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Inequalities for the Dirichlet and Neumann Eigenvalues of the Laplacian for Domains on Spheres*

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Abstract

In this paper we consider inequalities for the Dirichlet and Neumann eigenvalues of the Laplacian for domains in the n-dimensional sphere S^n. As background, we survey the corresponding (and more extensive) results for domains in $\mathbb{R}^n$. In particular, we consider inequalities between the Dirichlet and Neumann eigenvalues of a domain in S^n and we give the S^n analog of Aviles' result (also a special case of Levine and Weinberger's results): for a domain Ω with boundary $\partial\Omega$ having nonnegative mean curvature at each of its points the $(k+1)^{th}$ Neumann eigenvalue is always less than or equal to the k^{th} Dirichlet eigenvalue. For $\Omega \subset S^n$, this inequality is sharp for $k = 1$ and Ω a hemisphere (the boundary of which has 0 mean curvature everywhere). We also give an S^2 analog of a more specialized low eigenvalue inequality of Payne, and show how our main result extends to a comparison of the eigenvalues of a Robin problem with the corresponding Dirichlet eigenvalues.

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1 Introduction

We consider inequalities for and comparisons between Dirichlet and Neumann eigenvalues of the Laplacian for domains in the n-dimensional sphere S^n. For background and comparison, we also present corresponding (and in many cases more general) results for bounded domains in Euclidean space $\mathbb{R}^n$. We begin with a precise formulation of our problems, setting forth the notation we shall employ throughout. In what follows, Ω will always denote a bounded domain in $\mathbb{R}^n$ or S^n with smooth boundary (smooth enough to ensure the existence of an outward normal $\vec{\nu}$, and, further, to ensure the existence of continuous principal curvatures at each point of $\partial\Omega$).

The **Dirichlet problem** is the eigenvalue problem

$$-\Delta u = \lambda u \quad \text{in } \Omega, \quad \text{a bounded domain in } \mathbb{R}^n \text{ or } S^n, \tag{1.1a}$$

$$u = 0 \quad \text{on } \partial\Omega. \tag{1.1b}$$

It is well-known to have spectrum $\{\lambda_m\}_{m=1}^{\infty}$ consisting entirely of eigenvalues of finite multiplicity which we list (with multiplicities) as

$$(0 <)\lambda_1 < \lambda_2 \le \lambda_3 \le \lambda_4 \le \ldots \to \infty. \tag{1.2}$$

A corresponding orthonormal basis of real eigenfunctions will be denoted $u_1, u_2, u_3, u_4, \ldots$.

The **Neumann problem** is the eigenvalue problem

$$-\Delta v = \mu v \quad \text{in } \Omega, \quad \text{a bounded domain in } \mathbb{R}^n \text{ or } S^n, \tag{1.3a}$$

$$\frac{\partial v}{\partial \nu} = 0 \quad \text{on } \partial\Omega. \tag{1.3b}$$

For Ω with smooth boundary it is well-known to have spectrum $\{\mu_m\}_{m=0}^{\infty}$ consisting entirely of eigenvalues of finite multiplicity which we list (with multiplicities) as

$$0 = \mu_0 < \mu_1 \le \mu_2 \le \mu_3 \le \ldots \to \infty. \tag{1.4}$$

A corresponding orthonormal basis of real eigenfunctions will be denoted $v_0 = \text{const.}$, $v_1, v_2, v_3, \ldots$.

Note particularly that by convention we have chosen to index our Dirichlet eigenvalues from 1 and our Neumann eigenvalues from 0. This means that our k^{th} Dirichlet eigenvalue will be λ_k but our k^{th} Neumann eigenvalue will be μ_{k-1}.

Our main object in this paper will be to compare λ_k and μ_k, or perhaps λ_k and μ_{k+c}, where c is some fixed index shift. We would like our comparisons to hold for all $k \ge 1$. To begin with, we recall that the min-max principle (see, for example, [6]) gives the easy comparison

$$\mu_{k-1} \le \lambda_k \quad \text{for all} \quad k \ge 1. \tag{1.5}$$

It was suggested by Payne (see [14],[19, p.155],[2]) that

$$\mu_k \le \lambda_k \quad \text{for all} \quad k \ge 1 \tag{1.6}$$

and perhaps more, especially for convex domains.

2 Background

In surveying the relevant results of previous authors, we shall begin with results for $\mathbb{R}^n$ (the Euclidean case) and then proceed on to results for S^n. We shall further divide our discussion into "first nonzero eigenvalue results" (i.e., results for λ_1 and μ_1) and then general results. The general results for the case of $\Omega \subset S^n$ are our main contribution here, and are developed in the body of the paper.

A. Euclidean Case: First Nonzero Eigenvalues

For $\Omega \subset \mathbb{R}^n$ one has the Faber-Krahn result (first conjectured by Lord Rayleigh in 1877 and subsequently proved independently by Faber [9] and Krahn [12],[13] in the 1920's)

$$\lambda_1(\Omega) \geq \lambda_1(\Omega^\star) = \frac{j_\nu^2}{R_\star^2} \tag{2.1}$$

where $\nu = n/2 - 1$, j_ν denotes the first positive zero of the Bessel function J_ν, and $\Omega^\star$ is an n-ball of the same n-volume as Ω with $R_\star$ as its radius. The Faber-Krahn inequality is sharp if and only if Ω is itself a ball. In particular, if $n = 2$ we have $j_0 \approx 2.40483$ (see Abramowitz and Stegun [1] for various further details concerning Bessel functions).

