JOURNÉES ÉQUATIONS AUX DÉRIVÉES PARTIELLES

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Journées Équations aux dérivées partielles (1997), p. 1-8
<http://www.numdam.org/item?id=JEDP_1997____A10_0>

ON THE REGULARITY PROPERTIES OF NON-LINEAR WAVE EQUATIONS. JOINT WORK WITH S. KLAINERMAN.

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Before 1990, we knew, through the work of Klainerman [K1], Christodoulou and and others, that many wave equations arising in Mathematical Physics have special non-linear terms, satisfying the null condition. Also, such equations are better behaved than general equations as far as large time existence of solutions with small, smooth, data is concerned.

Since then, we have re-interpreted the null condition from the point of view of Fourier Analysis, to show that equations satisfying the null condition are also better behaved as far as the question of the optimal Sobolev space in which they are well posed locally in time is concerned. This work is progress towards proving global existence results for large data in the critical case, and better understanding the blow-up in the supercritical case.

Example 1. The Yang-Mills equations in 3+1 dimensions are sub-critical. There is a conserved energy, and our local existence result implies that the time of existence of a smooth solution depends only on the energy of the initial data (and the solution stays as smooth as it started in this interval). The argument is complicated by gauge dependance, and the fact that energy differs form the H^1 norm by a lower order term, see [K-M3]. The global existence result was already known, due to Eardley and Moncrief [E-M]. It was shown by M. Keel [Ke], along the same lines, that there is global regularity for Yang-Mills coupled with a critical power Higgs field. This is a new global existence result, accessible only through our new local estimates.

Example 2. Wave maps in 2+1 dimensions are critical. There is a conserved energy. The results of [K-M4], [K-S] imply that the time of existence depends on the "energy + ϵ" norm, thus one could hope that they come close to proving global regularity for wave maps. This result was obtained independently by Grillakis and announced at the 1994 ICM. It is expected that there is global regularity for these equations for target manifolds with negative curvature, and global regularity fails for target manifolds with positive curvature.

Example 3. Wave maps in 3+1 dimensions are supercritical. Some solutions blow up in finite time, due to Shatah [S]. The energy stays bounded even as time approaches blow-up time, but the $\|\phi(t,\cdot)\|_{H^{3/2+\epsilon}}$ must blow up for any solution ϕ that blows up. It would be very important to remove the ϵ and find the exact sharp space (presumably, not a Sobolev space) in which this equation is well posed locally in time.

Following is a summary of the main results and estimates. The theorems refer to null forms of the form $Q_0(\phi,\psi) = \phi_t\psi_t - \nabla_x\phi \cdot \nabla_x\psi$ and $Q_{ij}(\phi,\psi) = \partial_i\phi\partial_j\psi - \partial_j\phi\partial_i\psi$. We study systems of equations of the type

$$\Box\phi = Q(\phi,\phi) \tag{1}$$

in $n+1$ dimensions, where Q is a quadratic form in the derivatives of ϕ.

By "locally well posed in H^{s_0}" we mean that smooth Cauchy data with sufficiently small $\|\phi(0,\cdot)\|_{H^{s_0}}+\|\partial_t\phi(0,\cdot)\|_{H^{s_0-1}}$ (abbreviated, from now on as $\|\phi(0,\cdot)\|_{H^{s_0}}$) norm gives a solution which exists at least for $t \in [0,1]$ and satisfies

$$\|\phi(t,\cdot)\|_{H^s} \leq C_s\|\phi(0,\cdot)\|_{H^s}$$

and the same for differences of solutions. If we work with a norm higher than the scale-invariant one, we can rescale and obtain short time existence for large H^{s_0} data.

