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A PERTURBATION THEORY OF RESONANCES

SHMUEL AGMON*

1. INTRODUCTION

The notion of a resonance of an operator was introduced in quantum mechanics for Schrödinger operators. The notion had several definitions. It is now accepted to identify resonances of an operator with poles of the associated resolvent operator function taken in some generalized sense. The resonance poles are hidden spectral objects. They are uncovered by analytic continuation of the generalized resolvent through the continuous spectrum.

Problems on resonances arise in mathematical physics and in other fields such as geometry and number theory. There are many recent studies dealing with such problems (see [5] for many references). These studies indicate that resonances should be treated in some formal way like eigenvalues. The question arises whether one can push this analogy further and show that resonances of an operator are in fact eigenvalues of some closely related operators. In this paper we show that in some general abstract setup this is indeed the case - resonances can be equated with eigenvalues. We note that in the special case of a Schrödinger operator with a dilation analytic potential there is a well known procedure which identifies resonances with eigenvalues. However, our approach is different and it applies in various concrete situations where the dilation analyticity "trick" is not available.

Our study was motivated by a quest for a good perturbation theory for resonances. The resonance-eigenvalue connection established in this paper yields such a theory. The theory is as good, and it is essentially the same, as the classical perturbation theory for eigenvalues. (For other theories of perturbation of resonances see Howland [3], Albeverio and Høegh-Krohn [1], Gesztesy [2] and references given there).

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The plan of this paper is as follows. The main definitions and hypotheses are given in Section 2. The resonance-eigenvalue connection is established in Section 3. Some more details on this connection are given in Section 4. Finally, a perturbation theory for resonances of holomorphic families of operators is outlined in Section 5.

2. The setup

We consider a closed linear operator P in a given Banach space B. We assume that $\sigma(P)$ (the spectrum) $\neq \mathbb{C}$. We denote by D a domain in $\mathbb{C}$ such that $D \cap \sigma(P) \subset \sigma_{\text{dis}}(P)$. Thus the resolvent $R(\lambda) := (P-\lambda)^{-1}$ is a well defined meromorphic operator function in D with values in $\mathcal{L}(B)$. Its poles in D (the isolated eigenvalues of P) are of a finite rank. Next we introduce a notion of a generalized resolvent. To this end we assume that in addition to B there are given two Banach spaces B_0 and B_1 with $B_0 \subset B \subset B_1$ such that the injections:

$$J_0 : B_0 \mapsto B \text{ and } J : B \mapsto B_1 \tag{2.1}$$

are continuous. For $\lambda \in D \setminus \sigma_{\text{dis}}(P)$, we set

$$\tilde{R}(\lambda) = JR(\lambda)J_0. \tag{2.2}$$

Clearly $\tilde{R}(\lambda)$ is a meromorphic operator function in D with values in $\mathcal{L}(B_0, B_1)$. We refer to $\tilde{R}(\lambda)$ a the **generalized resolvent** of P. We shall assume that the following basic condition holds.

Hypothesis 2.1. *The operator function $\tilde{R}(\lambda)$ admits a meromorphic continuation with finite rank poles from D to a domain $D_+ \supset D$, where*

$$(D_+ \cap \sigma(P)) \setminus \sigma_{\text{dis}}(P) \neq \emptyset.$$

(The last restriction is of course the statement that $R(\lambda)$ does not admit such a meromorphic continuation to D_+).

We note that in the following the term "generalized resolvent" will apply to the meromorphic extension of $\tilde{R}(\lambda)$ from D to D_+. The function in D_+ will also be denoted by $\tilde{R}(\lambda)$. The domains D and D_+ will be fixed throughout.

We are in a position to define the notion of a resonance.

Definition. *A resonance of P is a pole λ_0 of $\tilde{R}(\lambda)$, $\lambda_0 \in D_+ \setminus D$, which verifies one of the following conditions. Either*
(i) $\lambda_0 \notin \sigma_{\text{dis}}(P)$, or
(ii) $\lambda_0 \in \sigma_{\text{dis}}(P)$ *but the relation (2.2) does not hold (identically) in any deleted neighborhood of λ_0.*

Remark. The definition of resonances clearly depends on the auxiliary spaces B_0, B_1. In this paper we don't investigate the "uniqueness problem" for resonances.

