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Trace formula in noncommutative Geometry and the zeros of the Riemann zeta function

Alain CONNES

Abstract *We give a spectral interpretation of the critical zeros of the Riemann zeta function, and a geometric interpretation of the explicit formulas of number theory as a trace formula on a noncommutative space. This reduces the Riemann hypothesis to the validity of the trace formula.*

It is an old idea, due to Polya and Hilbert that in order to understand the location of the zeros of the Riemann zeta function, one should find a Hilbert space $\mathcal{H}$ and an operator D in $\mathcal{H}$ whose spectrum is given by the non trivial zeros of the zeta function. The hope then is that suitable selfadjointness properties of D (of $i\left(D-\frac{1}{2}\right)$ more precisely) or positivity properties of $\Delta = D(1-D)$ will be easier to handle than the original conjecture. The main reasons why this idea should be taken seriously are first the work of A. Selberg in which a suitable Laplacian Δ is related in the above way to an analogue of the zeta function, and secondly the theoretical ([M][B][KS]) and experimental evidence ([O][BG]) on the fluctuations of the spacing between consecutive zeros of zeta. The number of zeros of zeta whose imaginary part is less than $E > 0$,

$$N(E) = \# \text{ of zeros } \rho\,,\ 0 < \operatorname{Im} \rho < E \tag{1}$$

has an asymptotic expression ([R]) given by

$$N(E) = \frac{E}{2\pi}\left(\log\left(\frac{E}{2\pi}\right) - 1\right) + \frac{7}{8} + o(1) + N_{\rm osc}(E) \tag{2}$$

where the oscillatory part of this step function is

$$N_{\rm osc}(E) = \frac{1}{\pi} \operatorname{Im} \log \zeta\left(\frac{1}{2} + iE\right) \tag{3}$$

assuming that E is not the imaginary part of a zero and taking for the logarithm the branch which is 0 at $+\infty$.

One shows (cf. [Pat]) that $N_{\rm osc}(E)$ is $O(\log E)$. In the decomposition (2) the two terms $\langle N(E)\rangle = N(E) - N_{\rm osc}(E)$ and $N_{\rm osc}(E)$ play an independent role. The first one $\langle N(E)\rangle$ which gives the average density of zeros just comes from Stirling's formula and is perfectly controlled. The second $N_{\rm osc}(E)$ is a manifestation of the randomness of the actual location of the zeros, and to eliminate the role of the density one returns to the situation of uniform density by the transformation

$$x_j = \langle N(E_j)\rangle \quad (E_j \text{ the } j^{\rm th} \text{ imaginary part of zero of zeta})\,. \tag{4}$$

Thus the spacing between two consecutive x_j is now 1 in average and the only information that remains is in the statistical fluctuation. As it turns out ([M][O]) these fluctuations are the same as the fluctuations of the eigenvalues of a random hermitian matrix of very large size.

H. Montgomery [M] proved (assuming RH) a weakening of the following conjecture (with $\alpha, \beta > 0$),

$$\begin{aligned} &\mathrm{Card}\,\{(i,j)\,;\, i,j \in 1,\ldots,M\,;\, x_i - x_j \in [\alpha,\beta]\} \\ &\qquad \sim M \int_\alpha^\beta \left(1 - \left(\frac{\sin(\pi u)}{\pi u}\right)^2\right) du \end{aligned} \tag{5}$$

This law (5) is precisely the same as the correlation between eigenvalues of hermitian matrices of the gaussian unitary ensemble ([M]). Moreover, numerical tests due to A. Odlyzko ([O][BG]) have confirmed with great precision the behaviour (5) as well as the analoguous behaviour for more than two zeros. In [KS], N. Katz and P. Sarnak proved an analogue of the Montgomery-Odlyzko law for zeta and L-functions of function fields over curves.

It is thus an excellent motivation to try and find a natural pair $(\mathcal{H}, D)$ where naturality should mean for instance that one should not even have to define the zeta function in order to obtain the pair (in order for instance to avoid the joke of defining $\mathcal{H}$ as the ℓ^2 space built on the zeros of zeta).

Let us first describe following [B] the direct atempt to construct the Polya-Hilbert space from quantization of a classical dynamical system. The original motivation for the theory of random matrices comes from quantum mechanics. In this theory the quantization of the classical dynamical system given by the phase space X and hamiltonian h gives rise to a Hilbert space $\mathcal{H}$ and a selfadjoint operator H whose spectrum is the essential physical observable of the system. For complicated systems the only useful information about this spectrum is that, while the average part of the counting function,

$$(6) \qquad N(E) = \#\text{ eigenvalues of } H \text{ in } [0, E]$$

is computed by a semiclassical approximation mainly as a volume in phase space, the oscillatory part,

$$(7) \qquad N_{\mathrm{osc}}(E) = N(E) - \langle N(E) \rangle$$

is the same as for a random matrix, governed by the statistic dictated by the symmetries of the system.

In the absence of a magnetic field, i.e. for a classical hamiltonian of the form,

$$(8) \qquad h = \frac{1}{2m} p^2 + V(q)$$

where V is a real-valued potential on configuration space, there is a natural symmetry of classical phase space,

$$(9) \qquad T(p, q) = (-p, q)$$

which preserves h, and entails that the correct ensemble on the random matrices is not the above GUE but rather the gaussian orthogonal ensemble: GOE. Thus the oscillatory part $N_{\mathrm{osc}}(E)$ behaves in the same way as for a random *real symmetric* matrix.

Of course H is just a specific operator in $\mathcal{H}$ and, in order that it behaves *generically* it is necessary (cf. [B]) that the classical hamiltonian system (X, h) be *chaotic* with isolated *periodic orbits* whose instability exponents (i.e. the logarithm of the eigenvalues of the Poincaré return map acting on the transverse space to the orbits) are different from 0.

One can then ([B]) write down an asymptotic semiclassical approximation to the oscillatory function $N_{\mathrm{osc}}(E)$

$$(10) \qquad N_{\mathrm{osc}}(E) = \frac{1}{\pi} \operatorname{Im} \int_0^\infty \operatorname{Trace}(H - (E + i\eta))^{-1}\, i d\eta$$

using the stationary phase approximation of the corresponding functional integral. For a system whose configuration space is 2-dimensional, this gives ([B]),

$$(11) \qquad N_{\mathrm{osc}}(E) \simeq \frac{1}{\pi} \sum_{\gamma_p} \sum_{m=1}^{\infty} \frac{1}{m} \frac{1}{2\mathrm{sh}\left(\frac{m\lambda_p}{2}\right)} \sin(S_{\mathrm{pm}}(E))$$

where the γ_p are the primitive periodic orbits, the label m corresponds to the number of traversals of this orbit, while the corresponding instability exponents are $\pm\lambda_p$. The phase $S_{\mathrm{pm}}(E)$ is up to a constant equal to $m\,E\,T_\gamma^\#$ where $T_\gamma^\#$ is the period of the primitive orbit γ_p.

The formula (11) gives very precious information ([B]) on the hypothetical "Riemann flow" whose quantization should produce the Polya-Hilbert space. The point is that the Euler product formula for the zeta function yields (cf. [B]) a similar asymptotic formula for $N_{\mathrm{osc}}(E)$ (3),

$$N_{\mathrm{osc}}(E) \simeq \frac{-1}{\pi} \sum_p \sum_{m=1}^{\infty} \frac{1}{m} \frac{1}{p^{m/2}} \sin\left(m\,E\,\log p\right). \tag{12}$$

Comparing (11) and (12) gives the following information,

(13) The periodic primitive orbits should be labelled by the prime numbers $p = 2, 3, 5, 7, \ldots$, their periods should be the $\log p$ and their instability exponents $\lambda_p = \pm \log p$.