A companion result for μ_1 was proved by Szegö (for simply connected domains in $\mathbb{R}^2$) [23] and Weinberger (in general) [24] in the 1950's. It reads

$$\mu_1(\Omega) \leq \mu_1(\Omega^\star) = \frac{p_{n/2}^2}{R_\star^2} \tag{2.2}$$

where $p_{n/2}$ denotes the first positive zero of the function $[t^{1-n/2}J_{n/2}(t)]'$, and $\Omega^\star$ and $R_\star$ are as above. This inequality, too, is sharp if and only if Ω is a ball. In particular, if $n = 2$ we have $p_1 = j'_{1,1} \approx 1.84118$ where $j'_{1,1}$ denotes the first positive zero of the derivative of the Bessel function J_1. Additional discussion may be found in [2].

It follows from the results above that

$$\mu_1(\Omega) < \lambda_1(\Omega) \quad \text{for all bounded domains } \Omega \subset \mathbb{R}^n. \tag{2.3}$$

Indeed,

$$\mu_1(\Omega) \leq \mu_1(\Omega^\star) < \lambda_1(\Omega^\star) \leq \lambda_1(\Omega) \tag{2.4}$$

and, even better,

$$\frac{\mu_1(\Omega)}{\lambda_1(\Omega)} \leq \frac{\mu_1(\Omega^\star)}{\lambda_1(\Omega^\star)} = \frac{p_{n/2}^2}{j_\nu^2} < 1 \quad \text{for all} \quad n \tag{2.5}$$

(recall that $\nu \equiv n/2 - 1$). For example, for $n = 2$ one has $\mu_1/\lambda_1 \lesssim .5862$. However, one should not be misled by this result into thinking that μ_k/λ_k can be bounded by a constant strictly less than 1 for all k: by Weyl asymptotics one can at best hope to get $\mu_k/\lambda_k < 1$ since $\mu_k/\lambda_k \to 1$ as $k \to \infty$.

B. Euclidean Case: General Results

The study of general eigenvalue comparisons in this context was initiated by Payne [18] in 1955. He proved that if Ω is a bounded, smooth, convex domain in $\mathbb{R}^2$ then

$$\mu_{k+1} < \lambda_k \quad \text{for all} \quad k \geq 1 \tag{2.6}$$

and also

$$\mu_1 < \lambda_1 - \frac{2}{(\rho h)_{\max}} \tag{2.7}$$

and a similar sharper result for μ_2, where ρ and h are functions on $\partial\Omega$ defined by

$$\begin{aligned} \rho &= \text{radius of curvature of } \partial\Omega \\ h(P) &= \text{ distance from an arbitrary (fixed) origin inside } \Omega \\ &\quad \text{to the tangent line to } \partial\Omega \text{ through } P \in \partial\Omega \\ &= \vec{OP} \cdot \vec{\nu} \text{ with } O \text{ an origin inside } \Omega \text{ and } \vec{\nu} \text{ the} \\ &\quad \text{outward unit normal to } \partial\Omega \text{ at } P. \end{aligned}$$

For example, for Ω a disk in $\mathbb{R}^2$ of radius a and with O chosen as its center, we have $h \equiv a \equiv \rho$ and hence $\mu_1 < \lambda_1 - \frac{2}{a^2}$ leading to the inequality $p_1^2 < j_0^2 - 2$.

It was only some 30 years later that Payne's result (2.6) was generalized to $\mathbb{R}^n$. Generalizations were found by Aviles [5] and Levine and Weinberger [15] independently and nearly simultaneously. Aviles proved that for Ω a bounded domain in $\mathbb{R}^n (n \geq 2)$ with smooth boundary $\partial\Omega$ which is everywhere of nonnegative mean curvature

$$\mu_k < \lambda_k \quad \text{for all } k \geq 1. \tag{2.8}$$

By the mean curvature of $\partial\Omega$ we mean the quantity

$$H = \sum_{i=1}^{n-1} \kappa_i \tag{2.9}$$

where the κ_i's are the principal curvatures of $\partial\Omega$ (which exist and are continuous due to our assumption that $\partial\Omega$ is sufficiently smooth). Properly speaking, $H/(n-1)$ should be the mean curvature, but H is a bit more convenient for us here. Beyond this, Levine and Weinberger achieved a more extensive generalization, which contains the result of Aviles as one case. In particular, they showed that for Ω a bounded, smooth, convex domain in $\mathbb{R}^n (n \geq 2)$

$$\mu_{k+n-1} < \lambda_k \quad \text{for all } k \geq 1 \tag{2.10}$$

and a family of results intermediate between this result and that of Aviles. These results assert that for an integer R with $1 \leq R \leq n$

