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Long-time Behavior of Nonlinear Elastic Waves

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Introduction. The equations of motion for the displacement of an isotropic, homogeneous, hyperelastic material form a quasilinear hyperbolic system in three space dimensions. We shall consider the initial value problem in the whole space for smooth data of small amplitude ε. First we shall describe a simplified proof, appearing in [10], of John's *almost global* existence result [6], that the initial value problem possesses smooth solutions up to a time of order $\exp(A/\varepsilon)$. Within the class of hyperelastic materials, we further distinguish a subclass whose nonlinearities fulfill a *null condition* leading to global existence of small solutions, [11]. The null condition turns out to be complementary to the genuine nonlinearity condition of John for which there is formation of singularities in finite time in the spherically symmetric case, [4].

What makes the elasticity system different from the case of the scalar wave equation, for which such a scenario is known [1], [7], [8], [9], is the presence of two speeds of propagation in the linear equation which results in absence of Lorentz invariance. The proof of the existence results combines *generalized energy* estimates and decay estimates as in the scalar case. However, here only the nonrelativistic invariance of the equation under translation, rotation, and change of scale is available. Thus, new decay estimates are obtained in order to compensate. First, we show how the generalized energy gives decay of the local L^2-norm, and then use this to control the L^∞-norm. This can be done without appealing to the explicit formula for the fundamental solution.

We begin with a brief description of the PDE's. We then give a more complete statement of the results. Finally, in order to simplify the exposition, the main ideas will be illustrated in the scalar case.

The equations of motion. Assume that R^3 is filled by an elastic material. The basic unknown of the problem $\varphi(t,x)$ is a smooth deformation $\varphi(t,\cdot) : R^3 \to R^3$ giving the position at time $t \geq 0$ of a particle which is at position x in the reference configuration. For hyperelastic materials, there is a potential energy density given by a function σ, called the stored energy function, of the deformation gradient $\nabla\varphi \equiv F$. It is required that $F > 0$. Isotropy and homogeneity, in turn, imply that the stored energy function depends on F only through the principal invariants $\imath_1, \imath_2, \imath_3$ of the Cauchy-Green strain matrix $B = FF^T$. (That is, $\imath_k$ is the elementary symmetric function of degree k in the eigenvalues of B.) The PDE's are derived by applying Hamilton's principle to

$$\int\int \left[\tfrac{1}{2}\left|\partial_t\varphi\right|^2 - \sigma(\imath_1,\imath_2,\imath_3)\right] dxdt.$$

For more information see [2] or [3].

We will only consider small displacements from the reference configuration. Therefore, it is convenient to use the variables $u(t,x) = \varphi(t,x) - x$, $G = F - I$, and $C = B - I = G + G^T + GG^T$ instead of φ, F, and B. We write $\jmath_1,\jmath_2,\jmath_3$ for the invariants of C which can be expressed as linear functions of $\imath_1,\imath_2,\imath_3$. The resulting Euler-Lagrange equations have the form

$$\frac{\partial^2 u^i}{\partial t^2} - \sum_{i=1}^{3}\frac{\partial}{\partial x^j}\frac{\partial\sigma}{\partial G_{ij}} = 0, \quad i = 1,2,3.$$

In three space dimensions, the global existence of small amplitude solutions to nonlinear hyperbolic systems hinges on the specific form of the quadratic portion of the nonlinearity in relation to the linear part. Therefore, we will only compute the PDE's to second order in u. (The higher order terms could be included here, but since we are only considering small displacements they do not affect the existence results.) Expanding the stored energy function and keeping only the terms of order ≤ 3 in G, we have

$$\sigma = \sigma_0 + \sigma_1\jmath_1 + \tfrac{1}{2}\sigma_{11}\jmath_1^2 + \sigma_2\jmath_2 + \tfrac{1}{6}\sigma_{111}\jmath_1^3 + \sigma_{12}\jmath_1\jmath_2 + \sigma_3\jmath_3 + \dots,$$

the constants σ_0, σ_1, etc., standing for the partial derivatives of σ at $\jmath_k = 0$. The values of these parameters reflect the properties of the material under consideration.

