JOURNÉES ÉQUATIONS AUX DÉRIVÉES PARTIELLES

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Journées Équations aux dérivées partielles (1995), p. 1-12

<http://www.numdam.org/item?id=JEDP_1995____A5_0>

THE SYMBOL OF A LAGRANGIAN DISTRIBUTION

D. Vassiliev

1. Main result

Let M be a smooth n-dimensional manifold without boundary, and let T^*M be the cotangent bundle. Local coordinates (and points) on M will be denoted by $x = (x_1, x_2, \dots, x_n)$ or $y = (y_1, y_2, \dots, y_n)$, and the dual coordinates on the fibres T_x^*M and T_y^*M will be denoted by $\xi = (\xi_1, \xi_2, \dots, \xi_n)$ and $\eta = (\eta_1, \eta_2, \dots, \eta_n)$ respectively. By $T'M$ we shall denote the cotangent bundle T^*M with the zero section ($\xi = 0$ or $\eta = 0$) excluded.

By

$$(x^*(t; y, \eta), \xi^*(t; y, \eta)), \qquad t \in (-\infty, +\infty), \quad (y, \eta) \in T'M, \tag{1.1}$$

we shall denote a smooth time-dependent homogeneous canonical transformation in $T'M$. We assume that

$$(x^*(0; y, \eta), \xi^*(0; y, \eta)) = (y, \eta). \tag{1.2}$$

For example, the canonical transformation (1.1) can be generated by some Hamiltonian $h(x, \xi)$ positively homogeneous in η of degree 1. In this case (1.1) is the solution of the Hamiltonian system of equations

$$\dot{x}^* = h_\xi(x^*, \xi^*), \qquad \dot{\xi}^* = -h_x(x^*, \xi^*) \tag{1.3}$$

with initial condition (1.2).

The phase functions in this paper are assumed to be positively homogeneous in η of degree 1 and with non-negative imaginary part. The fact that we allow our phase functions to be complex-valued is crucial, because otherwise we would not be able to use a global construction, see [1].

Throughout the paper we shall often denote partial derivatives by respective subscripts.

Definition 1.1. We say that the phase function

$$\varphi(t, x; y, \eta) \in C^\infty((-\infty, +\infty) \times M \times T'M)$$

is a phase function associated with the canonical transformation (1.1) if it satisfies the following conditions:

$$\begin{aligned}\varphi(t,x^*(t;y,\eta);y,\eta) &= 0\,,\\ \varphi_x(t,x^*(t;y,\eta);y,\eta) &= \xi^*(t;y,\eta)\,,\\ \det\varphi_{x\eta}(t,x^*(t;y,\eta);y,\eta) &\neq 0\,.\end{aligned}$$

By $\mathfrak{F}$ we denote the set of all phase functions associated with the canonical transformation (1.1).

The above conditions imply, in particular, that $\varphi_\eta(t,x^*(t;y,\eta);y,\eta)=0$.

Properties of phase functions of the type described above were studied in [1]. Let us mention briefly some of these properties:

(1) the class $\mathfrak{F}$ is non-empty;
(2) the class $\mathfrak{F}$ is contractible as a topological space;
(3) any phase function which is defined locally and satisfies locally the conditions of Definition 1.1 can be extended up to a phase function of the class $\mathfrak{F}$;
(4) it is possible to choose a phase function of the class $\mathfrak{F}$ which is locally linear with respect to x in some local coordinates.

Set for brevity

$$\mathfrak{C} \stackrel{\mathrm{def}}{=} \{(t,x;y,\eta)\ :\ x=x^*(t;y,\eta)\} \subset (-\infty,+\infty)\times M\times T'M\,.$$

For each phase function $\varphi\in\mathfrak{F}$ there exists a (open) connected simply connected conic neighbourhood $\mathcal{O}\subset(-\infty,+\infty)\times M\times T'M$ of the set $\mathfrak{C}$ such that

(1) $\varphi_\eta\neq 0$ on $\mathcal{O}\setminus\mathfrak{C}$,
(2) $\det\varphi_{x\eta}\neq 0$ on $\mathcal{O}$.