$$\mu_{k+R-1} < \lambda_k \quad \text{for all } k \geq 1 \tag{2.11}$$

if all $(n-R+1)$-fold sums of the n quantities

$$\kappa_1, \kappa_2, \ldots, \kappa_{n-1}, H \tag{2.12}$$

are nonnegative on $\partial\Omega$. Note that the $R = n$ case gives back (2.10) (and Payne's $n = 2$ result (2.6)), since a smooth domain is convex iff $\kappa_i \geq 0$ for all $1 \leq i \leq n-1$ at each of its boundary points. Similarly, Aviles' result (2.8) is recovered as the $R = 1$ case of (2.11). Levine and Weinberger also established various extensions to nonsmooth domains, including the result $\mu_{k+n-1} \leq \lambda_k$ for all $k \geq 1$ for an arbitrary bounded convex domain in $\mathbb{R}^n$.

To complete the story, we remark that relatively recently Friedlander [11] was able to establish Payne's conjecture (1.6) in the strict form

$$\mu_k < \lambda_k \quad \text{for all } k \geq 1 \tag{2.13}$$

for an arbitrary bounded smooth domain in $\mathbb{R}^n$. The proof, which we do not go into here, is based on estimating the number of negative eigenvalues of the Dirichlet-to-Neumann map for $\Delta + \lambda$, $\lambda > 0$. Some further observations on the method are found in Mazzeo [16]. While it would indeed be very interesting to extend Friedlander's method to domains $\Omega \subset S^n$ (and it does extend to bounded domains in hyperbolic space H^n, see Mazzeo's comments in [16]), there are impediments to doing so (see Mazzeo and the facts quoted below about geodesic balls in S^n which are larger than hemispheres) and we have nothing more to say about it here.

C. Spherical Case: First Nonzero Eigenvalues

For domains in S^n there are analogs of all the low eigenvalue results listed in A, except that in some cases Ω must be restricted to lie in a hemisphere. In particular, Sperner [22] (see also Friedland and Hayman [10]) proved the Faber-Krahn analog

$$\lambda_1(\Omega) \geq \lambda_1(\Omega^\star) \quad \text{for } \Omega \text{ arbitrary,} \tag{2.14}$$

where $\Omega^\star$ now represents the geodesic cap (ball) in S^n which has the same n-volume as Ω. Similarly, corresponding to the Szegö-Weinberger Euclidean result (2.2) we have

$$\mu_1(\Omega) \leq \mu_1(\Omega^\star) \quad \text{for } \Omega \text{ contained in a hemisphere,} \tag{2.15}$$

proved by Ashbaugh and Benguria [4] (see also [3]), extending an earlier result of Chavel [7],[8] which had more restrictive hypotheses.

The two preceding results can be combined to yield

$$\frac{\mu_1(\Omega)}{\lambda_1(\Omega)} \leq \frac{\mu_1(\Omega^\star)}{\lambda_1(\Omega^\star)} \quad \text{for } \Omega \text{ contained in a hemisphere,} \tag{2.16}$$

and this, together with the fact that $\mu_1 < \lambda_1$ for a geodesic ball smaller than a hemisphere, shows that

$$\mu_1(\Omega) < \lambda_1(\Omega) \quad \text{for } \Omega \text{ contained in a hemisphere but not a hemisphere.} \tag{2.17}$$

However, even if (2.15) and especially (2.16) could be shown to hold for arbitrary Ω, $\mu_1 < \lambda_1$ (or even $\mu_1 \leq \lambda_1$) cannot possibly hold for all Ω since for geodesic balls $\mu_1 = \lambda_1 = n$ at the hemisphere in S^n, and $\mu_1 > \lambda_1$ beyond the hemisphere (i.e., for geodesic balls which are larger than hemispheres). One might also note that for a geodesic ball approaching the full sphere S^n, $\mu_1 \to n$ whereas $\lambda_1 \to 0$.

3 The Mean Curvature Result

In this section we develop our main result that $\mu_k(\Omega) \leq \lambda_k(\Omega)$ for a domain $\Omega \subset S^n$ whose boundary is everywhere of nonnegative mean curvature (with strict inequality if the mean curvature is ever positive). Since the argument parallels that for $\Omega \subset \mathbb{R}^n$, relying mainly on the Rayleigh-Ritz inequality and a suitable choice of trial functions, we give it in a form that applies for both S^n and $\mathbb{R}^n$ simultaneously.

By Rayleigh-Ritz (we use real-valued functions in all cases)

$$\mu_k(\Omega) \leq \frac{\int_\Omega |\nabla\varphi|^2}{\int_\Omega \varphi^2} \tag{3.1}$$

where the trial function φ is not identically 0 and is orthogonal to the first k Neumann eigenfunctions, $v_0, v_1, \ldots, v_{k-1}$.