We impose the conditions $\sigma_0 = \sigma_1 = 0$ which means that the reference configuration is a stress-free state. ¿From the linear theory, the Lamé constants $\lambda = 4(\sigma_{11}+\sigma_2)$ and $\mu = -2\sigma_2$ are assumed to be positive. This makes the equations hyperbolic. Setting $\lambda + 2\mu = c_1^2$ and $\mu = c_2^2$, the linear portion of the equation is given by

$$Lu = \partial_t^2 u - c_2^2\Delta u - (c_1^2 - c_2^2)\nabla(\nabla\cdot u). \tag{1}$$

The constants c_1 and c_2 correspond to the speeds of spherical and rotational waves, respectively. For example, if $u(t,x) = \nabla\phi(t,x)$, then $Lu = (\partial_t^2 - c_1^2\Delta)u$, while if $u(t,x) = \nabla\wedge A(t,x)$, then $Lu = (\partial_t^2 - c_2^2\Delta)u$.

The quadratic nonlinear terms have the form $N(u,u)$ with

$$N(u,v)^i = D^{ijk}_{\ell mn}\partial_\ell(\partial_m u^j\partial_n v^k), \quad \text{(summation convention)} \tag{2}$$

for certain constants $D^{ijk}_{\ell mn}$, which depend on the parameters $\sigma_{11},\sigma_2,\sigma_{111},\sigma_{12},\sigma_3$. The coefficients $D^{ijk}_{\ell mn}$ have important symmetries which are used in the energy estimates.

Here are the rather unpleasant explicit formulas:

$$\begin{aligned}
-D^{ijk}_{\ell mn} \;=\;& 4(\sigma_{111}+3\sigma_{12}+\sigma_3)[{}_1C^{ijk}_{\ell mn}] \\
&+2(\sigma_{11}-\sigma_{12}+\sigma_2-\sigma_3)[2({}_2C^{ijk}_{\ell mn})+{}_3C^{ijk}_{\ell mn}] \\
&-2(\sigma_{12}+\sigma_3)[2({}_4C^{ijk}_{\ell mn})+{}_5C^{ijk}_{\ell mn}] \\
&-2(\sigma_2-\sigma_3)[{}_6C^{ijk}_{\ell mn}+{}_7C^{ijk}_{\ell mn}+{}_8C^{ijk}_{\ell mn}] \\
&-\sigma_3[{}_9C^{ijk}_{\ell mn}],
\end{aligned}$$

with the ${}_pC^{ijk}_{\ell mn}$ expressed in terms of Kronecker δ's,

$$\begin{array}{ll}
{}_1C^{ijk}_{\ell mn} = \delta^i_\ell \delta^j_m \delta^k_n & {}_5C^{ijk}_{\ell mn} = \delta^i_\ell \delta^j_n \delta^k_m \\
{}_2C^{ijk}_{\ell mn} = \frac{1}{2}(\delta^{ij}\delta^k_n\delta_{\ell m}+\delta^{ik}\delta^j_m\delta_{\ell n}) & {}_6C^{ijk}_{\ell mn} = \frac{1}{2}(\delta^{ij}\delta^k_m\delta_{\ell n}+\delta^{ik}\delta^j_n\delta_{\ell m}) \\
{}_3C^{ijk}_{\ell mn} = \delta^i_\ell \delta^{jk}\delta_{mn} & {}_7C^{ijk}_{\ell mn} = \frac{1}{2}(\delta^{ij}\delta^k_\ell\delta_{mn}+\delta^{ik}\delta^j_\ell\delta_{mn}) \\
{}_4C^{ijk}_{\ell mn} = \frac{1}{2}(\delta^i_m\delta^j_\ell\delta^k_n+\delta^i_n\delta^j_m\delta^k_\ell) & {}_8C^{ijk}_{\ell mn} = \frac{1}{2}(\delta^i_m\delta^{jk}\delta_{\ell n}+\delta^i_n\delta^{jk}\delta_{\ell m})
\end{array}$$
$$ {}_9C^{ijk}_{\ell mn} = \frac{1}{2}(\delta^i_m\delta^j_n\delta^k_\ell+\delta^i_n\delta^j_\ell\delta^k_m) \quad .$$

The (truncated) equations of motion are then

$$Lu = N(u,u).$$

The interested reader will find more details [11].