Let $\varsigma(t,x;y,\eta)\in C^\infty((-\infty,+\infty)\times M\times T'M)$ be a cut-off function satisfying the following four conditions:

(1) $\operatorname{supp}\varsigma\subset\mathcal{O}$;
(2) $\varsigma(t,x;y,\eta) = 0$ on the set $\{(t,x;y,\eta)\ :\ h(y,\eta)\leqslant 1\}$;
(3) $\varsigma(t,x;y,\eta) = 1$ on the intersection of a small conic neighbourhood of $\mathfrak{C}$ with the set $\{(t,x;y,\eta)\ :\ h(y,\eta)\geqslant 2\}$;
(4) $\varsigma(t,x;y,\lambda\eta) = \lambda\varsigma(t,x;y,\eta)$ for $h(y,\eta)\geqslant 2$, $\lambda\geqslant 1$.

Here for h we can take an arbitrary positive smooth function on $T'M$ positively homogeneous in η of degree 1, for example, the Hamiltonian introduced in the beginning of this section. The choice of a particular cut-off ς changes the resulting Lagrangian distribution (see below) only by a $C^\infty((-\infty,+\infty)\times M\times M)$ term.

Consider now the expression $\det^2\varphi_{x\eta}$. It is easy to see that it is a a 2-density with respect to x and a (-2)-density with respect to y. Consequently the argument of this expression (which is well-defined on $\mathcal{O}$) does not change under changes of local coordinates. Let us choose a particular continuous branch $\arg_0(\det^2\varphi_{x\eta})$ of the argument $\arg(\det^2\varphi_{x\eta})$, specified by the condition $\arg_0(\det^2\varphi_{x\eta})\big|_{t=0,x=y}=0$. Set

$$d_\varphi(t,x;y,\eta) = (\det^2\varphi_{x\eta})^{1/4} = |\det\varphi_{x\eta}|^{1/2}\,e^{i\arg_0(\det^2\varphi_{x\eta})/4}\,.$$

Obviously, d_φ is a (1/2)-density with respect to x and a $(-1/2)$-density with respect to y.

We denote by S^l the class of complex-valued functions $a \in C^\infty((-\infty,+\infty)\times M \times T'M)$ which admit an asymptotic expansion

$$a(t,x;y,\eta) \sim \sum_{k=0}^{\infty} a_{l-k}(t,x;y,\eta), \qquad |\eta| \to \infty,$$

with $a_{l-k}(t,x;y,\eta)$ positively homogeneous in η of degree $l-k$. We also use the notation $\bar{d}\eta = (2\pi)^{-n}\, d\eta_1\, d\eta_2 \ldots d\eta_n$.

DEFINITION 1.2. Let $\varphi \in \mathfrak{F}$ and $a \in S^l$. We say that

$$\mathcal{I}_{\varphi,a}(t,x,y) = \int e^{i\varphi(t,x;y,\eta)}\, a(t,x;y,\eta)\, \varsigma(t,x;y,\eta)\, d_\varphi(t,x;y,\eta)\, \bar{d}\eta \tag{1.4}$$

is an oscillatory integral associated with the canonical transformation (1.1). A distribution $\mathcal{I}(t,x,y)$ which can be written modulo $C^\infty((-\infty,+\infty)\times M\times M)$ as an oscillatory integral (1.4) associated with the canonical transformation (1.1) is called a *Lagrangian distribution* of order l associated with this transformation.

Note that our oscillatory integrals and Lagrangian distributions are half-densities both with respect to x and y.

As shown in [1] (see also Theorem 3.3 below), any oscillatory integral (1.4) can be rewritten (modulo C^∞) with an amplitude $\mathfrak{a}(t;y,\eta)$ independent of x. This amplitude $\mathfrak{a}$ is called the *(full) symbol* of our Lagrangian distribution. For a given Lagrangian distribution and a given phase function φ the (full) symbol is defined uniquely modulo $S^{-\infty}$. The leading homogeneous term $\mathfrak{a}_l$ (of degree l) of the symbol $\mathfrak{a}$ is called the *principal symbol.* The principal symbol does not depend on the choice of a particular phase function, and is determined by the Lagrangian distribution itself.