The set of all poles of $\tilde{R}(\lambda)$ in D_+ will be denoted by $\Lambda(P)$. It is clear that $\Lambda(P)$ is composed of the following two disjoint sets. (i) Resonances of P in D_+. (ii) Isolated eigenvalues of P in D_+.

We impose a second condition of P which is also basic for the theory. We assume

Hypothesis 2.2. *Consider P as an operator in B_1. Denote this operator by $\mathcal{P}$, i.e.*

$$\begin{aligned} \mathrm{Dom}(\mathcal{P}) &= \mathrm{Dom}(P) \subset B_1 \\ \mathcal{P}u &= Pu \quad \text{for} \quad u \in \mathrm{Dom}(\mathcal{P}). \end{aligned}$$

The following holds.
(i) *$\mathcal{P}$ is a closable operator in B_1.*
(ii) *Denote by P_1 the closure of $\mathcal{P}$ in B_1. Then the resolvent $R_1(\lambda) := (P_1 - \lambda)^{-1} \in \mathcal{L}(B_1)$ exists for λ in some open set O in D.*

We conclude this section with a simple lemma which will be very useful later on.

Lemma 2.3. *For any $\lambda_1 \in D_+ \setminus \Lambda(P)$ and $\lambda_2 \in O$ (O as above), the following formula holds:*

$$R_1(\lambda_2)\tilde{R}(\lambda_1) = (\lambda_2 - \lambda_1)^{-1}(\tilde{R}(\lambda_2) - \tilde{R}(\lambda_1)), \tag{2.3}$$

with the obvious interpretation of (2.3) when $\lambda_1 = \lambda_2$.

Proof. If $\lambda_1 \in O$ then (2.3) is (essentially) the resolvent equation for $R_1(\lambda)$ in O. The validity of (2.3) for any λ_1 in $D_+ \setminus \Lambda(P)$ follows from its validity in O by analytic continuation in λ_1.

3. The resonance-eigenvalue connection

In this section we shall establish that resonances of P are in fact eigenvalues of some closely related operators acting in different Banach spaces. To this end we introduce certain Banach spaces, depending on the operator P, which are intermediate spaces between B_0 and B_1. The construction is as follows. We pick a bounded domain Δ in $\mathbb{C}$, with a boundary Γ of class C^1, satisfying (i) $\bar{\Delta} \subset D_+$. (ii) $\Gamma \cap \Lambda(P) = \emptyset$. Having chosen Δ we denote by B_Γ the linear set of elements f in the Banach space B_1, admitting a representation of the form:

$$f = g + \int_\Gamma \tilde{R}(\zeta)\Phi(\zeta)d\zeta \tag{3.1}$$

where g is some element in B_0 and $\Phi(\zeta)$ is some continuous function on Γ with values in B_0 (the integration in (3.1) is over the positively oriented boundary with respect to Δ). Next we introduce a norm in B_Γ, setting for $f \in B_\Gamma$:

$$||f||_{B_\Gamma} = \inf_{g,\Phi}(||g||_{B_0} + ||\Phi||_{C(\Gamma;B_0)}) \tag{3.2}$$

where the infimum in (3.2) is taken over all $g \in B_0$ and $\Phi \in C(\Gamma; B_0)$ which verify (3.1) ($C(\Gamma; B_0)$ denotes the Banach space of continuous functions on Γ with values in B_0). It is easy to see that under the norm (3.2) B_Γ is complete. Hence B_Γ is a Banach space. We have the following inclusion relations with continuous injections:

$$B_0 \subset B_\Gamma \subset B_1. \tag{3.3}$$

Next we associate with any $\lambda \in D_+ \setminus \Lambda(P)$ a linear operator $T_\Gamma(\lambda) : B_\Gamma \to B_1$ defined as follows. For any $f \in B_\Gamma$, where f is given by (3.1), we set

$$T_\Gamma(\lambda)f = \tilde{R}(\lambda)g + \int_\Gamma (\zeta - \lambda)^{-1}(\tilde{R}(\zeta) - \tilde{R}(\lambda))\Phi(\zeta)d\zeta. \tag{3.4}$$

We have to show that $T_\Gamma(\lambda)$ is well defined on B_Γ (i.e. $T_\Gamma(\lambda)f$ is independent of the special representation of f).