Moreover, since each orbit is only counted once, the Riemann flow should not possess the symmetry T of (9) whose effect would be to duplicate the count of orbits. This last point excludes in particular the geodesic flows since they have the time reversal symmetry T.

However there are two important mismatches (cf. [B]) between the two formulas (11) and (12). The first one is the overall *minus sign* in front of formula (12), the second one is that though $2\mathrm{sh}\left(\frac{m\lambda_p}{2}\right) \sim p^{m/2}$ when $m \to \infty$, we do not have an equality for finite values of m.

These are two fundamental difficulties and in order to overcome them we shall use the well known strategy of extending the problem of finding the hypothetical Riemann flow to the case of arbitrary *global fields*. By specialising to the function field case we shall then obtain additional precious information. The basic example of a global field is the field $\mathbb{Q}$ of rational numbers and we shall take as a conceptual definition of such fields k, the fact that they are *discrete and cocompact* in a (non discrete) locally compact semisimple abelian ring A. As it turns out A then depends functorially on k and is called the Adele ring of k, often denoted by k_A. When the characteristic p of a global field k is > 0, the field k is the function field of a non singular algebraic curve Σ defined over a finite field $\mathbb{F}_q$ included in k as its maximal finite subfield, called the field of constants. One can then apply the ideas of algebraic geometry, first developed over $\mathbb{C}$, to the geometry of the curve Σ and obtain a geometric interpretation of the basic properties of the zeta function of k; the dictionary contains in particular the following lines

(14)

Functional equation	Riemann Roch theorem (Poincaré duality)
Explicit formulas of number theory	Lefchetz formula for the Frobenius
Riemann hypothesis	Castelnuovo positivity

Since $\mathbb{F}_q$ is not algebraically closed, the points of Σ defined over $\mathbb{F}_q$ do not suffice and one needs to consider $\bar{\Sigma}$, the points of Σ on the algebraic closure $\bar{\mathbb{F}}_q$ of $\mathbb{F}_q$, which is obtained by adjoining to $\mathbb{F}_q$ the roots of unity of order prime to q. This set of points is a countable union of periodic orbits under the action of the Frobenius automorphism, these orbits are parametrized by the set of places of k and their periods are indeed given by the analogues of the $\log p$ of (13). Being a countable set it does not qualify for analogue of the Riemann flow and it only aquires an interesting structure from algebraic geometry. The minus sign which was problematic in the above discussion admits here a beautiful resolution since the analogue of the Polya-Hilbert space is given, if one replaces $\mathbb{C}$ by $\mathbb{Q}_\ell$ the field of ℓ-adic numbers $\ell \neq p$, by the cohomology group

$$H^1_{\text{et}}(\bar{\Sigma}, \mathbb{Q}_\ell) \tag{15}$$

which appears with an overall minus sign in the Lefchetz formula $\text{Trace}/H^0 - \text{Trace}/H^1 + \text{Trace}/H^2$.

For the general case this suggests

(16) The Polya-Hilbert space $\mathcal{H}$ should appear from its negative $\ominus\mathcal{H}$.

The next thing that one learns from this excursion in characteristic $p > 0$ is that in that case one is not dealing with a flow but rather with a single transformation. In fact taking advantage of abelian covers of Σ and of the fundamental isomorphism of class field theory one finds that the natural group that should replace $\mathbb{R}$ for the general Riemann flow is the Idele class group:

$$C_k = \text{GL}_1(A)/k^* \, . \tag{17}$$

We can thus collect the information (13) (16) (17) that we have obtained so far and look for the Riemann flow as an action of C_k on an hypothetical space X.

There is a third approach to the problem of the zeros of the Riemann zeta function, due to G. Pólya [P] and M. Kac [K] and pursued further in [J] [BC]. It is based on statistical mechanics and the construction of a quantum statistical system whose *partition function* is the Riemann zeta function. Such a system was naturally constructed in [BC] and it does indicate using the first line of the dictionary of noncommutative Geometry what the space X should be in general:

$$X = A/k^* \tag{18}$$

namely the quotient of the space A of adeles, $A = k_A$ by the action of the multiplicative group k^*,

$$a \in A\ ,\ q \in k^* \to aq \in A\,. \tag{19}$$

This space X is a noncommutative space and for instance even at the measure theory level, the corresponding von Neumann algebra,

$$R_{01} = L^\infty(A) \rtimes k^* \tag{20}$$

where A is endowed with its Haar measure as an additive group, is the hyperfinite factor of type II_∞.

The idele class group C_k acts on X by

$$(j, a) \to ja \qquad \forall\, j \in C_k\ ,\ a \in X \tag{21}$$

and it was exactly necessary to divide A by k^* so that (21) makes good sense.

We shall come back later to the analogy between the action of C_k on R_{01} and the action of the Galois group of the maximal abelian extension of k.

What we shall do now is to construct the Hilbert space L^2_δ of functions on X with growth indexed by $\delta > 1$. Since X is a quotient space we shall first learn in the usual manifold case how to obtain the Hilbert space $L^2(M)$ of square integrable functions on M by working only on the universal cover $\widetilde{M}$ with the action of $\Gamma = \pi_1(M)$. Every function $f \in C_c^\infty(\widetilde{M})$ gives rise to a function $\widetilde{f}$ on M by

$$\widetilde{f}(x) = \sum_{\pi(\widetilde{x})=x} f(\widetilde{x}) \tag{22}$$

and all $g \in C^\infty(M)$ appear in this way. Moreover, one can write the Hilbert space inner product $\int_M \widetilde{f}_1(x)\, \widetilde{f}_2(x)\, dx$, in terms of f_1 and f_2 alone. Thus $\|\widetilde{f}\|^2 = \int \left|\sum_{\gamma\in\Gamma} f(\gamma x)\right|^2 dx$ where the integral is performed on a fundamental domain for Γ acting on $\widetilde{M}$. This formula defines a prehilbert space norm on $C_c^\infty(\widetilde{M})$ and $L^2(M)$ is just the completion of $C_c^\infty(\widetilde{M})$ for that norm. Note that any function of the form $f - f_\gamma$ has vanishing norm and hence disappears in the process of completion. In our case of $X = A/k^*$ we thus need to define the analoguous norm on the Schwartz space $\mathcal{S}(A)$ of functions on A. Since 0 is fixed by the action of k^* the expression $\sum_{\gamma\in k^*} f(\gamma x)$ does not make sense for $x = 0$ unless we require that $f(0) = 0$. Moreover, when $|x| \to 0$, the above sums approximate, as Riemann sums, the product of $|x|^{-1}$ by $\int f\, dx$ for the additive Haar measure, thus we also require $\int f\, dx = 0$. We can now define the Hilbert space $L^2_\delta(X)_0$ as the completion of

$$\mathcal{S}(A)_0 = \{f \in \mathcal{S}(A)\ ;\ f(0) = 0\ ,\ \int f\, dx = 0\} \tag{23}$$

for the norm $\|\ \|_\delta$ given by

$$\|f\|_\delta^2 = \int \left|\sum_{q\in k^*} f(qx)\right|^2 (1 + \log^2|x|)^{\delta/2}\, |x|\, d^*x \tag{24}$$

where the integral is performed on A^*/k^* and d^*x is the multiplicative Haar measure on A^*/k^*. The term $(1+\log^2|x|)^{\delta/2}$ is there to control the growth of the functions. The key point is that we use the measure $|x|\, d^*x$ instead of the additive Haar measure dx. Of course for a local field K one has $dx = |x|\, d^*x$ but this fails in the above global situation. Instead one has

$$dx = \lim_{\varepsilon\to 0} \varepsilon\, |x|^{1+\varepsilon}\, d^*x\,. \tag{25}$$

