle_{L^2(K,E)} - \langle f|_K, D_K g|_K \rangle_{L^2(K,E)} = \langle Df, g \rangle_{L^2(M,E)} - \langle f, Dg \rangle_{L^2(M,E)}.$$

Since $f \in W$, f can be approximated by a sequence $\{f_j\}$ of sections of $C_0^\infty(M, E)$ with respect to $\|\cdot\|_W$. Consequently this can be rewritten

$$= \lim_{j \rightarrow \infty} \langle Df_j, g \rangle_{L^2(M,E)} - \langle f_j, Dg \rangle_{L^2(M,E)}. \quad (2.5)$$

Here we have used $Df_j \rightarrow Df$ in $L^2(M, E)$, $g \in H^1(M, E)$, $(Dg)|_{M \setminus K} = 0$ and $\|f_j - f\|_{L^2(K,E)} \rightarrow 0$. Clearly (2.5) vanishes for each j . Since $\mathcal{D}(D_K)$ is the space of restrictions to K of elements of $\tilde{\mathcal{D}}(D_K)$ it follows that D_K is symmetric.

Now let $f, g \in \tilde{\mathcal{D}}(D_{K,S})$. Then we may write $f = f_1 + f_2$, $g = g_1 + g_2$, where $f_1, g_1 \in H^1(M, E)$ and $f_2, g_2 \in \ker(D|_W)$. We get

$$\begin{aligned} & \langle D_{K,S} f|_K, g|_K \rangle_{L^2(K,E)} - \langle f|_K, D_{K,S} g|_K \rangle_{L^2(K,E)} \\ &= \langle Df_1, g \rangle_{L^2(M,E)} - \langle f_1, Dg \rangle_{L^2(M,E)} - \langle f_2, Dg_1 \rangle_{L^2(M,E)}. \end{aligned} \quad (2.6)$$

Let $\{g_j\} \in C_0^\infty(M, E)$ be a sequence converging towards g with respect to the norm on W . Then since $Df_1 \in L^2(K, E)$, $(g_j)|_K \rightarrow g|_K$ in $L^2(K, E)$, $Dg_j \rightarrow Dg$ in $L^2(M, E)$ and $f_1 \in L^2(M, E)$ it follows that

$$\langle Df_1, g \rangle_{L^2(M,E)} - \langle f_1, Dg \rangle_{L^2(M,E)} = \lim_{j \rightarrow \infty} \langle Df_1, g_j \rangle_{L^2(M,E)} - \langle f_1, Dg_j \rangle_{L^2(M,E)} = 0.$$

In a similar way f_2 can be approximated by C_0^∞ -sections, giving that the third term in (2.6) vanishes since $Df_2 = 0$.

That τ anticommutes with D_K and $D_{S,K}$ is trivial.

That $D_{K,S}$ is closed follows since $D : H^1(M, E) \mapsto L^2(M, E)$ is bounded, and thus that the inverse image under D of the closed subspace $L^2(K, E)$ of $L^2(M, E)$ is closed in $H^1(M, E)$. It follows that D is closed on the domain

$$\{f \in H^1(M, E) \mid Df \in L^2(K, E)\}. \quad (2.7)$$

Since $\ker(D|_W)$ is finite dimensional, $D_{K,S}$ is a finite dimensional extension of a closed operator, and is therefore closed.

That also D_K is closed follows since D_K^+ is a direct summand in the closed realization of D defined on the domain

$$\{f \in W \mid Df \in L^2(K, E)\},$$

so that D_K^+ is closed. Further D_K^- is a direct summand in the domain (2.7), so that also D_K^- is closed.

This proves the lemma. □

In order to get self-adjointness we need an additional assumption:

Assumption 2.2. *For every $f \in L^{2,comp}(M, E)$, if $f \perp \ker(D|_W)$ the limit*

$$\lim_{\varepsilon \rightarrow 0^+} (D_M + i\varepsilon)^{-1} f$$

exists in $L^2(M, E)$.

Lemma 2.3. *Assumption 2.2 holds if the spectrum of D_M in a neighbourhood of 0 consists of a discrete point spectrum together with an absolutely continuous spectrum of finite multiplicity given by a finite sum of C^1 -families of generalized eigensections. In particular it holds for manifolds with cylindrical ends.*

In this case $\ker(D|_W)$ is the direct sum of $\ker(D|_{H^1(M,E)})$ and the space of generalized eigensections to the eigenvalue 0.

Proof: Since the continuous spectrum near 0 is given by a sum of C^1 -families of generalized eigensections it follows that the absolutely continuous part of the Fourier transform (with respect to a spectral representation of D_M) of a section with compact support is C^1 in a neighbourhood of 0. Let f_j be a sequence of C_0^∞ -sections convergent towards an element $f \in \ker(D|_W)$ in W . Considering the image in the spectral representation gives, that f is given as $f = f_1 + f_2$, where $f_1 \in \ker(D|_{H^1(M,E)})$ and f_2 is a generalized eigensection to the eigenvalue 0. On the other hand every generalized eigensection to the eigenvalue 0 can be constructed as an element of W in the spectral representation, so it follows that $\ker(D|_W)$ is the direct sum of $\ker(D|_{H^1(M,E)})$ and the space of generalized eigensections to the eigenvalue 0.