Suppose first that $\lambda \in O$ (O defined in Hypothesis 2.2). Applying $R_1(\lambda)$ to (3.1) and using (2.3) we have:

$$T_\Gamma(\lambda)f = R_1(\lambda)f \quad \text{for} \quad f \in B_\Gamma \tag{3.5}$$

which shows that $T_\Gamma(\lambda)f$ is well defined for any λ in the open set O. From (3.5) it follows by analytic continuation in λ that if $f = 0$ then $T_\Gamma(\lambda)f = 0$ for any $\lambda \in D_+ \setminus \Lambda(P)$. Thus the operator $T_\Gamma(\lambda)$ is well defined for all λ. Also, it follows readily from (3.4) that $T_\Gamma(\lambda) \in \mathcal{L}(B_\Gamma, B_1)$.

We shall now show that if $\lambda \in \Delta \setminus \Lambda(P)$ then $\mathrm{Ran}T_\Gamma(\lambda) \subset B_\Gamma$ and that $T_\Gamma(\lambda)$ is in fact an operator in $\mathcal{L}(B_\Gamma)$. To see this let $f \in B_\Gamma$ be given by (3.1) and rewrite (3.4) in the form

$$T_\Gamma(\lambda)f = \tilde{R}(\lambda)g_\lambda + \int_\Gamma \tilde{R}(\zeta)((\zeta - \lambda)^{-1}\Phi(\zeta))d\zeta \tag{3.6}$$

where

$$g_\lambda = g - \int_\Gamma (\zeta - \lambda)^{-1}\Phi(\zeta)d\zeta \in B_0.$$

Let $p(\zeta) \not\equiv 0$ be a polynomial of minimal order such that $p(\zeta)\tilde{R}(\zeta)$ is holomorphic in Δ. By Cauchy's formula:

$$\tilde{R}(\lambda)g_\lambda = (2\pi i p(\lambda))^{-1} \int_\Gamma \tilde{R}(\zeta)((\zeta - \lambda)^{-1}p(\zeta)g_\lambda)d\zeta. \tag{3.7}$$

Combining (3.6) and (3.7), we get:

$$T_\Gamma(\lambda)f = \int_\Gamma \tilde{R}(\zeta)((\zeta - \lambda)^{-1}\Psi_\lambda(\zeta))d\zeta \tag{3.8}$$

where

$$\Psi_\lambda(\zeta) = \Phi(\zeta) + (2\pi i p(\lambda))^{-1}p(\zeta)g_\lambda.$$

Hence: $T_\Gamma(\lambda)f \in B_\Gamma$. It is also readily checked that $T_\Gamma(\lambda)$ is in fact a continuous operator in B_Γ.

We are going to use the following notation. For any $\lambda \in \Delta \setminus \Lambda(P)$ we shall write $R_\Gamma(\lambda)$ for the operator $T_\Gamma(\lambda)$ when considered as an operator in $\mathcal{L}(B_\Gamma)$. It is clear from the above that $\Delta \ni \lambda \mapsto R_\Gamma(\lambda) \in \mathcal{L}(B_\Gamma)$ is a meromorphic operator function in Δ with poles contained in the set $\Lambda(P)$.

Proposition 3.1. *The following equation holds:*

$$R_\Gamma(\lambda_1)R_\Gamma(\lambda_2) = (\lambda_1 - \lambda_2)^{-1}(R_\Gamma(\lambda_1) - R_\Gamma(\lambda_2)) \tag{3.9}$$

for any $\lambda_1, \lambda_2 \in \Delta \setminus \Lambda(P)$.

Proof. (3.9) follows from the following more general relation which we shall need later on:

$$T_\Gamma(\lambda)R_\Gamma(\mu)f = (\lambda - \mu)^{-1}(T_\Gamma(\lambda)f - T_\Gamma(\mu)f) \tag{3.10}$$

for any $\lambda \in D_+ \setminus \Lambda(P)$, $\mu \in \Delta \setminus \Lambda(P)$ and $f \in B_\Gamma$.