It will be convenient for us to introduce the linear operator $\mathfrak{S}$ mapping the original amplitude $a(t,x;y,\eta)$ of an oscillatory integral into the corresponding symbol $\mathfrak{a}(t;y,\eta)$. The operator $\mathfrak{S}$ depends, of course, on the phase function φ. This operator admits an asymptotic expansion into a series of positively homogeneous in η terms:

$$\mathfrak{S} \sim \sum_{r=0}^{\infty} \mathfrak{S}_{-r},$$

where the operators $\mathfrak{S}_{-r}$ are positively homogeneous in η of degree $-r$.

The main result of this paper is

THEOREM 1.3. *The explicit formulae for the operators* $\mathfrak{S}_{-r}$ *are*

$$\mathfrak{S}_0 = (\,\cdot\,)|_{x=x^*}\,, \tag{1.5}$$

and

(1.6) $\mathfrak{S}_{-r} =$

$$\left.\left\{\left[i\, d_\varphi^{-1}\, \partial_\eta^T\, d_\varphi \left(\sum_{k=1}^{2r} \sum_{|\alpha|=k-1} \frac{(-\varphi_\eta)^\alpha}{\alpha!\,k} \left((\varphi_{x\eta})^{-1}\partial_x\right)^\alpha\right) (\varphi_{x\eta})^{-1}\partial_x\right]^r (\,\cdot\,)\right\}\right|_{x=x^*}$$

for $r \geqslant 1$.

The proof of Theorem 1.3 is given in sections 2, 3.

REMARK 1.4. For any phase function $\varphi \in \mathfrak{F}$ in a neighbourhood of the diagonal $\{x = y\}$ we have

$$\varphi(0,x;y,\eta) \;=\; \langle x-y,\eta\rangle \;+\; O(|x-y|^2)\,, \tag{1.7}$$

assuming that the local x- and y-coordinates are the same. Formula (1.7) implies that the restriction of a time-dependent Lagrangian distribution $\mathcal{I}(t,x,y)$ to $t=0$ is the Schwartz kernel of a pseudodifferential operator. If we choose the phase function in such a way that locally $\varphi(0,x;y,\eta) = \langle x-y,\eta\rangle$ then the symbol of the Lagrangian distribution $\mathcal{I}(0,x,y)$ defined as described above coincides with the dual symbol of the pseudodifferential operator $\int_M \mathcal{I}(0,x,y)(\,\cdot\,)\,dy$, and formulae (1.5), (1.6) at $t=0$ coincide with the standard formulae expressing the dual symbol of a pseudodifferential operator through its amplitude.

2. Special version of the Malgrange preparation theorem

In this section we describe one technical construction which allows us to factorize in a certain way a smooth function. This construction is a necessary prerequisite for the proof of Theorem 1.3.

The construction described below is in fact a simplified version of the well-known Malgrange preparation theorem [2, Theorem 7.5.7]. This simplified version will be sufficient for our purposes due to the special properties of our class of phase functions $\mathfrak{F}$.

The construction given in this section has its own advantages, which are not always evident in the traditional versions of the Malgrange preparation theorem.

(1) Our construction gives explicit formulae which can be used in the computation of the symbol.
(2) Our construction possesses certain invariancy properties, see subsection 2.4 below. This is important when we apply our formulae to amplitudes of oscillatory integrals (section 3) because we require a result which is invariant with respect to changes of local coordinates on the manifold.

2.1. Basic problem. Consider the Euclidean space $\mathbb{R}^n$ equipped with Cartesian coordinates $z = (z_1, \dots, z_n)$. Let $\mathcal{W} \subset \mathbb{R}^n$ be a neighbourhood of the origin, and let $a(z) \in C^\infty(\mathcal{W})$ be a complex-valued scalar function. Our first objective will be to find an n-component complex-valued column-function $g(z) = (g_1(z), \dots, g_n(z)) \in C^\infty(\mathcal{W})$ such that

$$a(z) = a(0) + \langle z, g(z) \rangle . \tag{2.1}$$

Here $\langle z, g(z) \rangle \stackrel{\text{def}}{=} \sum\limits_{j=1}^{n} z_j \, g_j(z)$.