It now follows that the Fourier transform of a section with compact support orthogonal to $\ker(D|_W)$ is C^1 in a neighbourhood of 0 and vanishes in 0. This implies that Assumption 2.2 holds. □

Theorem 2.4. *If Assumption 2.2 holds for M and (1.1) holds for M and K , D_K and $D_{K,S}$ are self-adjoint.*

Proof: Let P be the orthogonal projection on $\ker(D|_W)$ with respect to the inner product on W . We notice that this projection is also an orthogonal projection with respect to the L^2 -inner product on K .

For $f \in L^2(K, E)$ we define

$$(D_{K,S} + P)^{-1}f := (Pf + (1 - P) \lim_{\varepsilon \rightarrow 0_+} (D + i\varepsilon)^{-1}(1 - P)f)|_K.$$

Then by Assumption 2.2 $(D_{K,S} + P)^{-1}$ maps to $\mathcal{D}(D_{K,S})$. Further $(D_{K,S} + P)^{-1}$ is everywhere defined and closable since it is contained in the inverse of $(D_{K,S} + P)$, which is a closed operator. Every everywhere defined closable operator is closed, and every everywhere defined closed operator is bounded, so $(D_{K,S} + P)^{-1}$ is bounded. But then $D_{K,S} + P$ is a closed symmetric operator with a bounded right inverse. Every such operator is self-adjoint, so $D_{K,S} + P$ is self-adjoint. Since P is bounded it follows that $D_{K,S}$ is itself self-adjoint.

Let V be the orthogonal complement of $\ker(D|_{H^1(M, E_-)})$ in $\ker(D|_{W_-})$ and let Q be the projection on V with respect to the inner product on W . Further let \mathfrak{D} be the realization of D on the domain $\mathcal{D}(D_K) + \mathcal{D}(D_{K,S})$. Then it is easily tested

$$\ker(\mathfrak{D} + P) = \{f = f_+ \oplus f_- \in \mathcal{D}(D_K^+) \oplus \mathcal{D}(D_{K,S}^-) \mid D_K^+ f = -f_-, D_{K,S}^- f_- = 0, f_- \perp \ker(D_K^-), Pf_+ = 0\}.$$

If some $f \in \ker(\mathfrak{D} + P)$ satisfies that $f_- = 0$ it follows that $D_K f_+ = 0$ and thus that $f_+ \in \ker(D_{K,S})$. But then $D_{K,S} f_+ = P f_+ = 0$, so $f_+ = 0$. Consequently the map $J : \ker(\mathfrak{D} + P) \mapsto V$, given by $J(f_+ \oplus f_-) = f_-$, is injective. Now assume that $f_- \in V$ is given. Then the extension by zero of f_- to M is L^2 -orthogonal to $\ker(D|_{H^1(M, E)})$. Since $D|_{W_+}$ is Fredholm it follows, that $f_- \in \text{Im}(D|_{W_+})$. This gives that f_- is in the image of D_K^+ . Consequently there exists an f_+ such that $f_+ + f_- \in \ker(\mathfrak{D} + P)$. It follows that J is a bijection. Let J^{-1} be the inverse of J and let P_+ be the projection on $L^2(M, E_+)$. Then

$$(D_K + P)^{-1} := (1 - Q)(D_{K,S} + P)^{-1} + P_+ J^{-1} Q$$

is a bounded right inverse of $D_K + P$, so also $D_K + P$ is self-adjoint.

This proves the lemma. □

Notice that c) of Theorem 1.4 implies that in fact $\mathcal{D}(D_K)$ is isometrically isomorphic to the subspace $\tilde{\mathcal{D}}(D_K)$ of W . Using this isomorphism, the continuous imbedding of W in $H^{1,loc}$ and a cutoff function, we get a Calderón extension operator $\mathcal{C} : \mathcal{D}(D_K) \mapsto W_0^{2,1}(M, E)$. This gives by Rellichs lemma that D_K has a compact resolvent and therefore, that D_K is a Fredholm operator with a discrete spectrum with eigenvalues of finite multiplicity.

From the construction it is clear, that $\ker(D_K^+) \cong \ker(D|_{W_+})$ and that $\ker(D_K^-) \cong \ker(D|_{H^1(M, E_-)})$. Thus in particular

$$\text{Index}(D_K^+) = \text{Ind}_e(D^+). \tag{2.8}$$

Remark: (2.8) is remarked in [Ca3] for a special case. For smooth boundaries it would also be natural to prove (2.4) using the results of [Ca3] about the Dirac-Neuman operators together with the general theory for pseudodifferential boundary conditions [BW], [Gr2].

After checking that the space of extended harmonic sections on a manifold with cylindrical ends belongs to W it is easily seen, that $\text{Ind}_e(D_+)$ is exactly the index of Atiyah-Patodi-Singer (with the opposite augmentation) in the case of a manifold with cylindrical ends. This follows because the domain of the operator defined by Atiyah-Patodi-Singer is contained in $\mathcal{D}(D_K)$, and every self-adjoint operator is maximally symmetric. Thus since D_K is self-adjoint, it coincides with the operator defined by Atiyah-Patodi-Singer.