As our trial functions we consider

$$\varphi = cDu_k + \sum_{m=1}^{k} a_m u_m \tag{3.2}$$

where the u_m's are the Dirichlet eigenfunctions and D represents a first-order differential operator. For now, the key property of D is that it should commute with the Laplacian (=Laplace-Beltrami operator, in general). In particular, for $\Omega \subset \mathbb{R}^n$ we shall take $D = \frac{\partial}{\partial x_i} \equiv D_i$ for $i = 1, 2, \ldots, n$, where the x_i's are Cartesian coordinates for $\mathbb{R}^n$, and for $\Omega \subset S^n$ we shall take $D = x_i \frac{\partial}{\partial x_j} - x_j \frac{\partial}{\partial x_i} \equiv L_{ij}$ for $1 \leq i, j \leq n+1$, where we view S^n as contained in $\mathbb{R}^{n+1}$ and the x_i's are Cartesian coordinates for $\mathbb{R}^{n+1}$. The angular momentum operators L_{ij} are well-known to commute with the Laplacian. Also

$$\Delta_{S^n} = \frac{1}{2}\sum_{i,j=1}^{n+1} L_{ij}^2 = \sum_{i<j} L_{ij}^2 \tag{3.3}$$

(note that $L_{ii} = 0$ and $L_{ji} = -L_{ij}$); a related identity will be used later (see (3.14)). The coefficients c and a_m in φ must be chosen to make φ orthogonal to $v_0, v_1, \ldots, v_{k-1}$. Since these conditions give a system of k homogeneous linear equations in $k+1$ unknowns, we are assured of finding a nontrivial solution $(c, a_1, \ldots, a_k)$. If $c = 0$, φ is just a nontrivial linear combination of $u_1, \ldots, u_k$ and $\mu_k \leq \lambda_k$ follows easily by Rayleigh-Ritz (3.1). To see that equality cannot hold, we note that $\mu_k = \lambda_k$ would imply that φ is both a Dirichlet and a Neumann eigenfunction of $-\Delta$ on Ω. But then φ and $\partial\varphi/\partial\nu$ (and therefore also $\nabla\varphi$) would have to vanish on $\partial\Omega$, and this is impossible due to Hopf's boundary maximum principle (see [6, p.161], for example). Hence, $\mu_k(\Omega) < \lambda_k(\Omega)$ and we need consider this case no further.

For $c \neq 0$, we can divide through by it obtaining a new admissible trial function of the form (3.2) but with $c = 1$. Henceforth we proceed under this assumption, continuing to call our trial function φ. We shall not worry now about the possibility that $\varphi \equiv 0$; instead we proceed from the inequality

$$\mu_k(\Omega)\int_\Omega \varphi^2 \leq \int_\Omega |\nabla\varphi|^2 \tag{3.4}$$

which is true for our φ in any event. (Eventually we shall let D range over the set of operators enumerated above. We deal with the triviality issue at that stage.)

With

$$\varphi = \varphi_k = Du_k + \sum_{m=1}^{k} a_m u_m \tag{3.5}$$

we compute

$$\int_\Omega \varphi^2 = \int_\Omega (Du_k)^2 + 2\sum_{m=1}^{k} a_m \int_\Omega u_m(Du_k) + \sum_{m=1}^{k} a_m^2 \tag{3.6}$$

since $\int_\Omega u_l u_m = \delta_{lm}$, and, using the Divergence Theorem and the facts that $-\Delta u_m = \lambda_m u_m$ and that $u_m = 0$ on $\partial\Omega$,

$$\begin{aligned}
\int_\Omega |\nabla\varphi|^2 &= \int_\Omega |\nabla(Du_k)|^2 + 2\sum_{m=1}^{k} a_m \int_\Omega \nabla(Du_k)\cdot\nabla u_m + \sum_{l,m=1}^{k} a_l a_m \int_\omega \nabla u_l \cdot \nabla u_m \\
&= \int_\Omega |\nabla(Du_k)|^2 + 2\sum_{m=1}^{k} a_m \int_\Omega [-\Delta(Du_k)]u_m + \sum_{l,m=1}^{k} a_l a_m \int_\Omega u_l(-\Delta u_m) \\
&= \int_\Omega |\nabla(Du_k)|^2 + 2\lambda_k \sum_{m=1}^{k} a_m \int_\Omega u_m(Du_k) + \sum_{m=1}^{k} a_m^2\lambda_m
\end{aligned} \tag{3.7}$$

since by assumption D and Δ commute. Combining these identities with (3.4) we find

$$(\mu_k - \lambda_k)\int_\Omega \varphi^2 \le \int_\Omega |\nabla(Du_k)|^2 - \lambda_k \int_\Omega (Du_k)^2 - \sum_{m=1}^{k}(\lambda_k - \lambda_m)a_m^2. \tag{3.8}$$

Hence, since $\lambda_m \le \lambda_k$ for $m = 1, 2, \ldots, k$,

$$(\mu_k - \lambda_k)\int_\Omega \varphi^2 \le \int_\Omega |\nabla(Du_k)|^2 - \lambda_k \int_\Omega (Du_k)^2 \tag{3.9}$$

and by the Divergence Theorem

$$\begin{aligned}
(\mu_k - \lambda_k)\int_\Omega \varphi^2 &\le \int_\Omega \nabla\cdot[(Du_k)\nabla(Du_k)] + \int_\Omega (Du_k)[-\Delta(Du_k)] - \lambda_k \int_\Omega (Du_k)^2 \\
&= \int_{\partial\Omega} (Du_k)\frac{\partial}{\partial\nu}(Du_k)
\end{aligned} \tag{3.10}$$

where we have again used the fact that D and Δ commute and where $\partial/\partial\nu$ represents the outward normal derivative on $\partial\Omega$. Thus we arrive at

$$(\mu_k - \lambda_k)\int_\Omega \varphi^2 \le \frac{1}{2}\int_{\partial\Omega} \frac{\partial}{\partial\nu}(Du_k)^2. \tag{3.11}$$