Let us introduce the differential operators

$$Z_0 \stackrel{\text{def}}{=} 1, \qquad Z_k \stackrel{\text{def}}{=} \sum_{|\alpha|=k} \frac{z^\alpha}{\alpha!} \partial_z^\alpha , \qquad k = 1, 2, \dots, \tag{2.2}$$

where $\alpha = (\alpha_1, \dots, \alpha_n)$ are multiindices. The following formula easily follows from (2.2):

$$(k+1) Z_{k+1} = Z_1 Z_k - k Z_k = \sum_{j=1}^{n} z_j \, Z_k \, \partial_{z_j} , \qquad k = 0, 1, 2, \dots \tag{2.3}$$

Let $a_q(z)$ be a smooth function positively homogeneous in z of degree $q \in \mathbb{R}$. Then for $k = 1, 2, \dots$ we have

$$Z_k \, a_q = \frac{q(q-1)(q-2)\dots(q-k+1)}{k!} \, a_q \tag{2.4}$$

(when $k = 1$ this is the classical Euler identity). Formula (2.4) is established by induction in k with the help of the left equality (2.3). In the special case $q \in \mathbb{N}$ formula (2.4) can be rewritten as

$$Z_k \, a_q = \begin{cases} \frac{q!}{k!\,(q-k)!} \, a_q & \text{if} \quad k \leqslant q , \\ 0 & \text{if} \quad k > q . \end{cases} \tag{2.5}$$

Let us introduce the sequence of real numbers c_1, c_2, ..., defined as the solution of the following recursive system of linear algebraic equations:

$$\sum_{k=1}^{q} \frac{q!}{k!\,(q-k)!} \, c_k = 1 , \qquad q = 1, 2, \dots \tag{2.6}$$

Solving (2.6) we get $c_k = (-1)^{k-1}$.

Set

$$a_q(z) \stackrel{\text{def}}{=} \sum_{|\alpha|=q} (\partial_z^\alpha a)|_{z=0} \, \frac{z^\alpha}{\alpha!} .$$

According to Taylor's formula we have

$$a(z) \sim a(0) + \sum_{q=1}^{\infty} a_q(z) , \tag{2.7}$$

where the sign $\sim$ stands for the equality of formal Taylor expansions in powers of z. The $a_q(z)$ are polynomials homogeneous in z of degree q, so from (2.5), (2.6) we obtain

$$a_q(z) = \sum_{k=1}^{\infty} c_k\, Z_k\, a_q\,, \qquad q = 1, 2, \ldots \tag{2.8}$$

Substituting (2.8) into (2.7) we get

$$a(z) \sim a(0) + \sum_{k,q=1}^{\infty} c_k\, Z_k\, a_q \sim a(0) + \sum_{k=1}^{\infty} c_k\, Z_k\, a\,. \tag{2.9}$$

Here the interchange of order of summation is justified because the double sum contains only a finite number of nonzero terms of given degree q (degree in powers of z). Using (2.3) we can rewrite formula (2.9) as

$$a(z) \sim a(0) + \sum_{j=1}^{n} z_j \left(\sum_{k=1}^{\infty} c_k\, k^{-1}\, Z_{k-1}\, \partial_{z_j} a \right). \tag{2.10}$$

Let $\chi_k(z) \in C_0^\infty(\mathbb{R}^n)$, $0 \leqslant \chi_k(z) \leqslant 1$, $k = 1, 2, \ldots$, be cut-off functions which are identically equal to 1 in some neighbourhoods of the origin. Set

$$g' = \left(\sum_{k=1}^{\infty} c_k\, k^{-1}\, \chi_k\, Z_{k-1} \right) \partial_z a\,, \tag{2.11}$$

where ∂_z is the column-operator of first order partial differentiations in z. We shall assume that the sequence of cut-offs $\chi_1(z)$, $\chi_2(z)$, $\ldots$, is chosen in such a way that the series (2.11) converge uniformly over $\mathcal{W}$, as well as all their partial derivatives with respect to z. This can always be achieved by setting, for example, $\chi_k(z) = \chi(d_k|z|)$, where χ is an arbitrary function from $C_0^\infty(\mathbb{R})$ which is identically equal to 1 in some neighbourhood of zero, and d_1, d_2, $\ldots$, is a sequence of positive numbers tending to ∞; the necessary convergence properties hold if the sequence d_1, d_2, $\ldots$, tends to ∞ sufficiently quickly.