3. Inheritance of Self-Adjointness and Index.

Let the notation be like in the previous section, and assume that (1.1) is satisfied for M and K . In this section we will study what happens, when K is replaced by another subset $K' \subseteq M$, for which we do not know, whether (1.1) holds. We can not expect too much in complete generality, but everything worth mentioning holds under the following mild assumption:

Assumption 3.1. *The subset K' is the compact closure of an open subset of M such that the restriction*

$$R : \ker(D|_{W_+}) \oplus \ker(D|_{H^1(M, E_-)}) \mapsto L^2(K', E)$$

is injective.

Theorem 3.2. *Assume that Assumption 3.1 is satisfied. Then the operator $D_{K'}$ defined for K' like for K in (2.1) is selfadjoint, Fredholm, and $\text{Index}(D_{K'}^+) = \text{Ind}_e(D^+)$.*

Proof: If necessary K can be replaced by $K \cup K'$ without changing W , so we may assume, that $K' \subseteq K$. We notice, that

$$\mathcal{D}(D_{K'}) = \{f|_{K'} \mid f \in \mathcal{D}(D_K) \text{ and } D_K f \in L^2(K', E)\}.$$

From that and Assumption 3.1 it easily follows, that $D_{K'}$ is symmetric on $\mathcal{D}(D_{K'})$ and that $\ker(D_{K'}) \cong \ker(D_K)$ through extension and restriction. Assumption 3.1 further implies, that the H^1 -norm on $\mathcal{D}(D_K)$ is equivalent to the norm

$$\sqrt{\|\cdot\|_{L^2(K', E)}^2 + \|D_K f\|_{L^2(K, E)}^2}.$$

The restriction of this norm to $\ker(D_K)$ is further the same as the H^1 -norm of the restriction to K' . Consequently the orthogonal projection P on the kernel of D_K is defined equally well in $\mathcal{D}(K')$ and $\mathcal{D}(D_K)$. Again we may define a bounded right inverse of $D_{K'} + P$ by

$$(D_{K'} + P)^{-1} = R_{K'}(D_K + P)^{-1}E_K, \tag{3.1}$$

where $R_{K'}$ is the operator of restriction to K' and E_K is the operator of extension by 0 from $L^2(K', E)$ to $L^2(K, E)$. Thus $D_{K'}$ is self-adjoint. It is further Fredholm because (3.1) is a compact operator. Since it has the same kernel as D_K and since τ preserves $\mathcal{D}(D_{K'})$, it has the claimed index by (2.8).

This proves the theorem. \square

Example 3.3. *Let M be a manifold with cylindrical ends $Z \times [0, \infty)$, $K = M \setminus (Z \times (1, \infty))$ and let K' be the closure of an open subset U' of K , such that $U' \mapsto K \setminus \partial K$ induces a monomorphism in cohomology. Since every non-zero harmonic form on K satisfying the Atiyah-Patodi-Singer boundary conditions induces a non-zero cohomology class in $H^*(K, \mathbb{C})$, it follows by the monomorphism in cohomology that it can not vanish on K' .*

Example 3.4. *Assume that M and K are like in (3.3) and that E is some Dirac bundle such that the curvature term in the Weizenböck formula is non-negative as an operator on the fibres over $K \setminus K'$. Then Assumption 3.1 holds.*

Example 3.5. *If K' has more than one connected component and each component satisfies Assumption 3.1, the restriction of an element of $\ker(D'_K)$ to each component determines the element completely.*

4. Traces of Dirac Operators.

Let M , E , D , τ and K be like in the last section such that (1.1) holds. Further let $m \in \mathbb{R}$. The Dirac operator with mass term is given by

$$H = D_K + m\tau.$$

Since $m\tau$ is bounded and self-adjoint, H is self-adjoint on $\mathcal{D}(D_K)$.

For $k \in 2\mathbb{Z} + 1$ we define

$$\mathrm{tr}(H^k) = \eta(H; -k),$$

where $\eta(H; s)$ is the η -function of H , given by analyticity and that for $\mathrm{Re}(s) \gg 0$

$$\eta(H; s) = \sum_{\lambda \in \mathrm{spec}(H) \setminus \{0\}} \mathrm{sign}(\lambda) |\lambda|^{-s}. \quad (4.1)$$

Thus we immediately see, that the definition is a reasonable generalization of traces of odd powers of H . The traces of the even powers are defined using the ζ -function instead, and can not be treated in the same way as the traces of the odd powers. For the power $k = 0$ both the η - and the ζ -function stand to disposal, giving regularizations of the signature of H and the dimension of $L^2(K, E)$, respectively.

Lemma 4.1. *For $k \in 2\mathbb{Z} + 1$, $\mathrm{tr}(H^k)$ is well-defined and is given by*

$$\mathrm{tr}(H^k) = m^k \mathrm{Ind}_e(D^+). \quad (4.2)$$

In particular $\mathrm{tr}(H^k)$ depends only on m , k and M , but not on K .