The classical local existence theorem requires the Cauchy data $\phi(0,) \in H^{\frac{n+2}{2}+\epsilon}(\mathbf{R}^n)$. The first results of [K-M1] lower the requirement to $\phi(0,) \in H^{\frac{n+1}{2}}(\mathbf{R}^n)$ provided that the non-linear term is one of the null forms Q_0, Q_{ij}. This is not optimal for equations satisfying the null condition, and has been improved since then, but is strictly better than than the optimal results for general equations, not satyisfying the null condition, see the counterexamples of Hans Lindblad [L]. Also, these estimates are the only ones so fat to lead to non-trivial global existence results for Yang-Mills, see[K-M2, 3].

The estimates in the proof are most striking in 2+1 dimensions. The classical Strichartz inequality gives the (optimal) estimate for a solution of $\Box\phi = 0$

$$\|(\nabla\phi)^2\|_{L^3(\mathbf{R}^3)} \leq C\|\phi(0,\cdot)\|^2_{H^{3/2}(\mathbf{R}^2)}$$

However, for a null form we have

$$\|Q(\phi,\phi)\|_{L^2(\mathbf{R}^3)} \leq C\|\phi(0,\cdot)\|_{H^{5/4}(\mathbf{R}^2)}$$

The proof is based on writing the L^2 norm of the quadratic form as the L^2 norm of a convolution of measures supported on the light cone, on the Fourier transform side. The symbol of the null form kills the worst singularity in the convolution. This has been generalized to the variable coefficient case by C. Sogge [So]. Some ideas in the proof were also used in [Sc-So].

The rest of the theorems discussed here prove well posedness in the scale-invariant Sobolev norm + ϵ. The proofs make extensive use of the spaces $H_{s,\delta}$ used by Bourgain for KdV [B], and introduced by M. Beals in [Be].

$$\|\phi\|_{s,\delta} = \|w_+^s w_-^\delta \tilde{\phi}\|_{L^2(d\tau d\xi)}$$

where $w_+(\tau,\xi) = 1+|\tau|+|\xi|$, $w_-(\tau,\xi) = 1+||\tau|-|\xi||$, and $\tilde{\phi}$ denotes the space-time Fourier transform.

There are two advantages in working with these spaces. Functions in $H_{s,\delta}$ with $\delta > 1/2$ satisfy the same Strichartz estimates that solutions of $\Box\phi = 0$ with H^s Cauchy data would.

In 3+1 dimensions, for instance,

$$\|\phi\|_{L^\infty(dt)L^2(dx)} \le C\|\phi\|_{0,\delta} \tag{2a}$$

is the energy estimate, and all estimates obtained by interpolating it with the (false) end-point result $\|\phi\|_{L^2(dt)L^\infty(dx)} \le C\|\phi\|_{1,\delta}$ are true.

Also, the argument si simplified if one also notices that, for $\delta < 1/2$ and p defined by $\frac{1}{p} = \frac{1}{2} - \frac{\delta}{2}$,

$$\|\phi\|_{L^p(dt)L^2(dx)} \le C\|\phi\|_{0,\delta} \tag{2b}$$

See [T] for a general treatment of these spaces.

The second advantage of the spaces $H^{s,\delta}$ is that the solution to $\Box\phi = F$ with Cauchy data f_0, f_1 satisfies

$$\|\chi(t)\phi\|_{s,\delta} \le C\Big(\|F\|_{s-1,\delta-1} + \|f_0\|_{H^s} + \|f_1\|_{H^{s-1}}\Big)$$

where χ is a smooth cut-off function in time. In order to solve $\Box\phi = Q(\phi,\phi)$ for small time it suffices to solve the integral equation

$$\phi = \chi(t)\Big(W * Q + W(f_0) + \partial_t W(f_1)\Big) \tag{3}$$

This idea also goes back to Bourgain. See also [K-P-V], and [K-M7] for a more precise result.