To prove (3.10) we consider the resolvent equation for the operator P_1:

$$R_1(\lambda)R_1(\mu) = (\lambda - \mu)^{-1}(R_1(\lambda) - R_1(\mu)) \tag{3.11}$$

which holds for all λ, μ in O. From (3.11) and (3.5) it follows that

$$R_1(\lambda)T_\Gamma(\mu)f = (\lambda - \mu)^{-1}(T_\Gamma(\lambda)f - T_\Gamma(\mu)f) \tag{3.12}$$

for $\lambda, \mu \in O$ and any $f \in B_\Gamma$. By analytic continuation in μ it follows that (3.12) holds for all μ in $D_+ \setminus \Lambda(P)$. Restricting μ to Δ, using (3.5), yields (3.10) for $\lambda \in O$. Finally, an analytic continuation in λ establishes (3.10) for all $\lambda \in D_+ \setminus \Lambda(P)$ and $\mu \in \Delta \setminus \Lambda(P)$.

Proposition 3.2. *$R_\Gamma(\mu)$ is an injective operator in B_Γ for every $\mu \in \Delta \setminus \Lambda(P)$.*

Proof. Suppose by way of contradiction that $R_\Gamma(\mu)f = T_\Gamma(\mu)f = 0$ for some $\mu \in \Delta \setminus \Lambda(P)$, $f \in B_\Gamma$, $f \neq 0$. Applying (3.10) it follows that $T_\Gamma(\lambda)f = 0$ for all $\lambda \in D_+ \setminus \Lambda(P)$. Using (3.5) it follows that $R_1(\lambda)f = 0$ for all $\lambda \in O$. This, however, contradicts the injectivity of $R_1(\lambda)$. □

It follows from Proposition 3.1 and Proposition 3.2 that $R_\Gamma(\lambda)$ is the resolvent of an operator P_Γ in B_Γ:

$$R_\Gamma(\lambda) = (P_\Gamma - \lambda)^{-1} \quad \text{for} \quad \lambda \in \Delta \setminus \Lambda(P), \tag{3.13}$$

where P_Γ is a closed linear operator in B_Γ defined as follows:

$$\begin{aligned} &\mathrm{Dom}(P_\Gamma) = \mathrm{Ran} R_\Gamma(\lambda_0), \\ &P_\Gamma u = \lambda_0 u + f \end{aligned} \tag{3.14}$$

for $u = R_\Gamma(\lambda_0)f \in \mathrm{Dom}(P_\Gamma)$, $f \in B_\Gamma$. Here λ_0 is some fixed point in $\Delta \setminus \Lambda(P)$.

One should note that (3.14), (3.10) and (3.5) imply that P_1 is an extension of P_Γ in the sense that

$$\begin{aligned} &\mathrm{Dom}(P_\Gamma) \subset \mathrm{Dom}(P_1), \\ &P_\Gamma u = P_1 u \quad \text{for} \quad u \in \mathrm{Dom}(P_\Gamma). \end{aligned} \tag{3.15}$$

From (3.5) it follows, by analytic continuation, that

$$R_\Gamma(\lambda)f = \tilde{R}(\lambda)f \quad \text{for} \quad f \in B_0, \tag{3.16}$$

for any $\lambda \in \Delta \setminus \Lambda(P)$.

From (3.16) and (3.15) it follows in particular that

$$\mathrm{Ran}\tilde{R}(\lambda) \subset \mathrm{Dom}(P_\Gamma) \subset \mathrm{Dom}(P_1) \tag{3.17}$$

for any $\lambda \in \Delta \setminus \Lambda(P)$.

From (3.16) and (3.8) it follows that $R_\Gamma(\lambda)$ and $\tilde{R}(\lambda)$ possess the same poles in Δ. This yields

Theorem 3.3. *The operator P_Γ has a discrete spectrum in Δ given by*

$$\sigma(P_\Gamma) \cap \Delta = \Lambda(P) \cap \Delta. \tag{3.18}$$

In particular, all resonances of P in Δ are eigenvalues of P_Γ.

4. More on the resonance-eigenvalue connection

We continue with the discussion of the last section. We shall assume now that the domain Δ was chosen to contain a given resonance λ_0 of P. By Theorem 3.3 λ_0 is an isolated eigenvalue of P_Γ. We wish to explore this relation more closely.

Now, λ_0 is a pole of $\tilde{R}(\lambda)$ of order r and also a pole of $R_\Gamma(\lambda)$ of order n. From (3.16) it follows that $n \geq r$ (later we show that $n = r$). We consider the Laurent expansions about λ_0:

$$\tilde{R}(\lambda) = \sum_{j \geq -r} (\lambda - \lambda_0)^j S_j, \tag{4.1}$$

$$R_\Gamma(\lambda) = \sum_{j \geq -n} (\lambda - \lambda_0)^j S_j^\Gamma, \tag{4.1$_\Gamma$}$$

where $S_j \in \mathcal{L}(B_0, B_1)$ for $j \geq -r$, $S_j^\Gamma \in \mathcal{L}(B_\Gamma)$ for $j \geq -n$; S_{-r} and $S_{-n}^\Gamma \neq 0$.