Proof: First we notice, that the fact that the domain of D_K is the restriction of a subspace of $H^1(M, E)$ implies, that the heat kernel $e^{-tH^2}(x, y)$ is smooth on the interior of $K \times K$. For irregular boundaries, however, the heat kernel may have singularities at the boundary. Composing with a Calderón extension operator $\mathcal{C} : \mathcal{D}(D_K) \mapsto W_0^{2,1}(U, E)$ for some open set U with compact closure containing K , and the restriction R_K to K , we get an operator

$$\mathcal{C}e^{-tH^2} R_K : L^2(M, E) \mapsto W_0^{2,1}(M, E),$$

which coincides with e^{-tH^2} when integrated up against $L^2(K, E) \otimes L^2(K, E)$. Let $\psi \in C_0^\infty(M, E)$ be a cutoff function, which is identically equal to 1 on U , and let M_ψ denote the operator of multiplication by ψ . Then we may rewrite

$$\mathcal{C}e^{-tH^2} R_K = (M_\psi(D_M + i)^{-1}M_\psi) \left((D_M + i)\mathcal{C}e^{-tH^2} R_K \right).$$

Here D_M denotes the self-adjoint realization of D with domain $H^1(M, E)$. The operator $M_\psi(D_M + i)^{-1}M_\psi$ is in some Schatten class (i.e. some power of its absolute is of trace-class. See [Pe]). Since the Schatten classes make up ideals in $B(\mathcal{H})$ it follows that e^{-tH^2} is of some Schatten class. By the semigroup property it follows that e^{-tH^2} is of trace class for all $t > 0$. The trace class norms of e^{-tH^2} and He^{-tH^2} can be estimated by a constant times a power of the norm of He^{-tH^2} and $H^2e^{-tH^2}$, respectively, and can thus be seen to grow at most polynomially for $t \rightarrow 0$.

In the same way it follows that $(H + P_{\ker(H)})^{-1}$ is of some Schatten class, so that $\eta(H; s)$ is defined for high values of $\text{Re}(s)$.

Taking the Mellin transform in the standard way gives

$$\begin{aligned} \eta(H; s) &= \frac{1}{\Gamma(\frac{s+1}{2})} \int_0^\infty t^{\frac{s-1}{2}} \text{tr}(He^{-tH^2}) dt \\ &= \frac{1}{\Gamma(\frac{s+1}{2})} \int_0^\infty t^{\frac{s-1}{2}} \left\{ \text{tr}(D_K e^{-tD_K^2}) + m \text{tr}(\tau e^{-tD_K^2}) \right\} e^{-tm^2} dt. \end{aligned} \quad (4.3)$$

Here the first trace vanishes even locally because D_K anticommutes with τ . Thus if $m = 0$, $\eta(H; s)$ vanishes identically. Otherwise, since D_K commutes with D_K^2 and anticommutes with τ , contributions to the second trace coming from non-zero eigenvalues can be seen to vanish globally.

Consequently, for $m > 0$

$$\begin{aligned} \eta(H; s) &= \frac{m \text{Index}(D_K^+)}{\Gamma(\frac{s+1}{2})} \int_0^\infty t^{\frac{s-1}{2}} e^{-tm^2} dt \\ &= m^{-s} \text{Index}(D_K^+) = m^{-s} \text{Ind}_e(D^+). \end{aligned}$$

This proves the lemma for $m > 0$. For $m < 0$ we replace m by $-m$ and τ by $-\tau$ and exploit, that k is odd. \square

Remark: Lemma 4.1 holds for all values of $k \in \mathbb{R}$. The interpretation as a trace is however only meaningful for $k \in 2\mathbb{Z} + 1$. Further the local results only hold for $k \in 2\mathbb{N} + 1$.

In the following we assume $m > 0$. As above the generalization to $m < 0$ follows by replacing m by $-m$ and τ by $-\tau$. The expansion of the supertrace of the heat kernel of H in the interior of K is local, given by

$$m\text{tr}(\tau e^{-tH^2}(x, x)) \sim m \sum_{j=-\infty}^n a_j(x) t^{-\frac{j}{2}} \quad (4.4)$$

for $t \rightarrow 0$. Further we get the estimate

$$m\text{tr}(\tau e^{-tH^2}(x, x)) \leq mC e^{-tm^2} \quad ; t > t_0 > 0$$

locally uniformly in the interior of k . Consequently the heat trace can be computed locally. The formal analytic continuation of the η -function over the interior is given by

$$\eta_{int}(H; s) = \frac{m}{\Gamma(\frac{s+1}{2})} \left\{ \sum_{j=-N}^n \int_K \frac{2a_j(x)}{s-j+1} dx + \int_K \Theta_N(s; x) dx \right\},$$

where Θ_N is regular on any compact subset of \mathbb{C} for N high enough. This gives the result

$$\begin{aligned} \text{tr}(H^k) &= m \left(\int_K \frac{a_{-k+1}(x)}{\text{Res}(\Gamma, \frac{-k+1}{2})} dx + b_k \right) \\ &= m \left(\int_K (-1)^{\frac{k-1}{2}} \left(\frac{k-1}{2} \right)! a_{-k+1}(x) dx + b_k \right). \end{aligned}$$

Here b_k is a term coming only from the boundary and the boundary conditions imposed on D_K . Clearly the value of b_k is given by

$$b_k = m^{k-1} \text{Ind}_e(D^+) - (-1)^{\frac{k-1}{2}} \left(\frac{k-1}{2} \right)! \int_K a_{-k+1}(x) dx. \quad (4.5)$$

The aim of the next section will be to find suitable local formulas for b_k for a family of boundary conditions in the case, where ∂K is smooth.