Formulae (2.10), (2.11) imply

$$a(z) = a(0) + \langle z, g'(z) \rangle + O(|z|^\infty)\,, \tag{2.12}$$

where $O(|z|^\infty)$ denotes a function which has an infinite order zero at $z = 0$. Set

$$g = g' + g''\,, \tag{2.13}$$

$$g'' = \frac{a - a|_{z=0} - \langle z, g' \rangle}{z^T B\, z}\, B\, z\,, \tag{2.14}$$

where $z = (z_1, \ldots, z_n)$ is understood as a column, and $B = B(z) \in C^\infty(\mathcal{W})$ is an arbitrary positive Hermitian $n \times n$ matrix-function. Formulae (2.14), (2.12) imply that the column-function $g'' = g''(z)$ is infinitely smooth, and moreover,

$$g'' = O(|z|^\infty)\,. \tag{2.15}$$

Combining (2.12)–(2.14) we obtain (2.1).

2.2. Generalization of the basic problem. Now let us generalize our original problem (2.1). Let $f(z) = \{f_1(z), \dots, f_n(z)\} \in C^\infty(\mathcal{W})$ be a given smooth complex-valued n-component column-function such that

$$f = 0 \qquad \text{if and only if} \qquad z = 0\,, \tag{2.16}$$

$$\det J|_{z=0} \neq 0\,, \tag{2.17}$$

where

$$J = J(z) \stackrel{\text{def}}{=} \partial_z f^T\,. \tag{2.18}$$

We want to find a column-function $g(z) = (g_1(z), \dots, g_n(z)) \in C^\infty(\mathcal{W})$ such that

$$a(z) = a(0) + \langle f(z), g(z)\rangle\,. \tag{2.19}$$

Obviously, this problem coincides with (2.1) in the special case $f(z) \equiv z$.

Set

$$\partial_{f_j} \stackrel{\text{def}}{=} e_j^T J^{-1} \partial_z\,, \qquad j = 1, 2, \dots n, \tag{2.20}$$

where e_j is the j-th basis column (column with 1 in the j-th row and zeros elsewhere). By $\partial_f \stackrel{\text{def}}{=} J^{-1}\partial_z$ we shall denote the column of operators (2.20).

Let us prove that the operators (2.20) commute. In view of (2.20) we have $\partial_{f_j}\partial_{f_k} - \partial_{f_k}\partial_{f_j} = b\,\partial_z$, where $b = b(z)$ is a row-function defined by the formula

$$b(z) = \left(e_j^T J^{-1} \partial_z e_k^T - e_k^T J^{-1} \partial_z e_j^T\right) J^{-1}.$$

Let $\mathbf{z}$ be an arbitrary fixed point from $\mathcal{W}$ and let $\mathbf{J} \stackrel{\text{def}}{=} J(\mathbf{z})$ (constant matrix). Using the formula for the derivative of the inverse matrix and formula (2.18) we get

$$\begin{aligned} b(\mathbf{z}) &= \left\{\left(e_k^T \mathbf{J}^{-1} \partial_z e_j^T - e_j^T \mathbf{J}^{-1} \partial_z e_k^T\right) \mathbf{J}^{-1} J\right\}\Big|_{z=\mathbf{z}} \mathbf{J}^{-1} \\ &= \left\{\left(e_k^T \mathbf{J}^{-1} \partial_z e_j^T - e_j^T \mathbf{J}^{-1} \partial_z e_k^T\right) \mathbf{J}^{-1} \partial_z f^T\right\}\Big|_{z=\mathbf{z}} \mathbf{J}^{-1} \\ &= \left\{\left((e_k^T \mathbf{J}^{-1} \partial_z)(e_j^T \mathbf{J}^{-1} \partial_z) - (e_j^T \mathbf{J}^{-1} \partial_z)(e_k^T \mathbf{J}^{-1} \partial_z)\right) f^T\right\}\Big|_{z=\mathbf{z}} \mathbf{J}^{-1}\,. \end{aligned}$$

But $e_k^T \mathbf{J}^{-1} \partial_z$ and $e_j^T \mathbf{J}^{-1} \partial_z$ are scalar differential operators with constant coefficients, so they commute. This implies $b(\mathbf{z}) = 0$.

The fact that the operators ∂_{f_j} and ∂_{f_k} commute is not really surprising because formally they can be interpreted as derivatives with respect to new independent variables f_j and f_k. However in the general case when $f(z)$ is complex-valued and non-analytic we cannot make rigorously the change of variables $z \to f$, and for this reason we have given the detailed arguments above.