One has a natural representation of C_k on $L^2_\delta(X)_0$ given by

$$(U(j)\, f)\,(x) = f(j^{-1}\, x) \qquad \forall\, x \in A\ ,\ j \in C_k \tag{26}$$

and the result is independent of the choice of a lift of j in $J_k = \mathrm{GL}_1(A)$ because the functions $f - f_q$ are in the kernel of the norm. The conditions (23) which define $\mathcal{S}(A)_0$ are invariant under the action of C_k and give the following action of C_k on the 2-dimensional supplement of $\mathcal{S}(A)_0 \subset \mathcal{S}(A)$; this supplement is $\mathbb{C} \oplus \mathbb{C}(1)$ where $\mathbb{C}$ is the trivial C_k module (corresponding to $f(0)$) while the Tate twist $\mathbb{C}(1)$ is the module

$$(j, \lambda) \to |j|\, \lambda \tag{27}$$

coming from the equality

$$\int f(j^{-1}\,x)\,d\,x = |j| \int f(x)\,dx\,. \tag{28}$$

In order to analyse the representation (26) of C_k on $L^2_\delta(X)_0$ we shall relate it to the left regular representation of the group C_k on the Hilbert space $L^2_\delta(C_k)$ obtained from the following Hilbert space square norm on functions,

$$\|\xi\|^2_\delta = \int_{C_k} |\xi(g)|^2\,(1+\log^2|g|)^{\delta/2}\,d^*g \tag{29}$$

where we have normalized the Haar measure of the multiplicative group C_k, with module,

$$|\ | : C_k \to \mathbb{R}^*_+ \tag{30}$$

in such a way that (cf. [W3])

$$\int_{|g|\in[1,\Lambda]} d^*g \sim \log\Lambda \quad \text{when} \quad \Lambda \to +\infty\,. \tag{31}$$

The left regular representation V of C_k on $L^2_\delta(C_k)$ is

$$(V(a)\,\xi)\,(g) = \xi(a^{-1}\,g) \qquad \forall\, g, a \in C_k\,. \tag{32}$$

Note that because of the weight $(1+\log^2|x|)^{\delta/2}$, this representation is *not* unitary but it satisfies the growth estimate

$$\|V(g)\| = 0\,(\log|g|)^{\delta/2} \quad \text{when} \quad |g| \to \infty \tag{33}$$

which follows from the inequality (valid for $u, v \in \mathbb{R}$)

$$\rho(u+v) \leq 2^{\delta/2}\,\rho(u)\,\rho(v)\ ,\ \rho(u) = (1+u^2)^{\delta/2}\,. \tag{34}$$

We let E be the linear isometry from $L^2_\delta(X)_0$ into $L^2_\delta(C_k)$ given by the equality,

$$E(f)\,(g) = |g|^{1/2} \sum_{q\in k^*} f(qg) \qquad \forall\, g \in C_k\,. \tag{35}$$

By comparing (24) with (29) we see that E is an isometry and the factor $|g|^{1/2}$ is dictated by comparing the measures $|g|\,d^*g$ of (24) with d^*g of (29).

One has $E(U(a)\,f)\,(g) = |g|^{1/2} \sum_{k^*} (U(a)\,f)\,(qg) = |g|^{1/2} \sum_{k^*} f(a^{-1}\,qg) = |a|^{1/2}\,|a^{-1}\,g|^{1/2} \sum_{k^*} f(q\,a^{-1}\,g) = |a|^{1/2}\,(V(a)\,E(f))\,(g)$.

Thus,

$$E\,U(a) = |a|^{1/2}\,V(a)\,E\,. \tag{36}$$

This equivariance shows that the range of E in $L^2_\delta(C_k)$ is a closed invariant subspace for the representat.on V.

The following theorem and its corollary shows that the cokernel $\mathcal{H} = L^2_\delta(C_k)/\mathrm{Im}(E)$ of the isometry E plays the role of the Polya-Hilbert space. Since $\mathrm{Im}\,E$ is invariant under the representation V we let W be the corresponding representation of C_k on $\mathcal{H}$.

The abelian locally compact group C_k is (non canonically) isomorphic to $K \times N$ where

$$K = \{g \in C_k \; ; \; |g| = 1\}\ , \ N = \text{range}\ |\ | \subset \mathbb{R}^*_+\,. \tag{37}$$

For number fields one has $N = \mathbb{R}^*_+$ while for fields of non zero characteristic $N \simeq \mathbb{Z}$ is the subgroup $q^{\mathbb{Z}} \subset \mathbb{R}^*_+$.(Where $q = p^\ell$ is the cardinality of the field of constants).

We choose (non canonically) an isomorphism

$$C_k \simeq K \times N\,. \tag{38}$$

By construction the representation W satisfies (using (33)),

$$\|W(g)\| = 0(\log|g|)^{\delta/2} \tag{39}$$

and its restriction to K is unitary. Thus $\mathcal{H}$ splits as a canonical direct sum of pairwise orthogonal subspaces,

$$\mathcal{H} = \bigoplus_{\chi \in \widehat{K}} \mathcal{H}_\chi\ , \ \mathcal{H}_\chi = \{\xi \; ; \; W(g)\,\xi = \chi\,(g)\,\xi\ , \ \forall\, g \in K\} \tag{40}$$

where χ runs through the Pontrjagin dual group of K, which is the discrete abelian group $\widehat{K}$ of characters of K. Using the non canonical isomorphism (38), i.e. the corresponding inclusion $N \subset C_k$ one can now restrict the representation W to any of the sectors $\mathcal{H}_\chi$. When $\mathrm{char}(k) > 0$, then $N \simeq \mathbb{Z}$ and the condition (39) shows that the action of N on $\mathcal{H}_\chi$ is given by a single operator with *unitary* spectrum. (One uses the spectral radius formula

$|\mathrm{Spec}\, w| = \overline{\mathrm{Lim}}\, \|\, w^n \|^{1/n}$.) When $\mathrm{Char}(k) = 0$, we are dealing with an action of $\mathbb{R}_+^* \simeq \mathbb{R}$ on $\mathcal{H}_\chi$ and the condition (39) shows that this representation is generated by a closed unbounded operator D with purely imaginary spectrum. The resolvent $R_\lambda = (D - \lambda)^{-1}$ is given, for $\mathrm{Re}\,\lambda > 0$, by the equality

$$R_\lambda = \int_0^\infty W_\chi(e^s)\, e^{-\lambda s}\, ds \tag{41}$$

and for $\mathrm{Re}\,\lambda < 0$ by,

$$R_\lambda = \int_0^\infty W_\chi(e^{-s})\, e^{\lambda s}\, ds \tag{42}$$

while the operator D is defined by

$$D\,\xi = \lim_{\varepsilon \to 0} \frac{1}{\varepsilon}\, (W_\chi(e^\varepsilon) - 1)\, \xi\,. \tag{43}$$