5. Smoothly Imbedded Manifolds with Boundary.

Let K be a smooth compact manifold with a smooth boundary Z , and assume that K is smoothly imbedded in some complete Riemannian manifold N . N does not need to be non-parabolic at infinity. We will further assume, that $E \mapsto N$ together with its Hermitian structure, connection and structure of Clifford multiplication is natural, determined from the local geometry of N together with some topological data. Further we assume, that the topological data restrict to imbedded submanifolds of full dimension.

Some tubular neighbourhood T_Z of Z is diffeomorphic to $(-\varepsilon, \varepsilon) \times Z$ through the Fermi coordinates around Z . Further the interval $(-\varepsilon, \varepsilon)$ may be oriented such, that $(-\varepsilon, 0] \times Z$ is mapped to K and $(0, \varepsilon) \times Z$ is mapped to $N \setminus K$.

Let $\chi \in C^\infty(\mathbb{R})$ be a function, such that $0 \leq \chi(v) \leq 1$ for all $v \in \mathbb{R}$, $\chi(v) = 0$ for $v \leq \frac{\varepsilon}{3}$ and $\chi(v) = 1$ for $v \geq \frac{2\varepsilon}{3}$.

Let X be the manifold $K \cup T_Z$, supplied with the metric g , which coincides with the metric g^N on the image of K in X , and on T_Z is given by

$$g = (1 - \chi(v))g^N + \chi(v)(dv^2 + g^Z),$$

where v is the projection $T_Z \mapsto (-\varepsilon, \varepsilon)$ and g^Z is the pullback of g^N to Z . Then (X, g) is a manifold with boundary Z and a product metric in a neighbourhood of the boundary, which contains K as a smoothly imbedded submanifold with boundary. By the assumptions on E , there is an associated Dirac bundle $E^X \mapsto X$, for which all structures are of product type near ∂X and such that $E|_K^X = E|_K$.

Lemma 5.1. *The vector-field $\frac{\partial}{\partial v}$ is geodesic with respect to g .*

Proof: It is easily checked that g has a decomposition

$$g = \begin{pmatrix} 1 & 0 \\ 0 & g^\perp \end{pmatrix}$$

with respect to the decomposition

$$T(X \setminus K) = \text{span}\left\{\frac{\partial}{\partial v}\right\} \oplus TZ,$$

induced by the Fermi coordinates. Consequently, in local coordinates $(v, z_1, \dots, z_{n-1}) =: (x_1, \dots, x_n)$, where each z_j is a function of an open subset of Z , the inverse matrix of g takes the form

$$g^{-1} = \begin{pmatrix} 1 & 0 \\ 0 & h^{-1} \end{pmatrix}. \quad (5.1)$$

Now recall the formula for the Christoffel symbols in terms of the metric [Gi, p. 143]

$$\Gamma_{ij}^k = \frac{1}{2} \sum_{m=1}^n g^{km} \left(\frac{\partial g_{jm}}{\partial x_i} + \frac{\partial g_{im}}{\partial x_j} - \frac{\partial g_{ij}}{\partial x_m} \right). \quad (5.2)$$

Combining (5.1) with (5.2) gives

$$\Gamma_{11}^k = \frac{1}{2} \sum_{m=1}^n g^{km} \left(\frac{\partial g_{1m}}{\partial x_1} + \frac{\partial g_{1m}}{\partial x_1} - \frac{\partial g_{11}}{\partial x_m} \right) = 0.$$

Consequently $\nabla_{\frac{\partial}{\partial v}}^g \frac{\partial}{\partial v} = 0$, where ∇^g denotes the Levi-Civita connection with respect to g . This proves the lemma. \square

Corollary 5.2. *We have*

- T_Z is a tubular neighbourhood of ∂K with respect to g .
- The outward-pointing geodesic with respect to g normal to ∂K starting in a point $z \in \partial K$ gives the shortest path from z to ∂X .

□

Let M be the extension of X to a manifold with cylindrical ends. Then by the product structure there is an extension of E and its structures to a Dirac bundle on M , with product structure on the cylinder. We denote this Dirac bundle by E also.

We have already remarked, that M is non-parabolic at infinity with X in place of K , and that Assumption 2.2 holds for manifolds with cylindrical ends. We will further *assume* Assumption 3.1 with K in place of K' .

First we study $\text{tr}(D_X + m\tau)$. Since D_X has standard Atiyah-Patodi-Singer boundary conditions, a parametrix for $e^{-tD_X^2}(\cdot, \cdot)$ for $t \rightarrow 0$ is given uniformly in a neighbourhood of ∂X up to an error in $O(t^N)$ for any N , by the corresponding heat kernel for the Dirac operator \mathfrak{D} on the full cylinder $(-\infty, \varepsilon] \times Z$ with Atiyah-Patodi-Singer boundary conditions. The Dirac operator \mathfrak{D} takes the form

$$\mathfrak{D} = \gamma\left(\frac{\partial}{\partial u} + A\right),$$

where γ is Clifford multiplication by $\frac{\partial}{\partial u}$ on $(-\infty, \varepsilon] \times Z$ and u is the distance to ∂X . The operator $A = -\gamma D_Z$ is a Dirac operator on Z commuting with τ , which thus has a decomposition $A = A_+ \oplus A_-$ with respect to the eigenspaces of τ .