The null condition is satisfied when

$$D^{ijk}_{\ell mn}x^\ell x^m x^n = 0.$$

This turns out to be true provided we take

$$\sigma_{11} = \sigma_{12} \quad \text{and} \quad 3\sigma_{12}+2\sigma_{111} = 0,$$

leaving only σ_3 free. Nevertheless, within the class of isotropic, hyperelastic materials, the null condition can be fulfilled.

In the spherically symmetric case, $u(t,x) = \dfrac{x}{r}\psi(t,r)$ (ψ a scalar), the null condition reduces to $3\sigma_{12}+2\sigma_{111} = 0$ which is the opposite of John's genuine nonlinearity condition.

Long-time behavior of solutions. We now state the results on almost global and global existence of small amplitude solutions. For simplicity we take smooth initial in the Schwartz class to avoid detailed statements about regularity and decay at infinity. For completeness, we include a statement of John's blow-up theorem where it is essential that the data have compact support.

Theorem. *(i) Almost Global Existence (John* [6], *Klainerman-Sideris* [10]*): Let $f, g \in \mathcal{S}(R^3)$. There exist $A > 0$, $\varepsilon_0 > 0$ such that for all $0 < \varepsilon < \varepsilon_0$ the initial value* *problem*

$$Lu = N(u,u)$$
$$u(0) = \varepsilon f, \quad \partial_t u(0) = \varepsilon g$$

with the notation (1), (2), has a unique classical solution defined on $[0, T_\varepsilon) \times R^3$ *for* $T_\varepsilon > \exp(A/\varepsilon)$.

(ii) Global Existence (Sideris [11]*): Let* $f, g = \Delta h \in \mathcal{S}(R^3)$. *Suppose that the null condition is satisfied:*

$$\sigma_{11} = \sigma_{12} \quad \text{and} \quad 3\sigma_{12} + 2\sigma_{111} = 0.$$

There exists an $\varepsilon_0 > 0$ *such that for all* $0 < \varepsilon < \varepsilon_0$ *the initial value problem*

$$Lu = N(u,u)$$
$$u(0) = \varepsilon f, \quad \partial_t u(0) = \varepsilon g$$

has a unique global classical solution.

(iii) Formation of Singularities (John [5]*): Let* $f, g \in C_0^\infty(R^3)$ *be spherically symmetric. Suppose that the genuine nonlinearity condition holds:*

$$3\sigma_{12} + 2\sigma_{111} > 0.$$

Then there exists a constant $A' > A > 0$ *such that the lifespan* T_ε *of the solution in (i) is bounded above:* $T_\varepsilon < \exp(A'/\varepsilon)$.

Remarks on the existence proofs. To illustrate the ideas behind the proof, it is enough to look at a simple scalar model:

$$\Box\, u \equiv \partial_t^2 u - \Delta u = \partial_1(\partial_2 u \partial_3 u) \equiv N(u,u). \tag{3}$$

The generators of translations, rotations, and changes of scale are given by

$$\Gamma : \begin{cases} \partial = (\partial_0, \dots, \partial_3) = (\partial_t, \nabla) \\ \Omega = (\Omega_1, \Omega_2, \Omega_3) = x \wedge \nabla \\ S = t\partial_t + x \cdot \nabla = t\partial_t + r\partial_r. \end{cases}$$

The ∂ and Ω commute with $\Box$, while $[S, \Box] = \Box$. With the notation $\Gamma^a = \Gamma_{a_1} \cdots \Gamma_{a_k}$ for a generic derivative of order $|a| = k$, we find for solutions of (3)

$$\Box\, \Gamma^a u = N^a(u,u), \tag{4}$$

in which $N^a(u,u)$ represents a sum of terms of the form $\partial_\ell(\partial_m \Gamma^b u \partial_n \Gamma^c u)$, with $|b| + |c| \le |a|$.