Theorem 1. *Let $\chi \in \widehat{K}$, $\delta > 1$, $\mathcal{H}_\chi$ and D be as above. Then D has discrete spectrum, $\mathrm{Sp}\, D \subset i\,\mathbb{R}$ is the set of zeros of the L function with Grossencharacter $\widetilde{\chi}$ which have real part equal to $\frac{1}{2}$; $\rho \in \mathrm{Sp}\, D \Leftrightarrow L\left(\widetilde{\chi}, \frac{1}{2} + \rho\right) = 0$ and $\rho \in i\,\mathbb{R}$, where $\widetilde{\chi}$ is the unique extension of χ to C_k which is equal to 1 on N. Moreover the multiplicity of ρ in $\mathrm{Sp}\, D$ is equal to the largest integer $n < \frac{1+\delta}{2}$, $n \leq$ multiplicity of $\frac{1}{2} + \rho$ as a zero of L.*

Corollary 2. *For any Schwartz function $h \in \mathcal{S}(C_k)$ the operator $W(h) = \int W(g)\, h(g)\, d^* g$ in $\mathcal{H}$ is of trace class, and its trace is given by*

$$\mathit{Trace}\, W(h) = \sum_{\substack{L\left(\chi, \frac{1}{2} + \rho\right) = 0 \\ \rho \in i\,\mathbb{R}}} \widehat{h}(\chi, \rho)$$

where the multiplicity is counted as in Theorem 1 and where the Fourier transform $\widehat{h}$ of h is defined by,

$$\widehat{h}(\chi, \rho) = \int_{C_k} h(u)\, \widetilde{\chi}(u)\, |u|^\rho\, d^* u\,.$$

Note that we did not have to define the L functions before stating the theorem, which shows that the pair

$$(\mathcal{H}_\chi, D) \tag{44}$$

certainly qualifies as a Polya-Hilbert space.

The case of the Riemann zeta function corresponds to the trivial character $\chi = 1$ for the global field $k = \mathbb{Q}$ of rational numbers.

In general the zeros of the L functions can have multiplicity but one expects that for a fixed Grossencharacter χ this multiplicity is bounded, so that for a large enough value of δ the spectral multiplicity of D will be the right one. When the characteristic of k is > 0 this is certainly true.

If we modify the choice of non canonical isomorphism (38) this modifies the operator D by

$$D' = D - i\,s \tag{45}$$

where $s \in \mathbb{R}$ is determined by the equality

$$\widetilde{\chi}'(g) = \widetilde{\chi}(g)\,|g|^{i\,s} \qquad \forall\, g \in C_k\,. \tag{46}$$

The coherence of the statement of the theorem is insured by

$$L(\widetilde{\chi}', z) = L(\widetilde{\chi}, z + i\,s) \qquad \forall\, z \in \mathbb{C}. \tag{47}$$

When the zeros of L have multiplicity and δ is large enough the operator D is *not* semisimple and has a non trivial Jordan form. This is compatible with the almost unitary condition (39) but not with skew symmetry for D.

The proof of theorem 1 ([Co]) is based on the distribution theoretic interpretation by A. Weil [W2] of the idea of Tate and Iwasawa on the functional equation. Our construction should be compared with [Bg] and [Z].

As we expected from (16), the Polya-Hilbert space $\mathcal{H}$ appears as a cokernel. Since we obtain the Hilbert space $L^2_\delta(X)_0$ by imposing two linear conditions on $\mathcal{S}(A)$,

$$0 \to \mathcal{S}(A)_0 \to \mathcal{S}(A) \xrightarrow{L} \mathbb{C} \oplus \mathbb{C}(1) \to 0 \tag{48}$$

we shall define $L^2_\delta(X)$ so that it fits in an exact sequence of C_k-modules

$$0 \to L^2_\delta(X)_0 \to L^2_\delta(X) \to \mathbb{C} \oplus \mathbb{C}(1) \to 0\,. \tag{49}$$

We can then use the exact sequence of C_k-modules

$$0 \to L^2_\delta(X)_0 \to L^2_\delta(C_k) \to \mathcal{H} \to 0 \tag{50}$$

together with Corollary 2 to compute in a formal manner what the character of the module $L^2_\delta(X)$ should be. Using (49) and (50) we obtain,

$$(51)\qquad \text{"Trace"}\ (U(h)) = \widehat{h}(0) + \widehat{h}(1) - \sum_{\substack{L(\chi,\rho)=0\\ \mathrm{Re}\rho=\frac{1}{2}}} \widehat{h}(\chi,\rho) + \infty\, h(1)$$

where $\widehat{h}(\chi,\rho)$ is defined by Corollary 2 and

$$(52)\qquad U(h) = \int_{C_k} U(g)\, h(g)\, d^* g$$

while the test function h is in a suitable function space. Note that the trace on the left hand side of (51) only makes sense after a suitable regularisation since the left regular representation of C_k is not traçable. This situation is similar to the one encountered by Atiyah and Bott ([AB]) in their proof of the Lefchetz formula. In particular it is important to deal not with Hilbert spaces but rather with nuclear spaces in the sense of Grothendieck. The point being that the Schwartz kernel theorem is then available and one can at least talk about the integral of the diagonal values of the Schwartz kernels as a problem of product of distributions. In our context this is achieved by letting δ go to ∞, i.e. by considering

$$(53)\qquad \mathcal{S}(X) = \bigcap_\delta L^2_\delta(X)\,.$$

This space is *locally* nuclear for the action of C_k. In particular the Schwartz kernel theorem applies to the operators $U(h)$.

The distribution trace formula for flows on manifolds

In order to understand how the left hand side of (51) should be computed we shall first give a leisurly account of the much easier but analoguous computation of the distribution theoretic trace for flows on manifolds,which is a variation on the theme of [AB]. We just follow Guillemin Sternberg [GS] and extract from [GS] the relevant case for our discussion.

Recall that given a vector space E over $\mathbb{R}$, $\dim E = n$, a density is a map, $\rho \in |E|$,

$$(1)\qquad \rho : \wedge^n E \to \mathbb{C}$$

such that $\rho(\lambda v) = |\lambda|\, \rho(v) \qquad \forall\, \lambda \in \mathbb{R}$.

Given a linear map $T : E \to F$ we let $|T| : |F| \to |E|$ be the corresponding linear map,it depends contravariantly on T.

Given a manifold M and $\rho \in C_c^\infty(M, |TM|)$ one has a canonical integral,

$$\int \rho \in \mathbb{C}. \tag{2}$$

Given a vector bundle L on M one defines the generalized sections on M as the dual space of $C_c^\infty(M, L^* \otimes |TM|)$

$$C^{-\infty}(M, L) = \text{ dual of } C_c^\infty(M, L^* \otimes |TM|) \tag{3}$$

where L^* is the dual bundle. One has a natural inclusion,

$$C^\infty(M, L) \subset C^{-\infty}(M, L) \tag{4}$$

given by the pairing

$$\sigma \in C^\infty(M, L)\ ,\ s \in C_c^\infty(M, L^* \otimes |TM|) \to \int \langle s, \sigma \rangle \tag{5}$$

where $\langle s, \sigma \rangle$ is viewed as a density, $\langle s, \sigma \rangle \in C_c^\infty(M, |TM|)$.

One has a similar notion of generalized section with compact support.