Let φ_λ run over an orthonormal basis of eigensections of A , with corresponding eigenvalues λ , counted with multiplicity. It is well known [APS, 2.2.2] that

$$\text{tr}(\tau e^{-t\mathfrak{D}^2}((u, z), (u, z))) = \sum_{\lambda \in \text{sp}(A_+)} \text{sign}(\lambda) \frac{\partial}{\partial u} \left\{ \frac{1}{2} e^{2|\lambda|u} \text{erfc}\left(\frac{u}{\sqrt{t}} + |\lambda|\sqrt{t}\right) \right\} |\varphi_\lambda(z)|^2,$$

where erfc is the complementary error function given by

$$\text{erfc}(x) = \frac{2}{\sqrt{\pi}} \int_x^\infty e^{-u^2} du,$$

and $\text{sign}(0)$ is set to -1 (compared to [APS], we have augmented oppositely). Integrating over $[0, \infty)$ with respect to u gives like in [APS]

$$\begin{aligned} k(t, z) &:= \int_0^\infty \text{tr}(\tau e^{-t\mathfrak{D}^2}((u, z), (u, z))) du \\ &= - \sum_{\lambda \in \text{sp}(A_+)} \frac{\text{sign}(\lambda)}{2} \text{erfc}(|\lambda|\sqrt{t}) |\varphi_\lambda(z)|^2 \end{aligned} \quad (5.3)$$

and hence, differentiating with respect to t

$$k'(t, z) = \frac{1}{\sqrt{4\pi t}} \sum_{\lambda \in \text{sp}(A_+)} \lambda e^{-\lambda^2 t} |\varphi_\lambda(z)|^2.$$

The part of (5.3) coming from $u \in [\delta, \infty)$ vanishes uniformly with arbitrarily many derivatives in the limit for $t \rightarrow 0$ for every $\delta > 0$ and will therefore not affect the results.

We notice that the asymptotic expansion for $t \rightarrow 0$

$$k'(t, z) \sim \frac{1}{\sqrt{4\pi t}} \sum_{j=-\infty}^n t^{-\frac{j}{2}} c_j(z) \quad (5.4)$$

implies that $k(t, z)$ has an asymptotic expansion of the form

$$k(t, z) \sim \mathbf{d}_0(z) \log(t) + \sum_{j=-\infty}^n t^{-\frac{j}{2}} d_j(z), \quad (5.5)$$

where \mathbf{d}_0 and each d_j except from d_0 is local. Inserting this expansion in (4.3) we realize, that

$$\int_Z \mathbf{d}_0(z) dz = 0$$

since otherwise (4.3) would not be regular in $-2N + 1$. In order to pass from the global to the local level we will therefore have to subtract this term, which would otherwise lead to local divergence. Consequently we set

$$\tilde{k}(t, z) = k(t, z) - \mathbf{d}_0(z) \log(t).$$

The heat kernel, which we are really interested in, is the heat kernel \mathcal{K} of H . \mathcal{K} is given by

$$\mathcal{K}(t, z) = e^{-tm^2} k(t, z). \quad (5.6)$$

Again, in order to get local convergence, we have to subtract the logarithmic terms, which we know do not contribute globally. Thus we set

$$\tilde{\mathcal{K}}(t, z) = e^{-tm^2} \tilde{k}(t, z).$$

Since $\tilde{\mathcal{K}}(t, z)$ is uniformly exponentially decreasing for $t \rightarrow \infty$ it follows that the integral

$$\frac{1}{\Gamma(\frac{s+1}{2})} \int_0^\infty t^{\frac{s-1}{2}} \tilde{\mathcal{K}}(t, z) dt$$

is uniformly convergent in z for $\operatorname{Re}(s) \gg 0$. The local analytic continuation of this integral now is produced from the asymptotic expansion of $\tilde{\mathcal{K}}(t, z)$ for $t \rightarrow 0$, which again only depends of m^2 and the asymptotic expansion of $\tilde{k}(t, z)$ for $t \rightarrow 0$. This proves the following:

Proposition 5.3. *The term $b_{k,\varepsilon}$ corresponding to the term b_k in (4.5) for X is given by:*

$$b_{k,\varepsilon} = \int_Z b_k(z, \varepsilon) dz := (-1)^{\frac{k-1}{2}} \left(\frac{k-1}{2} \right)! \int_Z e_{-k+1}(z) dz, \quad (5.7)$$

where $e_l(z)$ is the l 'th coefficient in the asymptotic expansion

$$\tilde{K}(t, z) \sim \sum_{j=-\infty}^n e_j(z) t^{-\frac{j}{2}}. \quad (5.8)$$

□

Remark: The local expansion (5.8) does exist. The coefficients $e_j(z)$ are however only globally defined since they depend on $\ker(A_+)$.