At this point recall that we already know that $\mu_k(\Omega) < \lambda_k(\Omega)$ unless the constants c associated with each possible choice of D listed above (i.e., D_i for $i = 1, \dots, n$ for $\Omega \subset \mathbb{R}^n$, L_{ij} for $1 \le i < j \le n+1$ for $\Omega \subset S^n$) are all nonzero. In that case by our work above we may assume that (3.11) holds for all these choices of D, that is,

$$(\mu_k - \lambda_k) \int_\Omega \varphi_D^2 \le \frac{1}{2} \int_{\partial\Omega} \frac{\partial}{\partial\nu}(Du_k)^2 \tag{3.12}$$

(we deliberately leave the indexing of the D's vague here, so as to cover both the $\mathbb{R}^n$ and S^n cases in a unified fashion) and by summing these we obtain

$$\begin{aligned}(\mu_k - \lambda_k) \sum_D \int_\Omega \varphi_D^2 &\le \frac{1}{2} \int_{\partial\Omega} \frac{\partial}{\partial\nu} \left[\sum_D (Du_k)^2\right] \\ &= \frac{1}{2} \int_{\partial\Omega} \frac{\partial}{\partial\nu} |\nabla u_k|^2\end{aligned} \tag{3.13}$$

since it is known that (here f and g are arbitrary differentiable functions)

$$\sum_D (Df)(Dg) = \begin{cases} \sum_{i=1}^n (D_i f)(D_i g) = \nabla f \cdot \nabla g & \text{in } \mathbb{R}^n \\ \sum_{i<j} (L_{ij} f)(L_{ij} g) = \nabla f \cdot \nabla g & \text{in } S^n \end{cases} \tag{3.14}$$

and in particular $\sum_D (Df)^2 = |\nabla f|^2$ in $\mathbb{R}^n$ or S^n. Now since u_k vanishes on $\partial\Omega$ we have $|\nabla u_k|^2 = \left(\frac{\partial u_k}{\partial\nu}\right)^2$ there, and by introducing coordinates about $\partial\Omega$ based on those for $\partial\Omega$ and on the normal direction $\vec{\nu}$ we can write

$$|\nabla u_k|^2 = \left(\frac{\partial u_k}{\partial\nu}\right)^2 + |\nabla_{\|} u_k|^2 \tag{3.15}$$

in a neighborhood of $\partial\Omega$, from which we conclude

$$\frac{\partial}{\partial\nu} |\nabla u_k|^2 = 2 \frac{\partial u_k}{\partial\nu} \frac{\partial^2 u_k}{\partial\nu^2} \quad \text{on} \quad \partial\Omega \tag{3.16}$$

since $\nabla_{\|} u_k = \nabla_{\partial\Omega} u_k = 0$ on $\partial\Omega$. (Here $\nabla_{\|}$ denotes the gradient in $\partial\Omega$ and in the "parallel" hypersurfaces defined by fixing the "normal coordinate" ν at a constant value.) We therefore have

$$(\mu_k - \lambda_k) \sum_D \int_\Omega \varphi_D^2 \le \int_{\partial\Omega} \frac{\partial u_k}{\partial\nu} \frac{\partial^2 u_k}{\partial\nu^2}. \tag{3.17}$$

Finally, we employ one last decomposition fact concerning gradients and Laplacians, namely, that in a manifold M with coordinates based on those for a smooth hypersurface S and its normal direction field $\vec{\nu}$

$$\Delta_M = \frac{\partial^2}{\partial\nu^2} + H \frac{\partial}{\partial\nu} + \Delta_S \quad \text{on} \quad S \tag{3.18}$$

when acting on a function defined and C^2 on a neighborhood of S and then restricted to S (i.e., one takes the derivatives with respect to ν and then restricts to S). Here M denotes the ambient manifold, ν the normal coordinate to S, Δ_S the Laplacian in S induced by the metric for M, and H the "mean curvature function" for S (which we have **not** normalized by a dimensional factor). For our purposes here, M is $\mathbb{R}^n$ or S^n, as appropriate, and S is $\partial\Omega$.

Since

$$-\Delta u_k = \lambda_k u_k \tag{3.19}$$

and we are interested in $\frac{\partial^2 u_k}{\partial \nu^2}$ on $\partial\Omega$ (where $u_k \equiv 0$), we find, using (3.18),

$$\frac{\partial^2 u_k}{\partial \nu^2} = -H\frac{\partial u_k}{\partial \nu} \quad \text{on } \partial\Omega, \tag{3.20}$$

and hence, from (3.17),

$$(\mu_k - \lambda_k)\sum_D \int_\Omega \varphi_D^2 \le -\int_{\partial\Omega} H\left(\frac{\partial u_k}{\partial \nu}\right)^2 \le 0 \tag{3.21}$$

if $H \ge 0$ on $\partial\Omega$.