Given a smooth map $\varphi : X \to Y$, then if φ is *proper*, it gives a (contravariantly) associated map

$$\varphi^* : C_c^\infty(Y, L) \to C_c^\infty(X, \varphi^*(L))\ ,\ (\varphi^*\, \xi)(x) = \xi(\varphi(x)) \tag{6}$$

where $\varphi^*(L)$ is the pull back of the vector bundle L.

Thus, given a linear form on $C_c^\infty(X, \varphi^*(L))$ one has a (covariantly) associated linear form on $C_c^\infty(Y, L)$. In particular with L trivial we see that given a generalized density $\rho \in C^{-\infty}(X, |T|)$ one has a pushforward

$$\varphi_*(\rho) \in C^{-\infty}(Y, |T|) \tag{7}$$

with $\langle \varphi_*(\rho), \xi \rangle = \langle \rho, \varphi^*\, \xi \rangle \qquad \forall\, \xi \in C_c^\infty(X)$.

Next, if φ is a fibration and $\rho \in C_c^\infty(X, |T|)$ is a density then one can integrate ρ along the fibers, the obtained density on Y, $\varphi_*(\rho)$ is given as in (7) by

$$\langle \varphi_*(\rho), f \rangle = \langle \rho, \varphi^*\, f \rangle \qquad \forall\, f \in C^\infty(Y) \tag{8}$$

but the point is that it is not only a generalized section but a smooth section $\varphi_*(\rho) \in C_c^\infty(Y, |T|)$.

It follows that if $f \in C^{-\infty}(Y)$ is a generalized function, then one obtains a generalized function $\varphi^*(f)$ on X by,

$$(9) \qquad \langle \varphi^*(f), \rho \rangle = \langle f, \varphi_*(\rho) \rangle \qquad \forall\, \rho \in C_c^\infty(X, |T|)\,.$$

In general,the pullback $\varphi^*(f)$ continues to make sense provided the following transversality condition holds,

$$(10) \qquad d(\varphi^*(l)) \neq 0 \quad \forall l \in WF(f)\,.$$

where $WF(f)$ is the wave front set of f ([GS]). The next point is the construction of the generalized section of a vector bundle L on a manifold X associated to a submanifold $Z \subset X$ and a symbol,

$$(11) \qquad \sigma \in C^\infty(Z, L \otimes |N_Z^*|)\,.$$

where N_Z is the normal bundle of Z. The construction is the same as that of the current of integration on a cycle. Given $\xi \in C_c^\infty(X, L^* \otimes |T|)$, the product $\sigma\, \xi/Z$ is a density on Z, since it is a section of $|T_Z| = |T_X| \otimes |N_Z^*|$.One can thus integrate it over Z. When $Z = X$ one has $N_Z^* = \{0\}$ and $|N_Z^*|$ has a canonical section, so that the current associated to σ is just given by (5). When $Z = \mathrm{pt}$ is a single point $x \in X$ a generalized section of L given by a dirac distribution at x requires not only a vector $\xi_x \in L_x$ but also a dual density, i.e. a volume multivector $v \in |T_x^*|$.

Now let $\varphi : X \to Y$ with Z a submanifold of Y and σ as in (11).

Let us assume that φ is transverse to Z, so that for each $x \in X$ with $y = \varphi(x) \in Z$ one has

$$(12) \qquad \varphi_*(T_x) + T_{\varphi(x)}(Z) = T_y Y\,.$$

Let

$$(13) \qquad \tau_x = \{X \in T_x \,,\ \varphi_*(X) \in T_y(Z)\}\,.$$

Then φ_* gives a canonical isomorphism,

$$(14) \qquad \varphi_* : T_x(X)/\tau_x \simeq T_y(Y)/T_y(Z) = N_y(Z)\,.$$

And $\varphi^{-1}(Z)$ is a submanifold of X of the same codimension as Z with a natural isomorphism of normal bundles

$$N_{\varphi^{-1}(Z)} \simeq \varphi^* N_Z \,. \tag{15}$$

In particular, given a (generalized) δ-section of a bundle L with support Z and symbol $\sigma \in C^\infty(Z, L \otimes |N_Z^*|)$ one has a corresponding symbol on $\varphi^{-1}(Z)$ given by

$$\varphi^* \sigma(x) = \sigma(\varphi(x)) \in (\varphi^* L)_x \otimes |N_x^*| \tag{16}$$

using the isomorphism (15) i.e. $N_x^* \simeq N_{\varphi(x)}^*$.
Now for any δ-section associated to Z, σ, the wave front set is contained in the conormal bundle of the submanifold Z which shows that if φ is transverse to Z the pull back $\varphi^* \delta_{Z,\sigma}$ of the distribution on Y associated to Z, σ makes sense ,it is equal to $\delta_{\varphi^{-1}(Z), \varphi^*(\sigma)}$.

Let us now formulate the Schwartz kernel theorem. One considers a continuous linear map,

$$T : C_c^\infty(Y) \to C^{-\infty}(X)\,, \tag{17}$$

the statement is that one can write it as

$$(T\,\xi)\,(x) = \int k(x,y)\, \xi(y)\, dy \tag{18}$$

where $k(x,y)\,dy$ is a generalized section,

$$k \in C^{-\infty}(X \times Y \ , \ \mathrm{pr}_Y^*(|T|))\,. \tag{19}$$

Let $f : X \to Y$ be a smooth map, and $T = f^*$ the operator

$$(T\,\xi)\,(x) = \xi\,(f(x)) \qquad \forall\, \xi \in C_c^\infty(Y)\,. \tag{20}$$

Let us show that the corresponding k is the δ-section associated to the submanifold of $X \times Y$ given by

$$\mathrm{Graph}(f) = \{(x, f(x))\ ;\ x \in X\} = Z \tag{21}$$

and identify its symbol, $\sigma \in C^\infty(Z, \mathrm{pr}_Y^*(|T|) \otimes |N_Z^*|)$.

Given $\xi \in T_x^*(X)$, $\eta \in T_y^*(Y)$ one has $(\xi,\eta) \in N_Z^*$ iff it is orthogonal to $(v, f_* v)$ for any $v \in T_x(X)$, i.e. $\langle v,\xi\rangle + \langle f_* v,\eta\rangle = 0$ so that

$$\xi = -f_*^t \eta\,. \tag{22}$$

Thus one has a canonical isomorphism $T_y^*(Y) \simeq N_Z^*$, $\eta \xrightarrow{j} (-f_*^t \eta, \eta)$. The transposed $(j^{-1})^t$ is given by $(j^{-1})^t(Y) =$ class of $(0,Y)$ in N_Z, $\forall\, Y \in T_y(Y)$. Thus, there is a canonical choice of symbol σ,

$$\sigma = |j^{-1}| \in C^\infty(Z, \mathrm{pr}_Y^*(|T|) \otimes |N_Z^*|)\,. \tag{23}$$

We denote the corresponding δ-distribution by

$$k(x,y)\,dy = \delta(y - f(x))\,dy\,. \tag{24}$$

One then checks the formula,

$$\int \delta(y-f(x))\,\xi(y)\,dy = \xi(f(x)) \qquad \forall\,\xi \in C_c^\infty(Y)\,. \tag{25}$$

Let us now consider a manifold M with a flow F_t

$$F_t(x) = \exp(t\,v)\,x \qquad v \in C^\infty(M, T_M) \tag{26}$$

and the following map f,

$$f : M \times \mathbb{R} \to M\ ,\ f(x,t) = F_t(x)\,. \tag{27}$$

We apply the above discussion with $X = M \times \mathbb{R}$, $Y = M$. The graph of f is the submanifold Z of $X \times Y$,

$$Z = \{(x,t,y)\ ;\ y = F_t(x)\}\,. \tag{28}$$

One lets φ be the diagonal map,

$$\varphi(x,t) = (x,t,x)\ ,\ \varphi : M \times \mathbb{R} \to X \times Y \tag{29}$$

and the first issue is the transversality $\varphi \pitchfork Z$.