The relation (3.16) implies that

$$S_j f = S_j^\Gamma f \quad \text{for} \quad f \in B_0 \tag{4.2}$$

and all $j \geq -n$ (if $n > r$ we set $S_j = 0$ for $-n \leq j < -r$). From (4.2) and (3.17) it follows that

$$\text{Ran} S_j \subset \text{Ran} S_j^\Gamma \subset \text{Dom}(P_\Gamma) \subset \text{Dom}(P_1). \tag{4.3}$$

We now claim that

$$\text{Ran} S_j = \text{Ran} S_j^\Gamma \tag{4.4}$$

for $-n \leq j \leq -1$. (Note that this proves that $n = r$). In this connection we recall that S_j and S_j^Γ are finite rank operators for $j \leq -1$, and that (by spectral theory)

$$\text{Ran} S_j^\Gamma \subset \text{Ran} S_{j+1}^\Gamma \tag{4.5}$$

for $-n \leq j \leq -2$.

Proof of (4.4) (sketch). Let $f \in B_\Gamma$ and suppose that f is given by (3.1). Using (3.4) one finds by integration over a small circle γ centered at λ_0 that

$$\int_\gamma (\lambda - \lambda_0)^{-j-1} R_\Gamma(\lambda) f d\lambda = \int_\gamma (\lambda - \lambda_0)^{-j-1} \tilde{R}(\lambda) F(\lambda) d\lambda \tag{4.6}$$

for $-n \leq j \leq -1$ where $F(\lambda)$ is some analytic function in Δ with values in B_0. From (4.6) it follows that

$$\mathrm{Ran} S_j^\Gamma \subset \bigoplus_{-n \leq k \leq j} \mathrm{Ran} S_k \tag{4.7}$$

for $-n \leq j \leq -1$. Combining (4.7), (4.5) and (4.3) yields (4.4). □

Recall that by standard spectral theory $-S_{-1}^\Gamma$ is a projection operator in B_Γ which projects on the set of generalized eigenvectors ("root vectors") of P_Γ at λ_0 and that

$$(4.8)\qquad \begin{aligned} &(P_\Gamma - \lambda_0) S_j^\Gamma = S_{j-1}^\Gamma \quad \text{for} \quad j > -n,\ j \neq 0, \\ &(P_\Gamma - \lambda_0) S_{-n}^\Gamma = 0. \end{aligned}$$

Taking account of (4.3), (4.4), (4.5), (4.8) and (3.15), we obtain

Theorem 4.1. *Let $\lambda_0 \in D_+$ be a resonance of P, λ_0 a pole of $\tilde{R}(\lambda)$ of order r. Let $S_j \in \mathcal{L}(B_0, B_1)$ be the coefficients in the Laurent expansion of $\tilde{R}(\lambda)$ about λ_0 given by (4.1). The following holds.*
(i) $\mathrm{Ran} S_j$ *is a finite dimensional invariant subspace of* P_1 *for any* $j \leq -1$.
(ii) *If* $r \geq 2$ *then* $\mathrm{Ran} S_{j-1} \subset \mathrm{Ran} S_j$ *for* $-r+1 \leq j \leq -1$.

$$(\mathrm{iii})\qquad \begin{aligned} &(P_1 - \lambda_0) S_j = S_{j-1} \quad \textit{for} \quad j > -r,\ j \neq 0, \\ &(P_1 - \lambda_0) S_{-r} = 0. \end{aligned}$$

Definition. *Let λ_0 be a resonance of P. Denote by S_{-1} the residue of $\tilde{R}(\lambda)$ at λ_0 as above. Then*
(i) *The integer* $\dim \mathrm{Ran} S_{-1}$ *is called the multiplicity (or the algebraic multiplicity) of* λ_0.
(ii) *An element* $u \in \mathrm{Ran} S_{-1}$ *such that*

$$(P_1 - \lambda_0) u = 0$$

is called a resonance vector of P at λ_0. Any element in $\mathrm{Ran} S_{-1}$ *is called a generalized resonance vector at λ_0.*

Remark. *In this paper we adopt the convention that a resonance vector is also a generalized resonance vector and that an eigenvector is also a generalized eigenvector.*

Finally, we elaborate on the relation between resonances of P and eigenvalues of P_Γ given in Theorem 3.3.