Now consider the family $\{K_t\}_{t \in [0, \varepsilon]}$ of imbedded submanifolds of X with boundary given by taking

$$K_t = \{x \in X \mid \text{dist}(x, K) \leq t\}.$$

Further let $b_{k,t}$ be the boundary term for K_t corresponding to b_k in (4.5). Obviously (4.5) gives for $t, t' \in [0, \varepsilon]$, $t > t'$

$$b_{k,t} - b_{k,t'} = (-1)^{\frac{k-1}{2}} \left(\frac{k-1}{2}\right)! \int_{K_t \setminus K_{t'}} a_{-k+1}(x) dx.$$

The volume form dx on X can on the collar be written

$$dx = \Theta(z, t) dz dt,$$

where dz is the volume form on Z , dt is the volume form on $[0, \varepsilon]$ and $\Theta(z, t)$ is some non-vanishing real-valued function. It follows

$$\frac{\partial b_{k,t}}{\partial t} = (-1)^{\frac{k-1}{2}} \left(\frac{k-1}{2}\right)! \int_Z a_{-k+1}(z, t) \Theta(z, t) dz.$$

Let

$$A_{-k+1}(z, t) = \int_0^t a_{-k+1}(z, s) ds.$$

Then we may define

$$\begin{aligned} b_k(z, t) &= b_k(z, \varepsilon) - (-1)^{\frac{k-1}{2}} \left(\frac{k-1}{2}\right)! \int_t^\varepsilon a_{-k+1}(z, s) \Theta(z, s) ds \\ &= b_k(z, \varepsilon) - (-1)^{\frac{k-1}{2}} \left(\frac{k-1}{2}\right)! \left([A_{-k+1}(z, t) \Theta(z, t)]_t^\varepsilon - \int_t^\varepsilon A_{-k+1}(z, s) \Theta'(z, s) ds \right). \end{aligned}$$

We can exploit, that $A_{-k+1}(z, 0) = 0$ and that $\Theta(z, \varepsilon) = 1$ to rewrite

$$b_k(z, 0) = b_k(z, \varepsilon) - (-1)^{\frac{k-1}{2}} \left(\frac{k-1}{2}\right)! \left(A_{-k+1}(z, \varepsilon) - \int_0^\varepsilon A_{-k+1}(z, t) \Theta'(z, t) dt \right).$$

Thus we have proved:

Theorem 5.4. *The term b_k for K is given by*

$$b_k = (-1)^{\frac{k-1}{2}} \left(\frac{k-1}{2}\right)! \int_Z e_{-k+1}(z) dz$$

$$-(-1)^{\frac{k-1}{2}} \left(\frac{k-1}{2}\right)! \int_Z \left(A_{-k+1}(z, \varepsilon) - \int_0^\varepsilon A_{-k+1}(z, t) \Theta'(z, t) dt \right) dz. \quad (5.9)$$

□

6. Explicit Formulae in Terms of m .

In the last section we deliberately avoided writing out formulae in terms of m in order to simplify the presentation. In this section we will complete the computations by giving the full formulae. Thereby we become able to separate local terms from global terms.

We will in the following let $\{a_j(x)\}$, $\{c_j(z)\}$, $\{d_j(z)\}$ and $\mathbf{d}_0(z)$ have the same meaning as in the last section, defined in (4.4), (5.4), and (5.5), respectively. We further define $\{a_j^0\}$ by

$$\mathrm{tr}(\tau e^{-tD_K^2})(x, x) \sim \sum_{j=-\infty}^n a_j^0(x) t^{-\frac{j}{2}}.$$

Let $r(z, s)$ be the analytic continuation in s of the integral

$$r(z, s) := \int_0^\infty t^{\frac{s-1}{2}} \mathrm{tr}(A_+ e^{-tA_+^2})(z, z) dt.$$

In $s = 0$, $r(z, s)$ has a simple pole, so we may define coefficients by

$$r(z, s) = \frac{r_{-1}(z)}{s} + r_0(z) + o(1) \quad ; \text{ for } s \rightarrow 0.$$

Notice that $\frac{1}{\sqrt{\pi}} r_0(z)$ is a density, which integrates up to the η -invariant of A_+ , so that $r_0(z)$ is not locally determined.

Lemma 6.1. *We have the following formulae*

$$a_j(x) = \sum_{q=0}^{\lfloor \frac{n-j}{2} \rfloor} \frac{(-1)^q m^{2q}}{q!} a_{j+2q}^0(x), \quad (6.1)$$

$$\mathbf{d}_0(z) = \frac{c_1(z)}{\sqrt{4\pi}}, \quad (6.2)$$

$$d_j(z) = \frac{-c_{j+1}(z)}{j\sqrt{\pi}} \quad ; \quad j \neq 0, \quad (6.3)$$

$$d_0(z) = \frac{-1}{\sqrt{4\pi}} r_0(z) + \frac{1}{2} \sum_{0=\lambda \in \mathrm{sp}(A_+)} |\varphi_\lambda(z)|^2. \quad (6.4)$$

Proof: The formula (6.1) follows since

$$\mathrm{tr}(\tau e^{-t(D_k^2+m^2)}(x, x)) \sim \left(\sum_{j=-\infty}^n a_j^0(x) t^{-\frac{j}{2}} \right) \sum_{q=0}^{\infty} \frac{(-1)^q m^{2q}}{q!} t^q.$$