This will conclude the proof if we can be sure that at least one of the φ_D's is nontrivial (since then $\sum_D \int_\Omega \varphi_D^2 > 0$ follows). In the Euclidean case one can argue as in Levine and Weinberger [15] using the Gauss-Bonnet formula that H cannot vanish identically on $\partial\Omega$; hence by continuity of H there is a neighborhood in $\partial\Omega$ on which $H > 0$ and $\frac{\partial u_k}{\partial \nu}$ does not vanish identically (else u_k would have to vanish identically in Ω). It then follows from (3.21) that $\mu_k < \lambda_k$ if $H \ge 0$ on $\partial\Omega$, and also $\sum_D \int_\Omega \varphi_D^2 > 0$. (To see this directly, note that since φ_D is Du_k on $\partial\Omega$ and hence $\sum_D \varphi_D^2 = \sum_D (Du_k)^2 = |\nabla u_k|^2 = \left(\frac{\partial u_k}{\partial \nu}\right)^2$ on $\partial\Omega$, it is clear by continuity and the argument above that $\sum_D \int_\Omega \varphi_D^2 > 0$.) We thus recover the strict inequality $\mu_k(\Omega) < \lambda_k(\Omega)$ established by Aviles [5] and Levine and Weinberger [15].

In the case of $\Omega \subset S^n$ there are additional complications for (at least) two reasons: (a) There are bounded domains Ω, for example the hemisphere, for which $H \equiv 0$ on $\partial\Omega$ (this has to do with existence of closed geodesics, as can be seen clearly in the case of S^2). (b) There are domains, for example geodesic caps (balls) and annular regions, for which $Du_m = L_{ij}u_m \equiv 0$ on Ω for certain choices of i, j, and m. The second of these is obviated by summing over our operators $D = L_{ij}$ as usual. Noting that $\sum_D \varphi_D^2 = \left(\frac{\partial u_k}{\partial \nu}\right)^2$ on $\partial\Omega$ and that $\int_{\partial\Omega}\left(\frac{\partial u_k}{\partial \nu}\right)^2 > 0$ will imply $\sum_D \int_\Omega \varphi_D^2 > 0$ by continuity, we see that it suffices to show that $\frac{\partial u_k}{\partial \nu}$ cannot be identically 0 on $\partial\Omega$. But this is clear from Hopf's boundary maximum principle (see, for example, [6, p.161]). Beyond this, if we assume that $H \ge 0$ and $H > 0$ for at least one point of $\partial\Omega$, then it follows similarly by continuity that

$$(\mu_k - \lambda_k)\sum_D \int_\Omega \varphi_D^2 \le -\int_{\partial\Omega} H\left(\frac{\partial u_k}{\partial \nu}\right)^2 < 0 \tag{3.22}$$

and hence that $\sum_D \int_\Omega \varphi_D^2 < 0$ and $\mu_k < \lambda_k$ for $k = 1, 2, 3, \ldots$. This completes the proof of our main theorem.

Theorem 3.1. *Let $\Omega \subset S^n$ be a domain with smooth boundary $\partial\Omega$ with nonnegative mean curvature everywhere (i.e., $H \geq 0$ on $\partial\Omega$). Then the Neumann ($\mu_k, k = 0, 1, 2, \ldots$) and Dirichlet ($\lambda_k, k = 1, 2, 3, \ldots$) eigenvalues of $-\Delta$ on Ω satisfy*

$$\mu_k(\Omega) \leq \lambda_k(\Omega) \quad \text{for } k = 1, 2, 3, \ldots \tag{3.23}$$

The inequality is strict if H is ever positive on $\partial\Omega$. Moreover, the inequality is sharp since it is saturated for Ω a hemisphere and $k = 1$ (in which case $\mu_1 = \lambda_1 = n$ and $H \equiv 0$ on $\partial\Omega$).

Remarks. Inequality (3.23) also reduces to an equality for infinitely many other values of the index k when Ω is a hemisphere. For example, when $n = 2$ equality occurs at $k = 1, 3, 6, 10, \ldots$ (the triangular numbers). In general, equality occurs at $k = \binom{n+m-1}{n}$ for $m = 1, 2, 3, \ldots$. These equalities imply identities between Dirichlet and Neumann eigenfunctions of the hemisphere (and hence spherical harmonics as well) that one could extract from the details of our proof above. We leave this to the ambitious reader. We note that a useful summary of information on the Dirichlet and Neumann eigenvalues of the Laplacian on a hemisphere of S^n (or S^n itself) can be found in Section 4.3 of [21].