Theorem 4.2. *With the same notation as above, let $\mathcal{E}$ denote the space of resonance vectors (resp. generalized resonance vectors) of P at λ_0. Then $\mathcal{E}$ coincides with the space of eigenvectors (resp. generalized eigenvectors) of P_Γ at λ_0. Also, $\mathcal{E} \subset$* $\mathrm{Dom}(P_1)$ *and $P_1 = P_\Gamma$ on $\mathcal{E}$.*

Theorem 4.2 follows readily from (4.4), (4.8) and Theorem 4.1.

Remark 4.3. If $\lambda_0 \in \Delta$ is an isolated eigenvalue of P then the same arguments used in this section show that Theorem 4.2 holds with $\mathcal{E}$ replaced by the space of eigenvectors (resp. generalized eigenvectors) of P.

5. Perturbation theory

We turn to perturbation problems. With P the operator studied before, we consider a family of operators $\mathcal{P}(t)$ in B, defined for t in a connected open neighborhood Ω of the origin in $\mathbb{C}$, of the form:

$$\mathcal{P}(t) = P + V(t) \tag{5.1}$$

where $V(t) : B \to B$ is a closable linear operator for any t in Ω, verifying the following conditions.
(i) $\mathrm{Dom}(V(t)) = \mathrm{Dom}(P)$, $\forall t$.
(ii) $V(0) = 0$.
(iii) $V(t)u$ is a holomorphic function of t in Ω (with values in B) for any $u \in \mathrm{Dom}(P)$.

Our first observation is that $\mathcal{P}(t)$ is a holomorphic family of operators of type A in the sense of Kato ([4]) if t is restricted to some sufficiently small neighborhood of the origin.

To prove this we need only to show that $\mathcal{P}(t)$ is a closed operator for all t sufficiently small. Now, since the resolvent set of P is not empty it follows by standard arguments (closed graph theorem) that $V(t)$ is a P bounded operator for each t. More precisely, we find that

$$||V(t)u||_B \leq \varepsilon(t)(||Pu||_B + ||u||_B) \tag{5.2}$$

for any $u \in \mathrm{Dom}(P)$ where $\varepsilon(t) \to 0$ as $t \to 0$. By a well known theorem it follows that if $\varepsilon(t) < 1$ then $P + V(t)$ is a closed operator. Hence the result.

Replacing Ω, if necessary, by a smaller domain we may assume without loss of generality that $\mathcal{P}(t)$ is a holomorphic family in Ω of type A. let λ_0 be a simple isolated eigenvalue of P. A classical result in perturbation theory asserts that for t sufficiently small $\mathcal{P}(t)$ has a unique simple eigenvalue $\lambda(t)$ near λ_0, $\lambda(t)$ being an analytic function of t. Moreover the theory furnishes "explicit" formulas for the derivatives of $\lambda(t)$ at λ_0. When λ_0 is a degenerate eigenvalue similar results hold for $\hat{\lambda}(t)$ which is the mean of the "eigenvalue group" of $\mathcal{P}(t)$ near λ_0 (see [4]).

The question arises whether similar perturbation results hold for resonances of $\mathcal{P}(t)$. We shall show that this is indeed the case under a suitable restriction on the perturbation $V(t)$. We introduce the following assumption.

Hypothesis 5.1. *There exists a family of closable operators $V_1(t) : B_1 \to B_1$, defined for $t \in \Omega$, with $V_1(0) = 0$, such that the following holds.*
(i) $\mathrm{Dom}(V_1(t)) = \mathrm{Dom}(P_1)$ *and* $\mathrm{Ran} V_1(t) \subset B_0$, $\forall t$.
(ii) $V_1(t)u$ *is a holomorphic function with values in B_0 for $t \in \Omega$, for any* $u \in \mathrm{Dom}(P_1)$.
(iii) $V(t)u = V_1(t)u$ *for* $u \in \mathrm{Dom}(P)$, $\forall t$.