As a generalization of (2.2), (2.11), (2.14) set

$$F_0 \stackrel{\text{def}}{=} 1\,, \qquad F_k \stackrel{\text{def}}{=} \sum_{|\alpha|=k} \frac{f^\alpha}{\alpha!}\,\partial_f^\alpha\,, \qquad k = 1, 2, \dots, \tag{2.21}$$

$$(2.22) \qquad g' = \left(\sum_{k=1}^{\infty} c_k \, k^{-1} \, \chi_k \, F_{k-1} \right) \partial_f a \,,$$

$$(2.23) \qquad g'' = \frac{a - a|_{z=0} - \langle f, g' \rangle}{f^T B \, \overline{f}} \, B \, \overline{f} \,.$$

Here we have the right to use the notation ∂_f^α because we know that the operators (2.20) commute.

The $g(z)$ constructed in accordance with formulae (2.13), (2.18), (2.20)–(2.23) satisfies the required equality (2.19). Let us substantiate this claim. As in subsection 2.1, it is sufficient to establish that

$$(2.24) \qquad a(z) = a(0) + \langle f(z), g'(z) \rangle + O(|z|^\infty) \,,$$

cf. (2.12). Moreover, it is sufficient to establish (2.24) under the assumption that $a(z)$ and $f(z)$ are polynomials, because any $a(z), f(z) \in C^\infty(\mathcal{W})$ can be approximated with arbitrary accuracy in powers of z by polynomials. But in the case when $a(z)$, $f(z)$ are polynomials the construction (2.18), (2.20)–(2.22) is reduced to (2.2), (2.11) by change of independent variables $z \to f$.

Note that in the above argument the change $z \to f$ leads, generally speaking, to complex independent variables f. The necessity of dealing with complex variables forced us to consider polynomials as an intermediate step.

2.3. Further generalization with "trivial" parameters u**.** Let us generalize the problem (2.19) further by introducing additional real parameters $u = (u_1, \ldots, u_m)$. Thus, we study the function $a(z;u)$, where $z \in \mathbb{R}^n$, $u \in \mathbb{R}^m$. The function a is defined in some neighbourhood of the set $\{z = 0\}$. Here as well as in the next subsection we do not specify more precisely the domain of definition of the function a in all its variables because in our applications (section 3) it is clear from the context. We also have a given complex-valued n-component column-function $f(z;u) = \{f_1(z;u), \ldots, f_n(z;u)\}$ with the same domain of definition as $a(z;u)$, and such that (2.16), (2.17) hold. We want to find a $g(z;u) = (g_1(z;u), \ldots, g_n(z;u))$ such that

$$(2.25) \qquad a(z;u) = a(0;u) + \langle f(z;u), g(z;u) \rangle \,.$$

It is easy to see that our previous construction (2.13), (2.18), (2.20)–(2.23) gives the required $g(z;u)$. The only difference is that now we have everywhere dependence on the additional parameters u. In particular, the cut-offs $\chi_k = \chi_k(z;u)$ appearing in (2.22) depend on the additional parameters u; as usual, $0 \leqslant \chi_k \leqslant 1$ and $h_k = 1$ in some neighbourhoods of the set $\{z = 0\}$. These cut-offs should (and can) be chosen in such a way that the series (2.22) converge uniformly over any compact set in the $(n+m)$-dimensional domain of definition of the function $a(z;u)$, as well as all their partial derivatives with respect to z and u.

2.4. Invariancy properties. The operators Z_k are invariant under linear changes of coordinates z, that is under changes of the type $\widehat{z} = A\,z$ where A is a constant non-degenerate $n \times n$ matrix. This fact is easily checked for $k = 0, 1$, and for $k \geqslant 2$ it is established by induction in k with the help of the left equality (2.3).

An immediate consequence is that the operators F_k are invariant under linear changes of the column-function f, namely, under changes of the type $\widehat{f} = A\,f$ where A is a non-degenerate $n \times n$ matrix-function independent of z. This implies (see formula (2.22)) that under a linear change $\widehat{f} = A\,f$ the column-function g' also changes linearly: $\widehat{g}' = (A^{-1})^T g'$.