4 A First Nonzero Eigenvalue Result for S^2

Just as Payne was able to do for μ_1 and λ_1 when $n = 2$ in the Euclidean case, we can do for μ_1 and λ_1 for $\Omega \subset S^2$. That is, we can prove an S^2 analog of the more detailed bound (2.7) of Payne. To do this one adds three things to the ingredients we've used already. One is that $\varphi = Du_1$ is already orthogonal to $v_0 =$ const. so that no coefficients a_m are needed and the $c = 0$ case never arises. The second is that $\partial\Omega$ is one-dimensional so that the mean curvature function H can be identified explicitly as $\kappa = \cot\rho(P)$, where $\rho(P)$ is the (geodesic) radius of the osculating polar cap at $P \in \partial\Omega$, where the polar cap is the one lying on the same side of $\partial\Omega$ as Ω. (To be more specific, we can use the tangent great circle at P to determine the hemisphere that goes with the "side of $\partial\Omega$ on which Ω lies" locally at P. Then of the two possible centers for the osculating circle to $\partial\Omega$ at P, we choose the one that lies in this hemisphere and measure $\rho(P)$ as the geodesic distance between P and this center. For a geodesic cap of radius ρ, for example, this gives back $\rho(P) = \rho$ at each $P \in \partial\Omega$.) Finally, given the more explicit forms that the first two items allow, we can use an S^2 analog of Rellich's identity to actually estimate the right-hand side in

$$\mu_1 - \lambda_1 \leq \frac{-\int_{\partial\Omega} H \left(\frac{\partial u_1}{\partial\nu}\right)^2}{\sum_D \int_\Omega \varphi_D^2} = \frac{-\int_{\partial\Omega} \cot\rho(P) \left(\frac{\partial u_1}{\partial\nu}\right)^2}{\int_\Omega |\nabla u_1|^2}, \tag{4.1}$$

since the denominator here is just λ_1 and an S^2 analog of Rellich's identity is (see Molzon [17])

$$
\begin{aligned}
\lambda_1 \int_\Omega u_1^2 \cos\theta &= \frac{1}{2}\int_{\partial\Omega} \left(\frac{\partial u_1}{\partial \nu}\right)^2 \frac{\partial}{\partial \nu}(1-\cos\theta) \\
&= \frac{1}{2}\int_{\partial\Omega} h \left(\frac{\partial u_1}{\partial \nu}\right)^2
\end{aligned} \tag{4.2}
$$

where θ is the angle from the north pole of S^2 and

$$
\begin{aligned}
h &= \frac{\partial}{\partial \nu}(1-\cos\theta) \\
&= (\sin\theta)\hat{\theta}\cdot\vec{\nu}
\end{aligned} \tag{4.3}
$$

with $\hat{\theta} = (\cos\theta\cos\phi, \cos\theta\sin\phi, -\sin\theta)$ with respect to the standard Euclidean basis for $\mathbb{R}^3$ (θ and ϕ are the usual angular variables from spherical coordinates for $\mathbb{R}^3$). For Ω a convex domain in S^2 we can be sure that $h > 0$ along $\partial\Omega$ if we place our north pole inside Ω. This we shall always assume henceforth. We also assume that $\kappa = \cot\rho(P) \geq 0$ on $\partial\Omega$, i.e., that $0 \leq \rho(P) \leq \pi/2$ for all $P \in \partial\Omega$. Note that our function h is a spherical analog of Payne's h in the case of $\Omega \in \mathbb{R}^2$.

We now have

$$
\mu_1 \leq \lambda_1 - \frac{2\int_{\partial\Omega} \cot\rho(P)\left(\frac{\partial u_1}{\partial \nu}\right)^2 \int_\Omega u_1^2\cos\theta}{\int_{\partial\Omega} h\left(\frac{\partial u_1}{\partial \nu}\right)^2} \tag{4.4}
$$

and therefore, if $\int_\Omega u_1^2\cos\theta \geq 0$, we obtain

$$
\mu_1 \leq \lambda_1 - \frac{2\int_\Omega u_1^2\cos\theta}{(h\tan\rho(P))_{\max}} \leq \lambda_1 - \frac{2(\cos\theta)_{\min}}{(h\tan\rho(P))_{\max}}, \tag{4.5}
$$

which is the S^2 analog of Payne's result that we sought. The last inequality in (4.5) holds even if $(\cos\theta)_{\min} \leq 0$ (since we already know that $\mu_1 \leq \lambda_1$), but of course it is only of real interest if $(\cos\theta)_{\min} > 0$, i.e., if Ω lies in the northern hemisphere.

Remarks. (1) For the case of a polar cap in S^2 with center at the north pole, (4.5) reduces to

$$
\mu_1 \leq \lambda_1 - \frac{2\cos^2\rho}{\sin^2\rho} \quad \text{for } 0 < \rho \leq \pi/2 \tag{4.6}
$$

where ρ denotes the radius of the polar cap. Note that (4.6) is sharp at the hemisphere ($\theta_1 = \pi/2$).