Let $\lambda_0 \in D_+$ be a resonance of P. We propose to study resonances of $\mathcal{P}(t)$ near λ_0 (for small t). To this end we pick a domainD' satisfying: (i) $D' \subset\subset D$. (ii) $D' \cap O \neq \emptyset$ where O is the set in Hypothesis 2.2. Then we choose an open set O' such that $O' \subset\subset D' \cap O$. Next we pick a domain D'_+, containing the resonance λ_0, verifying: $D'_+ \subset\subset D_+$ and $D'_+ \supset D'$. Finally we fix some domain Δ, with a boundary Γ of class C^1, satisfying: (i) $\bar{\Delta} \subset D_+$. (ii) $\Delta \supset D'_+$. (iii) $\Gamma \cap \Lambda(P) = \emptyset$.

With P and Δ fixed, we denote by B_Γ and P_Γ the Banach space and the operator introduced in Section 3. We recall that $B_0 \subset B_\Gamma \subset B_1$ and that $\mathrm{Dom}(P_\Gamma) \subset \mathrm{Dom}(P_1)$. Next we consider two families of operators $\mathcal{P}_1(t) : B_1 \to B_1$ and $\mathcal{P}_\Gamma(t) : B_\Gamma \to B_\Gamma$, defined for any $t \in \Omega$, as follows:

$$\begin{aligned} &\mathrm{Dom}(\mathcal{P}_1(t)) = \mathrm{Dom}(P_1), \\ &\mathcal{P}_1(t)u = P_1 u + V_1(t)u \quad \text{for} \quad u \in \mathrm{Dom}(P_1). \end{aligned} \tag{5.3}$$

$$\begin{aligned} &\mathrm{Dom}(\mathcal{P}_\Gamma(t)) = \mathrm{Dom}(P_\Gamma), \\ &\mathcal{P}_\Gamma(t)u = P_\Gamma u + V_1(t)u \quad \text{for} \quad u \in \mathrm{Dom}(P_\Gamma). \end{aligned} \tag{5.3$_\Gamma$}$$

We have

Proposition 5.2. *$\mathcal{P}_1(t)$ and $\mathcal{P}_\Gamma(t)$ are holomorphic families of type A in Ω_0, with $\mathcal{P}_1(0) = P_1$, $\mathcal{P}_\Gamma(0) = P_\Gamma$, where $\Omega_0 \subset \Omega$ is some domain in $\mathbb{C}$ containing the origin.*

The proof of the proposition is similar to the proof of the same result for the family $\mathcal{P}(t)$.

Applying well known perturbation results for holomorphic families of closed operators ([4]) we find that for t in some sufficiently small disc ω centered at the origin the following holds:
(i) $\mathcal{P}(t)$ has a discrete spectrum in D'.
(ii) $\mathcal{P}_1(t)$ has no spectrum in O'.
(iii) $\mathcal{P}_\Gamma(t)$ has a discrete spectrum in D'_+.

For any $t \in \omega$ we introduce the resolvents:

$$\begin{aligned} R(t;\lambda) &:= (\mathcal{P}(t) - \lambda)^{-1}, \\ R_1(t;\lambda) &:= (\mathcal{P}_1(t) - \lambda)^{-1}, \\ R_\Gamma(t;\lambda) &:= (\mathcal{P}_\Gamma(t) - \lambda)^{-1}. \end{aligned} \tag{5.4}$$

$R(t;\lambda)$ is meromorphic in λ in D'; $R_1(t;\lambda)$ is holomorphic in O' and $R_\Gamma(t;\lambda)$ is meromorphic in λ in D'_+ (all poles are of finite rank). We also introduce the generalized resolvent of $\mathcal{P}(t)$ defined by

$$\tilde{R}(t;\lambda) = J R(t;\lambda) J_0 \tag{5.5}$$

where J, J_0 are the injection operators (2.1). $\tilde{R}(t;\lambda)$ takes its values in $\mathcal{L}(B_0, B_1)$. It is meromorphic in λ in D'.