Suppose now that the matrix-function B appearing in the definition of g'' (see (2.14) or (2.23)) changes according to the law $\widehat{B} = (A^{-1})^T \cdot B \cdot A^{-1}$. Then the whole column-function $g = g' + g''$ changes according to the law $\widehat{g} = (A^{-1})^T g$.

In particular, let $m \geqslant n$ and let $(u_1, \ldots, u_n) \subset u$ be local coordinates on some manifold. If with respect to these local coordinates f is a vector, a and the χ_k are functions, and B is a covariant tensor, then our g will be a covector.

Moreover, suppose that z are also local coordinates on some manifold, and that with respect to these local coordinates f, a, χ_k, B behave as functions. Then our g behaves as a function with respect to z. This fact follows from the invariancy of the operator ∂_f under changes of coordinates z.

2.5. Non-uniqueness of g. The n-component column-function g in formulae (2.1), (2.19), (2.25) is defined non-uniquely. One reason for this is that in the right-hand side of (2.3) we could have performed the factorization with respect to z in other ways, and another reason is that we can always add to g a nontrivial column-function which is orthogonal to f and has an infinite order zero at $z = 0$. However $g|_{z=0}$ is uniquely defined.

3. Proof of the main result

In this section we give the proof of Theorem 1.3.

An immediate consequence of the results of the previous section is

LEMMA 3.1. *Let $a(t, x; y, \eta) \in C^\infty((-\infty, +\infty) \times M \times T'M)$ be a function positively homogeneous in η of degree l, and let $\varphi \in \mathfrak{F}$. Then there exists a covector field*

$$\vec{g}\,(t, x; y, \eta) = \{g_1(t, x; y, \eta), \ldots, g_n(t, x; y, \eta)\} \in C^\infty((-\infty, +\infty) \times M \times T'M)$$

positively homogeneous in η of degree l such that

$$a(t, x; y, \eta) \;=\; a(t, x^*(t; y, \eta); y, \eta) \;+\; \langle \varphi_\eta(t, x; y, \eta), \vec{g}\,(t, x; y, \eta)\rangle\,. \tag{3.1}$$

Moreover, the covector field $\vec{g}$ can be constructed effectively in the form (2.13), (2.18), (2.20)–(2.23) with

$$z = x - x^*\,, \qquad f = \varphi_\eta\,, \qquad u = (t; y, \eta)\,, \tag{3.2}$$

some cut-off functions $\chi_k = \chi_k(t,x;y,\eta) \in C^\infty((-\infty,+\infty) \times M \times T'M)$ *positively homogeneous in* η *of degree 0 which are identically equal to 1 in some neighbourhoods of the set* $\mathfrak{C}$*, and an arbitrary positive Hermitian covariant tensor* $B = B(y) \in C^\infty(M)$.

The crucial point of Lemma 3.1 is the fact that the first term in the right-hand side of (3.1) is independent of x. Note also that in the formulation of this lemma the words "covector" and "covariant tensor" are used in relation to the coordinates y.

The expression for $\vec{g}$ given by (2.13), (2.18), (2.20)–(2.23), (3.2) possesses important invariancy properties: it behaves as a function (i.e., does not change) under changes of local coordinates x, and as a covector under changes of local coordinates y. (Here we assume that the cut-offs χ_k are functions in the full sense of the word, that is they are independent of the choice of local coordinates x and y.) These invariancy properties follow from the results of subsection 2.4. In Lemma 3.1 we placed an arrow over g to stress the fact that it is a covector field in the full sense of the word.

Obviously, φ_η behaves as a function under changes of local coordinates x, and as a vector under changes of local coordinates y. Consequently the expression $\langle \varphi_\eta, \vec{g} \rangle$ appearing in (3.1) is independent of the choice of local coordinates x and y, i.e. it is a function in the full sense of the word.