(2) To finish the analogous argument in Payne's case one simply uses the usual Rellich identity [20] (or see, for example, [6, p.200], [17],[18])

$$
\lambda_1 = \frac{1}{2}\int_{\partial\Omega} h\left(\frac{\partial u_1}{\partial \nu}\right)^2 \tag{4.7}
$$

with h as defined following (2.7). Using the same approach, one can also obtain an n-dimensional Euclidean analog of Payne's special inequality for $\mu_1 - \lambda_1$. One simply combines (4.7), which is valid in $\mathbb{R}^n$ of any dimension, with (3.21). On the other hand, the Rellich-type identity of Molzon [17] contains an additional term containing the curvature for all dimensions other than 2. This leads to an additional term (giving a tighter inequality) in the n-dimensional analog of (4.5) for $\Omega \subset S^n$. By dropping the curvature term one finds that inequality (4.5) continues to hold for $\Omega \subset S^n$ for all dimensions $n \geq 2$.

5 Robin Boundary Conditions

As noted in the Euclidean case by Levine [14], the techniques used above can also be used to bound the eigenvalues of the Robin problem,

$$-\Delta v = \mu v \qquad \text{in } \Omega, \text{ a bounded domain in } \mathbb{R}^n \text{ or } S^n, \tag{5.1a}$$

$$\frac{\partial v}{\partial \nu} = -\alpha(x) v \qquad \text{on } \partial\Omega, \tag{5.1b}$$

by Dirichlet eigenvalues. It turns out that the function $\alpha(x)$ on $\partial\Omega$ enters our calculation on the same footing as the mean curvature function so that direct pointwise comparisons between $\alpha(x)$ and $H(x)$ on $\partial\Omega$ yield eigenvalue inequalities. In particular, letting $\mu_k[\alpha]$ for $k = 0, 1, 2, \ldots$ denote the Robin eigenvalues of Ω (counting multiplicities), we obtain the following theorem.

Theorem 5.1. *Let $\Omega \subset S^n$ be a domain with smooth boundary $\partial\Omega$ which has mean curvature function H (in the sense of (3.18)) obeying $H(x) \geq \alpha(x)$ for all $x \in \partial\Omega$. Then the Robin eigenvalues of Ω for α, $\mu_k[\alpha]$ for $k = 0, 1, 2, \ldots$, obey*

$$\mu_k[\alpha] \leq \lambda_k \quad \text{for } k = 1, 2, 3, \ldots \tag{5.2}$$

where the λ_k are the Dirichlet eigenvalues of Ω (with multiplicities).

Remarks. (1) The case $\alpha \equiv 0$ gives back Theorem 3.1 for the Neumann eigenvalues.

(2) This theorem allows us to consider domains Ω for which the mean curvature function is not always nonnegative, but at the expense of using "weaker" boundary conditions since as α decreases the Robin eigenvalues drop lower (cf. the Rayleigh quotient (5.3) below). In particular, in S^2 we can treat spherical caps beyond the hemisphere using this theorem.

(3) We could also formulate conditions under which (5.2) will be strict, as was done in Theorem 3.1. For example, α continuous on $\partial\Omega$ and such that $\alpha \leq H$ and $\alpha(x) < H(x)$ for at least one point $x \in \partial\Omega$ is sufficient to ensure strict inequality.

(4) The ideas from Section 4 can be applied directly to the Robin problem (in $\mathbb{R}^n$ or S^n) with similar results. Because the right-hand side of our new inequality involves only $H - \alpha$ and Dirichlet information, so that (3.21) changes only by μ_k being replaced by $\mu_k[\alpha]$ and H being replaced by $H - \alpha$ (see (5.5) below), one can get corresponding inequalities for $\mu_1[\alpha] - \lambda_1$ which will involve a factor of $\left(\frac{H-\alpha}{h}\right)_{\min}$ (in place of $(H/h)_{\min}$ for $\mu_1 - \lambda_1$) in the "correction term". Similarly, Payne's $\mu_2 - \lambda_1$ result can be promoted to a result for $\mu_2[\alpha] - \lambda_1$ for $\Omega \subset \mathbb{R}^2$.

Proof. Since the proof follows the general outline of the proof of Theorem 3.1, we only highlight the most pertinent formulas here. First, the Rayleigh quotient (compare to (3.1) above) for the Robin problem (5.1) is given by

$$R[\varphi] = \frac{\int_\Omega |\nabla\varphi|^2 + \int_{\partial\Omega} \alpha(x)\varphi^2}{\int_\Omega \varphi^2}. \tag{5.3}$$

Proceeding as above yields

$$\begin{aligned}(\mu_k[\alpha] - \lambda_k)\int_\Omega \varphi_D^2 &\le \frac{1}{2}\int_{\partial\Omega} \frac{\partial}{\partial\nu}(Du_k)^2 + \int_{\partial\Omega} \alpha(x)\varphi_D^2 \\ &= \frac{1}{2}\int_{\partial\Omega} \frac{\partial}{\partial\nu}(Du_k)^2 + \int_{\partial\Omega} \alpha(x)(Du_k)^2\end{aligned} \tag{5.4}$$

since $\varphi_D = Du_k$ on $\partial\Omega$. Summing over D then gives, much as before,

$$(\mu_k[\alpha] - \lambda_k)\sum_D \int_\Omega \varphi_D^2 \le -\int_{\partial\Omega} [H(x) - \alpha(x)]\left(\frac{\partial u_k}{\partial\nu}\right)^2 \tag{5.5}$$

in place of (3.21), which completes the proof. □

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