Since $\mathcal{P}(t)$ is a restriction of $\mathcal{P}_1(t)$ to B and $P_\Gamma(t)$ is a restriction of $\mathcal{P}_1(t)$ to B_Γ (see (3.15)), we find that for any $t \in \omega$ and $\lambda \in O'$ the following relations hold:

$$R_1(t;\lambda)f = R(t;\lambda)f \;\text{ for } f \in B, \qquad R_1(t;\lambda)f = R_\Gamma(t;\lambda)f \;\text{ for } f \in B_\Gamma. \tag{5.6}$$

Since $B_0 \subset B$ and $B_0 \subset B_\Gamma$, it follows from (5.6) and (5.5) that for any $t \in \omega$ and $\lambda \in O'$:

$$\tilde{R}(t;\lambda)f = R_\Gamma(t;\lambda)f \quad \text{for} \quad f \in B_0. \tag{5.7}$$

Finally, since $R_\Gamma(t;\lambda)$ is meromorphic in λ in D'_+ (recall that $D'_+ \supset D' \supset O'$), it follows that $\tilde{R}(t;\lambda)f$ admits a meromorphic continuation from O' to D'_+ given by the r.h.s. of (5.7). This shows that Hypothesis 2.1. holds for the operator $\mathcal{P}(t)$, any $t \in \omega$. Also, the first relation (5.6) shows that Hypothesis 2.2 holds for the operators $\mathcal{P}(t)$.

Let $\tilde{D}_+$ be a domain in $\mathbb{C}$ such that $\tilde{D}_+ \subset\subset D_+$, $\tilde{D}_+ \supset \bar{\Delta}$. The above considerations show that by choosing ω sufficiently small we may further assume that $\tilde{R}(t;\lambda)$ admits a meromorphic continuation in λ from O' to $\tilde{D}_+$, for any $t \in \omega$. Now, using the relation (5.7) it is not difficult to show (by arguments similar to those used in the proof of (4.4)) that $\tilde{R}(t;\lambda)$ and $R_\Gamma(t;\lambda)$ possess the same poles in D'_+ for any $t \in \omega$ and that if $\mu \in D'_+$ is a pole of $\tilde{R}(t;\lambda)$ and $R_\Gamma(t;\lambda)$ for some $t \in \omega$, then:

$$\mathrm{RanRes}\tilde{R}(t;\lambda)|_{\lambda=\mu} = \mathrm{RanRes}R_\Gamma(t;\lambda)|_{\lambda=\mu}.$$

This yields the following

Theorem 5.3. *Let $\mathcal{P}(t)$ and $\mathcal{P}_\Gamma(t)$, $t \in \omega$, be as above. The following holds.*
(i) *The operator functions (in λ) $\tilde{R}(t;\lambda)$ and $R_\Gamma(t;\lambda)$ possess the same poles in D'_+.*
(ii) *Let $\mu \in D'_+$ be a pole of $\tilde{R}(t;\lambda)$ and $R_\Gamma(t;\lambda)$. Denote by $\mathcal{E}$ the space of generalized eigenvectors of $\mathcal{P}_\Gamma(t)$ at the eigenvalue μ. Then $\mathcal{E}$ coincides with the space of generalized resonance vectors of $\mathcal{P}(t)$ at the resonance μ. (If μ is an isolated eigenvalue of $\mathcal{P}(t)$ then $\mathcal{E}$ coincides with the space of generalized eigenvectors of $\mathcal{P}(t)$ at μ.)*
(iii) *$\mathcal{E}$ is a finite dimensional invariant subspace for the operators $\mathcal{P}_1(t)$ and $\mathcal{P}_\Gamma(t)$. The two operators coincide on $\mathcal{E}$.*

It follows from Theorem 5.3 that perturbation problems for resonances of the family $\mathcal{P}(t)$ can be translated into perturbation problems for isolated eigenvalues of the family $\mathcal{P}_\Gamma(t)$. Thus perturbation theory for resonances (in our setup) is reduced to classical perturbation theory for eigenvalues. Using this reduction we can obtain perturbation series formulas for resonances which are essentially the same as those obtained for eigenvalues in the classical theory.

In conclusion we remark that the perturbation theory for resonances described in this paper is applicable to many concrete differential problems. Here are some examples of operators P to which the theory is applicable (with a suitable choice of perturbations).
(i) *P the operator $-\Delta + V$ on $\mathbb{R}^n \setminus \Omega$ where V is an exponentially decaying potential and Ω is a compact obstacle.*
(ii) *P the operator $-\Delta + V_1 + V_2$ on $\mathbb{R}^n$, where V_1 is a periodic potential and V_2 an exponentially decaying potential.*
(iii) *P the Laplace-Beltrami operator on a non-compact hyperbolic manifold with a finite volume.*

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