COROLLARY 3.2. *Suppose that the amplitude* $a(t,x;y,\eta)$ *in the oscillatory integral (1.4) is positively homogeneous in* η *of degree* l. *Then the oscillatory integral (1.4) coincides modulo* C^∞ *with an oscillatory integral with the same phase function and amplitude*

$$(3.3) \qquad \mathbf{a} \stackrel{\text{def}}{=} a^* + i\, d_\varphi^{-1} \operatorname{div}_\eta(d_\varphi \vec{g}\,) = a^* + i\, d_\varphi^{-1} \sum_{j=1}^{n} \partial_{\eta_j}(d_\varphi\, g_j),$$

where $a^* \equiv a(t,x^*(t;y,\eta);y,\eta)$, *and* $\vec{g}$ *is the covector field from Lemma 3.1.*

PROOF OF COROLLARY 3.2. Let us substitute (3.1) into the oscillatory integral (1.4), replace $\varphi_\eta\, e^{i\varphi}$ by $-i\,\nabla_\eta(e^{i\varphi})$ and integrate by parts with respect to η. This transforms (1.4) into

$$(3.4) \quad \mathcal{I}_{\varphi,a}(t,x,y) = \int e^{i\varphi(t,x;y,\eta)}\, \mathbf{a}(t,x;y,\eta)\, \varsigma(t,x;y,\eta)\, d_\varphi(t,x;y,\eta)\, đ\eta + \int e^{i\varphi(t,x;y,\eta)}\, b(t,x;y,\eta)\, d_\varphi(t,x;y,\eta)\, đ\eta\,,$$

where $\mathbf{a}$ is given by formula (3.3), and $b \stackrel{\text{def}}{=} i\langle \vec{g}, \nabla_\eta \varsigma\rangle$. Since for sufficiently large $|\eta|$ we have $\varsigma = 1$ in a neighbourhood of the set $\mathfrak{C}$, the function b is identically zero in a neighbourhood of $\mathfrak{C}$, consequently the second term in the right-hand side of (3.4) is a C^∞-half-density. □

It is easy to see that the expression $i\, d_\varphi^{-1} \operatorname{div}_\eta(d_\varphi \vec{g}\,)$ appearing in (3.3) is invariant under changes of local coordinates x and y. Note also that the two terms in the right-hand side of (3.3) have different degrees of homogeneity: a^* is positively homogeneous in η of degree l (as the original amplitude a), whereas $i\,d_\varphi^{-1}\,\mathrm{div}_\eta(d_\varphi\,\vec{g}\,)$ is positively homogeneous in η of degree $l-1$.

Corollary 3.2 allows us to eliminate the variable x from the leading homogeneous term of the amplitude because a^* depends only on $(t;y,\eta)$. This opens the way to the complete elimination of the variable x from the amplitude of an oscillatory integral.

THEOREM 3.3. *Any Lagrangian distribution (1.4) of order l associated with the canonical transformation (1.1) can be written modulo C^∞ in the form*

$$\mathcal{I}_{\varphi,\mathfrak{a}}(t,x,y) \;=\; \int e^{i\varphi(t,x;y,\eta)}\,\mathfrak{a}(t;y,\eta)\,\varsigma(t,x;y,\eta)\,d_\varphi(t,x;y,\eta)\,đ\eta \tag{3.5}$$

with an amplitude $\mathfrak{a}\in S^l$ independent of x.

PROOF OF THEOREM 3.3. Applying Corollary 3.2 to the leading (of degree l) homogeneous term of the amplitude a we represent (modulo C^∞) our original oscillatory integral as a sum of two — one with amplitude independent of x and positively homogeneous in η of degree l, and the other with amplitude dependent on x and of the class S^{l-1}. Treating the latter in a similar way and repeating this procedure infinitely many times we obtain a full expansion for the required amplitude $\mathfrak{a}(t;y,\eta)$ into homogeneous in η terms. $\square$

It remains to note that combining formulae (2.13), (2.15), (2.18), (2.20)–(2.22), (3.2), (3.3) and recalling that $c_k=(-1)^{k-1}$ we get the required formulae (1.5), (1.6). Theorem 1.3 is proved.

4. Case of a manifold with a boundary

The results of this paper admit a natural extension to the case when M is a manifold with a boundary under the additional assumptions that the canonical transformation (1.1) is generated by a Hamiltonian $h(x,\xi)$ and that this Hamiltonian is analytic with respect to ξ. In this situation one has to consider the Lagrangian distributions associated with all the real and complex reflected trajectories.

The relevant results for the case of a manifold with a boundary were sketched in [3], and are described in greater detail in Chapter 2 of [4].

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