In order to show that the equation (1) is well posed in H^s, for $s > 3/2$, in 3+1 dimensions, it suffices to prove an inequality of the form[1]

[1] The original argument used the spaces $H_{s,\delta}$ with $\delta > 1/2$. See [K-T] for the use of $\delta = 1/2$, based on estimates (2b), which simplifies the argument.

$$\|Q_0(\phi,\psi)\|_{s-1,-1/2} \le C\|\phi\|_{s,1/2}\|\psi\|_{s,1/2} \tag{4}$$

The symbol of the null form Q_0 is $\tau\lambda - \xi\cdot\eta = \frac{1}{2}\Big((\tau+\lambda)^2 - |\xi+\eta|^2 - \tau^2 + |\xi|^2 - \lambda^2 + |\eta|^2\Big)$. Using just the square root of this, the left hand side of (4) is dominated by the sum of three terms, a typical one being

$$\|D_-^{-1/2}D_+^{s-1}(D_+D_-^{1/2}\phi)(D_+^{1/2}\psi)\|$$

where D_{+-} has symbol w_{+-}. After distributing derivatives, the worst term is

$$\|D_-^{-1/2}(D_+^sD_-^{1/2}\phi)(D_+^{1/2}\psi)\|$$

By duality, it suffices to show $H_{0,1/2}\cdot H_{s-1/2,1/2}\subset L^2$. This is true, and follows from (2). The original argument of [K-M4] used convolutions of measures.

For most other equations satisfying the null condition this type of argument does not work, although the final result is true. See [K-M 5] for the equation of wave maps written with respect to a frame. To get the result, one has to bootstrap twice.

Another interesting case, worked out in [K-M6] is the model equation (1) with $Q = Q_{ij}$:

$$\Box\phi = Q_{ij}(\phi,\phi) \tag{5}$$

The analogue of (4) is not true. There is an estimate for the symbol $|\xi\times\eta| \le |\xi||\eta|\xi+\eta|(w_-(\tau,\xi)+w_-(\lambda,\eta)+w_-(\tau+\lambda,\xi+\eta)$, but after distributing derivatives as above one has to bound a troublesome term

$$\|D_-^{-1/2}(D_-^{1/2}D_+^{1/2}\phi)(D_+^s\psi)\|_{L^2}$$

By duality, this would correspond to an estimate

$$D^{-(s-1/2)}\Big(H_{0,1/2}\cdot H_{0,1/2}\Big)\subset L^2$$

This is false, the counterexample is an adaptation of an old construction due to A. Knapp. There are other useful estimates along these lines which are true, see [K-M5], [K-T]. In 3+1 dimensions the (barely false) end-point estimates are are $D^{-1/2}\Big(H_{1/4,\delta}\cdot H_{1/4,\delta}\Big)\subset L^2$ and $D^{-1}\Big(H_{1/2,\delta}\cdot H_{1/2,\delta}\Big)\subset L^1(dt)L^\infty(dx)$

Back to (5), we are foced to make stronger assumptions on our norms. The original argument of [K-M6] insisted (modulo an ϵ) that, in addition to $\phi\in H_{s,1/2}$ ϕ should also satisfy

$$|\int D_-^{1/2}D_+^{1/2}\phi\cdot b_1b_2dxdt|\leq C\|b_1\|_B\|b_2\|_B$$

for all b, where $\|b\|_B=\|\int_0^\infty\tilde{b}(\tau,\xi)\|_{L^2(d\xi)}$

Another closely related way out is to make the stronger requirement $|w_+^{1/2}w_-^{1/2}\tilde{\phi}|\leq\tilde{F}$ for some $F\in L^1(dt)L^\infty(dx)$. Either way, the iteration converges in the intersection of $H^{s,1/2}$ with one of the above.

Finally, we mention that equation (5) is purely a model, but the related, more difficult, system of equations of the type

$$\Box\phi=Q_{ij}(D^{-1}\phi,\phi)+D^{-1}Q_{ij}(\phi,\phi) \tag{6}$$

is essentially Yang-Mills. Such equations are treated in the critical 4+1 dimensions in [K-M8] (a special case) and [K-T] (general case).

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