We thus need to consider (12) for each (x,t) such that $\varphi(x,t) \in Z$, i.e. such that $x = F_t(x)$. One looks at the image by φ_* of the tangent space $T_x\,M \times \mathbb{R}$ to $M \times \mathbb{R}$ at (x,t). One lets ∂_t be the natural vector field on $\mathbb{R}$. The image of $(X, \lambda\,\partial_t)$ is $(X, \lambda\,\partial_t, X)$ for $X \in T_x\,M, \lambda \in \mathbb{R}$. Dividing the tangent space of $M \times \mathbb{R} \times M$ by the image of φ_* one gets an isomorphism,

$$(X, \lambda\,\partial_t, Y) \to Y - X \tag{30}$$

with $T_x\,M$.The tangent space to Z is $\{(X', \mu\,\partial_t, (F_t)_*\,X' + \mu\,v_{F_t(x)});\ X' \in T_x\,M, \mu \in \mathbb{R}\}$. Thus the transversality condition means that every element of $T_x\,M$ is of the form

$$(F_t)_*\,X - X + \mu\,v_x \qquad X \in T_x\,M\ ,\ \mu \in \mathbb{R}\,. \tag{31}$$

One has

$$(F_t)_*\,\mu\,v_x = \mu\,v_x \tag{32}$$

so that $(F_t)_*$ defines a quotient map, the Poincaré return map

$$P : T_x/\mathbb{R}\,v_x \to T_x/\mathbb{R}\,v_x = N_x \tag{33}$$

and the transversality condition (31) means exactly,

$$1 - P \qquad \text{is invertible.} \tag{34}$$

Let us make this hypothesis and compute the symbol σ of the distribution,

$$\tau = \varphi^*(\delta(y - F_t(x))\,dy)\,. \tag{35}$$

First, as above, let $W = \varphi^{-1}(Z) = \{(x,t)\ ;\ F_t(x) = x\}$. The codimension of $\varphi^{-1}(Z)$ in $M \times \mathbb{R}$ is the same as the codimension of Z in $M \times \mathbb{R} \times M$ so it is dim$M$ which shows that $\varphi^{-1}(Z)$ is 1-dimensional. If $(x,t) \in \varphi^{-1}(Z)$ then $(F_s(x), t) \in \varphi^{-1}(Z)$. Thus,if we assume that v does not vanish at x, the map,

$$(x,t) \xrightarrow{q} t \tag{36}$$

is locally constant on the connected components of $\varphi^{-1}(Z)$.
This allows to identify the transverse space to $W = \varphi^{-1}(Z)$ as the product,

$$N_{x,t}^W \simeq N_x \times \mathbb{R} \tag{37}$$

where to $(X, \lambda\,\partial_t) \in T_{x,t}(M \times \mathbb{R})$ we associate the pair $(\widetilde{X}, \lambda)$ of the class of X in $N_x = T_x/\mathbb{R}\,v_x$ and $\lambda \in \mathbb{R}$.

The symbol σ of the distribution (35) is a smooth section of $|N^{W*}|$ tensored by the pull back $\varphi^*(L)$ where $L = \mathrm{pr}_Y^* \, |T_M|$, and one has

$$\varphi^*(L) \simeq |p^* \, T_M| \tag{38}$$

where

$$p(x,t) = x \qquad \forall \, (x,t) \in M \times \mathbb{R} \, . \tag{39}$$

To compute σ one needs the isomorphism,

$$N^W_{(x,t)} \overset{\varphi_*}{\to} T_{\varphi(x,t)}(M \times \mathbb{R} \times M)/T_{\varphi(x,t)}(Z) = N^Z \, . \tag{40}$$

The map $\varphi_* : N^W_{x,t} \to N^Z$ is given by

$$\varphi_*(X, \lambda \, \partial_t) = (1 - (F_t)_*) \, X - \lambda \, v \qquad X \in N_x \;, \; \lambda \in \mathbb{R} \tag{41}$$

and the symbol σ is just

$$\sigma = |\varphi_*^{-1}| \in |p^* \, T_M| \otimes |N^{W*}| \, . \tag{42}$$

This makes sense since $\varphi_*^{-1} : p^* \, T_M \to N^W$.

Let us now consider the second projection,

$$q(x,t) = t \in \mathbb{R} \tag{43}$$

and compute the pushforward $q_*(\tau)$ of the distribution τ.

By construction $\delta(y - F_t(x)) \, dy$ is a generalized section of $\mathrm{pr}_Y^* \, |T|$, so that τ is a generalized section of $p^* \, |T| = \varphi^* \, \mathrm{pr}_Y^* \, |T|$.

Thus $q_*(\tau)$ is a generalized function.

We first look at the contribution of a periodic orbit, the corresponding part of $\varphi^{-1}(Z)$ is of the form,

$$\varphi^{-1}(Z) = V \times \Gamma \subset M \times \mathbb{R} \tag{44}$$

where Γ is a discrete cocompact subgroup of $\mathbb{R}$, while $V \subset M$ is a one dimensional compact submanifold of M.

To compute $q_*(\tau)$, we let $h(t) \, |dt|$ be a 1-density on $\mathbb{R}$ and pull it back by q as the section on $M \times \mathbb{R}$ of the bundle $q^* \, |T|$,

$$\xi(x,t) = h(t) \, |dt| \, . \tag{45}$$

We now need to compute $\int_{\varphi-1(Z)} \xi\,\sigma$. We can look at the contribution of each component: $V \times \{T\}$, $T \in \Gamma$.

One gets

$$T^{\#} \frac{1}{|1-P_T|}\, h(T)\,. \tag{46}$$

Where $T^{\#}$ is the length of the primitive orbit or equivalently the covolume of Γ in $\mathbb{R}$ for the Haar measure $|dt|$. We can thus write the contributions of the periodic orbits as

$$\sum_{\gamma_p}\sum_{\Gamma} \mathrm{Covol}(\Gamma)\, \frac{1}{|1-P_T|}\, h(T)\,. \tag{47}$$

Where the test function h vanishes at 0.

The next case to consider is when the vector field v_x has an isolated 0, $v_{x_0} = 0$. In that case, the transversality condition (31) becomes

$$1-(F_t)_* \quad \text{invertible (at } x_0)\,. \tag{48}$$

One has $F_t(x_0) = x_0$ for all $t \in \mathbb{R}$ and now the relevant component of $\varphi^{-1}(Z)$ is $\{x_0\} \times \mathbb{R}$. The transverse space N^W is identified with T_x and the map $\varphi_* : N^W \simeq N^Z$ is given by:

$$\varphi_* = 1-(F_t)_*\,. \tag{49}$$

Thus the symbol σ is the scalar function $|1-(F_t)_*|^{-1}$. The generalized section $q_*\,\varphi^*(\delta(y-F_t(x))\,dy)$ is the function, $t \to |1-(F_t)_*|^{-1}$. We can thus write the contribution of the zeros of the flow as,

$$\sum_{zeros} \int \frac{h(t)}{|1-(F_t)_*|}\, dt \tag{50}$$

where h is a test function vanishing at 0.