The formula (6.2) is an easy consequence of (5.4). In a similar way (6.3) follows easily from (5.4). In order to compute (6.4) we first set

$$\bar{k}(z, t) = k(z, t) - \frac{1}{2} \sum_{0=\lambda \in \mathrm{sp}(A_+)} |\varphi_\lambda(z)|^2.$$

Then the integral

$$\Psi(s, z) := \int_0^\infty t^{\frac{s-1}{2}} \bar{k}(z, t) dt \quad (6.5)$$

is absolutely convergent for $\mathrm{Re}(s) \gg 0$. By using the asymptotic expansion (5.5) it follows that in a neighbourhood of $s = -1$, the analytic continuation of (6.5) is of the form

$$\Psi(s, z) = \frac{-4\mathbf{d}_0(z)}{(s+1)^2} + \frac{2(d_0(z) - \frac{1}{2} \sum_{0=\lambda \in \mathrm{sp}(A_+)} |\varphi_\lambda(z)|^2)}{s+1} + O(1).$$

On the other hand we can integrate by part and get

$$\Psi(s, z) = \frac{-2}{s+1} \int_0^\infty t^{\frac{s+1}{2}} k'(z, t) dt = \frac{-1}{\sqrt{\pi}(s+1)} \int_0^\infty t^{\frac{(s+1)-1}{2}} \mathrm{tr}(A_+ e^{-tA_+}^2)(z, z) dt.$$

The contribution of the analytic continuation of the integral to the residue of $\Psi(s, z)$ in -1 is thus the regular part of the analytic continuation of the integral in -1 , which is easily seen to coincide with $r_0(z)$. This gives

$$\frac{-1}{\sqrt{\pi}} r_0(z) = 2(d_0(z) - \frac{1}{2} \sum_{0=\lambda \in \mathrm{sp}(A_+)} |\varphi_\lambda(x)|^2).$$

Formula (6.4) immediately follows. □

The coefficients $\{e_j(z)\}$ were defined in (5.8). Like in (6.1) it follows

$$e_j(z) = \sum_{q=0}^{\lfloor \frac{n-j}{2} \rfloor} \frac{(-1)^q m^{2q}}{q!} d_{j+2q}(z).$$

We notice that if we had replaced $\tilde{\mathcal{K}}$ by \mathcal{K} , there would have been a logarithmic term for every even j .

It follows by (5.7) that

$$b_k(z, \varepsilon) = (-1)^{\frac{k-1}{2}} \left(\frac{k-1}{2} \right)! \sum_{q=0}^{\lfloor \frac{n+k-1}{2} \rfloor} \frac{(-1)^q m^{2q}}{q!} d_{-k+1+2q}(z). \quad (6.6)$$

Combining this with (4.5) and (6.1) renders

$$\frac{m^{k-1} \text{Ind}_e(D^+)}{(-1)^{\frac{k-1}{2}} \left(\frac{k-1}{2}\right)!} = \sum_{q=0}^{\lfloor \frac{n+k-1}{2} \rfloor} \frac{(-1)^q m^{2q}}{q!} \left(\int_X a_{-k+1+2q}(x) dx + \int_Z d_{-k+1+2q}(z) dz \right).$$

Using that this holds for all m we realize, that

$$\text{Ind}_e(D^+) = \int_X a_0(x) dx + \int_Z d_0(z) dz, \quad (6.7)$$

which is just the Atiyah-Patodi-Singer index theorem for X . For the remaining coefficients we get

$$\int_X a_{2p}(x) dx + \int_Z d_{2p}(z) dz = 0 \quad ; p \in \mathbb{Z}, p \leq \frac{n}{2}, p \neq 0. \quad (6.8)$$

That means, that $d_{2p}(z)$ is a locally defined boundary contribution neutralizing the contribution from a_{2p} .

We can define coefficients A_j^0 like the coefficients A_j

$$A_j^0(z, t) = \int_0^t a_j^0(z, s) ds.$$

Again a formula similar to (6.6) holds. The above arguments can be repeated with the additional term, and we get the following

Theorem 6.2. *We have the following index theorem for D_K^+*

$$\text{Ind}(D_K^+) = \int_K a_0(x) dx + \int_Z \left(d_0(z) - \left(A_0^0(z, \varepsilon) - \int_0^\varepsilon A_0^0(z, t) \Theta'(z, t) dt \right) \right) dz. \quad (6.9)$$

Further, for $p \in 2\mathbb{Z}$, $p \leq \frac{n}{2}$ and $p \neq 0$, we have the following vanishing formulae:

$$0 = \int_K a_{2p}(x) dx + \int_Z \left(d_{2p}(z) - \left(A_{2p}^0(z, \varepsilon) - \int_0^\varepsilon A_{2p}^0(z, t) \Theta'(z, t) dt \right) \right) dz. \quad (6.10)$$

□

Here (6.9) is an index theorem generalizing the Atiyah-Patodi-Singer index theorem with the augmentation $\ker(\tau - 1)$. The formula (6.10) gives the almost local boundary contributions promised in the introduction.

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MATHEMATISCHES INSTITUT DER UNIVERSITÄT BONN, BERINGSTRASSE 1, D-53115 BONN, GERMANY
 gorm@styx.math.uni-bonn.de
<http://styx.math.uni-bonn.de/~gorm/>