We can thus collect the contributions 47 and 50 as

$$\sum_{\gamma} \int_{I_\gamma} \frac{h(u)}{|1-(F_u)_*|}\, d^*u \tag{51}$$

where h is as above ,I_γ is the isotropy group of the periodic orbit γ , the haar measure d^*u on I_γ is normalised so that the covolume of I_γ is equal to one and we still write $(F_u)_*$ for its restriction to the transverse space of γ.

To understand what $(F_t)_*$ looks like at a zero of v we can replace $v(x)$ for x near x_0 by its tangent map. For simplicity we take the one dimensional case, with $v(x) = x\,\frac{\partial}{\partial x}$, acting on $\mathbb{R} = M$.
One has $F_t(x) = e^t\,x$. Since F_t is linear the tangent map $(F_t)_*$ is

(52) $$(F_t)_* = e^t$$

and (50) becomes

(53) $$t \to \frac{1}{|1 - e^t|}\,.$$

Thus for this flow the distribution trace formula is

(54) $$\text{“Trace”}\ (U(h)) = \int \frac{h(u)}{|1-u|}\ d^*u$$

where we used the multiplicative notation so that $\mathbb{R}_+^*$ acts on $\mathbb{R}$ by multiplication,while $U(h) = \int U(v)h(v)\ d^*v$ and d^*v is the haar measure of the group $\mathbb{R}_+^*$.

One can treat in a similar way the action , by multiplication ,of the group of non zero complex numbers on the manifold $\mathbb{C}$.
We shall now investigate the more general case of an arbitrary local field.

The action $(\lambda, x) \to \lambda\,x$ of K^* on K

We let K be a local field and consider the map,

(1) $$f : K \times K^* \to K\ ,\ \ f(x, \lambda) = \lambda\,x$$

together with the diagonal map,

(2) $$\varphi : K \times K^* \to K \times K^* \times K\ ,\ \ \varphi(x, \lambda) = (x, \lambda, x)$$

as in (27) (29) above.
When K is Archimedean we are in the framework of manifolds and we can associate to f a δ-section with support $Z =$ Graph (f),

(3) $$\delta_Z = \delta(y - \lambda\,x)\,dy\,.$$

Using the projection $q(x, \lambda) = \lambda$ from $K \times K^*$ to K^* we then consider as above the generalized function on K^* given by,

$$q_*(\varphi^* \delta_Z)\,. \tag{4}$$

The formal computation of this generalized function of λ is

$$\int \delta(x - \lambda\, x)\, dx = \int \delta((1-\lambda)x)\, dx = \int \delta(y)\, d((1-\lambda)^{-1}\, y)$$

$$= |1-\lambda|^{-1} \int \delta(y)\, dy = |1-\lambda|^{-1}\,.$$

We want to justify it by computing the convolution of the Fourier transforms of $\delta(x-y)$ and $\delta(y - \lambda\, x)$ since this is the correct way of defining the product of two distributions in this local context . Let us first compute the Fourier transform of $\delta(ax+by)$ where $(a,b) \in K^2 (\neq 0)$. The pairing between K^2 and its dual K^2 is given by

$$\langle (x,y), (\xi, \eta) \rangle = \alpha(x\,\xi + y\,\eta) \in U(1)\,. \tag{5}$$

where α is a suitable character of the additive group K.

Let $(c,d) \in K^2$ be such that $ad - bc = 1$ and consider the linear invertible transformation of K^2,

$$L \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}\,. \tag{6}$$

The Fourier transform of $\varphi \circ L$ is given by

$$(\varphi \circ L)^\wedge = |\det L|^{-1}\, \widehat{\varphi} \circ (L^{-1})^t\,. \tag{7}$$

Here one has $\det L = 1$ and $(L^{-1})^t$ is

$$(L^{-1})^t = \begin{bmatrix} d & -c \\ -b & a \end{bmatrix}\,. \tag{8}$$

One first computes the Fourier transform of $\delta(x)$, the additive Haar measure dx is normalized so as to be selfdual, and in one variable $\delta(x)$ and 1 are Fourier of each other, thus

$$(\delta \otimes 1)^\wedge = 1 \otimes \delta\,. \tag{9}$$

Using (7) one gets that the Fourier transform of $\delta(ax+by)$ is $\delta(-b\,\xi + a\,\eta)$. Thus we have to compute the convolution of the two generalized functions, $\delta(\xi+\eta)$ and $\delta(\xi+\lambda\,\eta)$. Now

$$\int f(\xi,\eta)\,\delta(\xi+\eta)\,d\xi\,d\eta = \int f(\xi,-\xi)\,d\xi$$

and

$$\int f(\xi,\eta)\,\delta(\xi+\lambda\,\eta)\,d\xi\,d\eta = \int f(-\lambda\,\eta,\eta)\,d\eta$$

thus we are dealing with two measures carried respectively by two distinct lines. Their convolution evaluated on $f \in C_c^\infty(K^2)$ is $\int f(\alpha+\beta)\,d\mu(\alpha)\,d\nu(\beta) = \int\int f((\xi,-\xi)+(-\lambda\,\eta,\eta))\,d\xi\,d\eta = \int\int f(\xi-\lambda\,\eta,-\xi+\eta)\,d\xi\,d\eta = \left(\int\int f(\xi',\eta')\right.$ $\left.d\xi'\,d\eta'\right)\times|J|^{-1}$ where J is the determinant of the matrix $\begin{bmatrix} 1 & -\lambda \\ -1 & 1 \end{bmatrix} = L$, so that $\begin{bmatrix} \xi' \\ \eta' \end{bmatrix} = J\begin{bmatrix} \xi \\ \eta \end{bmatrix}$. One has $J = 1-\lambda$ and thus the convolution of the generalized functions $\delta(\xi+\eta)$ and $\delta(\xi+\lambda\,\eta)$ gives as expected the constant function

$$|1-\lambda|^{-1}\,1\,. \tag{10}$$

Correspondingly, the product of the distribution $\delta(x-y)$ and $\delta(y-\lambda\,x)$ gives $|1-\lambda|^{-1}\,\delta_0$ so that,

$$\int \delta(x-y)\,\delta(y-\lambda\,x)\,dx\,dy = |1-\lambda|^{-1}\,. \tag{11}$$

In this local case the Fourier transform alone was sufficient to make good sense of the relevant product of distributions. In fact this would continue to make sense if we replace $\delta(y-\lambda\,x)$ by $\int h(\lambda^{-1})\,\delta(y-\lambda\,x)\,d^*\lambda$ where $h(1)=0$. In the global case it is necessary to use microlocalisation instead of the Fourier transform.

The global case

We shall now consider the action of C_k on X and write down the analogue of (51) for the distribution trace formula.

Both X and C_k are defined as quotients and we let

(1) $$\pi : A \to X\,,\ c : \mathrm{GL}_1(A) \to C_k$$

be the corresponding quotient maps.

As above we consider the graph Z of the action

(2) $$f : X \times C_k \to X\,,\ f(x,\lambda) = \lambda x$$

and the diagonal map

(3) $$\varphi : X \times C_k \to X \times C_k \times X \qquad \varphi(x,\lambda) = (x,\lambda,x)\,.$$

We first investigate $\varphi^{-1}(Z)$, i.e. the pairs $(x,\lambda) \in X \times C_k$ such that $\lambda x = x$. Let $x = \pi(\tilde{x})$ and $\lambda = c(j)$. Then the equality $\lambda x = x$ means that $\pi(j\tilde{x}) = \pi(\tilde{x})$ thus there exists $q \in k^*$ such that with $\tilde{j} = qj$, one has

(4) $$\tilde{j}\tilde{x} = \tilde{x}\,.$$

Recall now that A is the restricted direct product $A = \prod_{\text{res}} k_v$ of the local fields k_v obtained by completion of k with respect to the place v. The equality (4) means that $\tilde{j}_v \tilde{x}_v = \tilde{x}_v$, thus, if $\tilde{x}_v \neq 0$ for all v it follows that $\tilde{j}_v = 1$ $\forall v$ and $\tilde{j} = 1$. This shows that the projection of $\varphi^{-1}(Z) \cap C_k \backslash \{1\}$ on X is the union of the hyperplanes

(5) $$\cup H_v\,;\ H_v = \pi(\tilde{H}_v)\,,\ \tilde{H}_v = \{x\,;\ x_v = 0\}\,.$$

Each $\tilde{H}_v$ is closed in A and is invariant under multiplication by elements of k^*. Thus each H_v is a closed subset of X and one checks that it is the closure of the orbit under C_k of any of its generic points

(6) $$x\,,\ x_u = 0 \quad \Longleftrightarrow \quad u = v\,.$$

For any such point x, the isotropy group I_x is the image in C_k of the multiplicative group k_v^*,

(7) $$I_x = k_v^*$$

by the map $\lambda \in k_v^* \to (1,\ldots,1,\lambda,1,\ldots)$. This map already occurs in class field theory (cf [W1]) to relate the local Galois theory to the global one.

Both groups k_v^* and C_k are commensurable to $\mathbb{R}_+^*$ by the module homomorphism, which is proper with cocompact range,

$$G \xrightarrow{|\ |} \mathbb{R}_+^* \,. \tag{8}$$

Since the restriction to k_v^* of the module of C_k is the module of k_v^*, it follows that

$$I_x \text{ is a cocompact subgroup of } C_k \,. \tag{9}$$

This allows to normalize the respective Haar measures in such a way that the covolume of I_x is 1. This is in fact insured by the canonical normalisation of the Haar measures of modulated groups ([W3]),

$$\int_{|g|\in[1,\Lambda]} d^*g \sim \log \Lambda \quad \text{when} \quad \Lambda \to +\infty \,. \tag{10}$$

It is important to note that though I_x is cocompact in C_k, the orbit of x is not closed and one needs to close it, the result being H_v. This is one of the subtle features of the space X (cf [Co]). We can now in view of the results of the two preceding sections, write down the contribution of each H_v to the distributional trace;

Since $\tilde{H}_v$ is a hyperplane, we can identify the transverse space N_x to H_v at x with the quotient

$$N_x = A/\tilde{H}_v = k_v \tag{11}$$

namely the additive group of the local field k_v. Given $j \in I_x$ one has $j_u = 1\ \forall u \neq v$, and $j_v = \lambda \in k_v^*$. The action of j on A is linear and fixes x, thus the action on the transverse space N_x is given by

$$(\lambda, a) \to \lambda a \quad \forall a \in k_v. \tag{12}$$

We can thus proceed with some faith and write down the contribution of H_v to the distributional trace in the form,

$$\int_{k_v^*} \frac{h(\lambda)}{|1-\lambda|} \, d^*\lambda \tag{13}$$

where h is a test function on C_k which vanishes at 1. We now have to take care of a discrepancy in notation with the first section (formula 26), where we used the symbol $U(j)$ for the operation

$$\big(U(j)f\big)(x) = f(j^{-1}x) \tag{14}$$

whereas we use j in the above discussion. This amounts to replace the test function $h(u)$ by $h(u^{-1})$ and we thus obtain as a formal analogue of (51) the following expression for the distributional trace

$$\text{(15)} \qquad \text{“Trace”}\ (U(h)) = \sum_v \int_{k_v^*} \frac{h(u^{-1})}{|1-u|}\, d^*u\,.$$

The above discussion is *not a* rigorous justification of this formula. To justify it one should define the appropriate distribution theory and use *microlocalisation* on X. The Fourier transform is available on A, since A is its own Pontrjagin dual by means of the pairing

$$\text{(16)} \qquad \langle a, b\rangle = \alpha(ab)$$

where $\alpha : A \to U(1)$ is a nontrivial character which vanishes on $k \subset A$. Note that such a character is *not canonical*, but that any two such characters are related by k^*,

$$\text{(17)} \qquad \alpha'(a) = \alpha(qa) \quad \forall a \in A\,.$$

It follows that the corresponding Fourier transformations on A are related by

$$\text{(18)} \qquad \hat{f}' = \hat{f}_q\,.$$

This is yet another reason why it is natural to mod out by functions of the form $f - f_q$, i.e. to consider the quotient space X.

Now the right-hand side of (15) is, when restricted to the hyperplane $h(1) = 0$, the distribution obtained by André Weil [W3] as the synthesis of the explicit formulas of number theory for all L-functions with Grössencharacter. In particular we can rewrite it as

$$\text{(19)} \qquad \hat{h}(0) + \hat{h}(1) - \sum_{L(\chi,\rho)=0} \hat{h}(\chi,\rho) + \infty\, h(1)$$

where this time the restriction $\mathrm{Re}(\rho) = \frac{1}{2}$ has been eliminated.

Thus, equating (51) of section 1 and (19) for $h(1) = 0$ would yield the desired information on the zeros. Of course, besides the above microlocalisation problem, this would require (as in [AB]) to prove that the distributional trace coincides with the ordinary operator theoretic trace on the cokernel of

E. This is achieved for the usual set-up of the Lefchetz fixed point theorem by the use of families.

A very important property of the right hand side of (15) (and of (51) in general) is that if the test function h is positive,

(20) $$h(u) \geq 0 \quad \forall\, u \in C_k$$

then the right-hand side is *positive*. This indicated from the very start that in order to obtain the Polya-Hilbert space from the Riemann flow, it is *not* quantization that should be involved but simply the passage to the L^2 space, $X \to L^2(X)$. Indeed the positivity of (51) is typical of *permutation matrices* rather than of quantization. This distinction plays a crucial role in the above discussion of the trace formula, in particular the expected trace formula is not a semi-classical formula but a Lefchetz formula in the spirit of [AB].

Our construction of the Polya-Hilbert space bears some resemblance to [Z] and in fact one should clarify their relation. What matters in our case is that there is a natural Hilbert space that singles out the critical zeros, as well as a (formal) computation of the trace. The essential quality of a nuclear space is to be small enough so that the Schwartz kernel theorem applies. The space $\mathcal{S}(X)$ seems optimal for this purpose. The tuning of the measure $|x|d^*x$ is intimately related to the behaviour of the statistical system [BC] at critical inverse temperature $\beta = 1$.

The occurence of type III factors in [BC] indicates that the classification of hyperfinite type III factors [C] should be viewed as a refinement of local class field theory for Archimedean places. For global fields of zero characteristic, the Idele class group has a non trivial connected component of the identity and this connected group has so far received no interpretation from Galois theory. The above mentionned refinement should provide such an interpretation.

I was informed by P. Sarnak that the above work is in the line of unpublished ideas of Paul Cohen.

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