Solutions are constructed by controlling the generalized energy norm based on the Γ_i. First, define the usual energy

$$E_1(u(t)) = \tfrac{1}{2} \int_{R^3} [|\partial_t u(t,x)|^2 + |\nabla u(t,x)|^2] dx,$$

and then the higher order generalized energy

$$E_k(u(t)) = \sum_{|a|\leq k-1} E_1(\Gamma^a u(t)).$$

To estimate E_k, we proceed as usual: multiply (4) by $\partial_t \Gamma^a u$, integrate over R^3, and sum on $|a| \leq k$ to get

$$\frac{d}{dt} E_k(u(t)) = \sum_{|a|\leq k} \int_{R^3} N^a(u,u)\partial_t \Gamma^a u dx.$$

The terms on the right with derivatives of order $k+1$ are handled using integration by parts, pulling a time derivative outside the integral, and then absorbing them on the left, as is possible for small solutions. Applying Cauchy-Schwartz, the outcome is an inequality of the form

$$\frac{d}{dt} E_k(u(t)) \leq C \left(\sum_{\substack{|b|+|c|\leq k \\ |b|+2\leq k \\ |c|+1\leq k}} \|\partial \nabla \Gamma^b u \nabla \Gamma^c u\|_{L^2} \right) E_k(u(t))^{1/2}.$$

In a moment we will describe decay estimates which yield the bound

$$\frac{C}{1+t} E_k(u(t))$$

for the terms $\|\partial \nabla \Gamma^b u \nabla \Gamma^c u\|_{L^2}$. Once this is proven however, it follows that $E_k^{1/2}$ remains of order ε up to a time of order $\exp(A/\varepsilon)$.

Now we briefly discuss the two decay estimates we use. Let $\alpha(z) = (1+z^2)^{1/2}$. Then for small solutions of (3), we have for $k \geq 6$

$$\alpha(r)|\nabla \Gamma^a u(t,x)| \leq C E_k(u(t))^{1/2}, \quad |a|+3 \leq k \tag{5}$$

$$\alpha(r)\alpha(t-r)|\nabla \partial \Gamma^a u(t,x)| \leq C E_k(u(t))^{1/2}, \quad |a|+4 \leq k \tag{6}$$

$$\|\alpha(t-r)\partial \nabla \Gamma^a u(t,x)\|_{L^2(R^3)} \leq C E_k(u(t))^{1/2}, \quad |a|+2 \leq k. \tag{7}$$

The generalized energy of order k controls a weighted L^∞-norm of order $k-2$ in (5), (6) and a weighted L^2-norm of order k in (7). The weight $\alpha(t-r)$ gives decay away from the light cone $t=r$, and the factor $\alpha(r)$ gives decay away from the origin, $r > t/2$. Taken together, they give t^{-1} decay in the whole space.

The first of these (5) follows from an inequality of Sobolev type which gives

$$|\alpha(r)\nabla \Gamma^a u(t,x)| \leq C \sum_{|a|+1\leq k} \|\nabla \Gamma^a u(t,x)\|_{L^2}, \quad |a|+3 \leq k.$$

Note that the right-hand side is bounded by E_k. This also holds with the degenerate factor $\alpha(t-r)$ inserted on both sides. Thus, (6) can obtained from (7).

The local L^2 decay (7) is derived in several steps. One begins by showing pointwise bounds

$$(8) \qquad \alpha(t-r)[|\Delta u(t,x)| + |\partial_t \nabla u(t,x)|] \leq C\left[\sum_{|a|\leq 1} |\partial \Gamma^a u(t,x)| + t|\Box u(t,x)|\right].$$

This hold for arbitrary C^2 functions, and it is based on the algebraic properties of the operators Γ, Δ, and $\Box$. In fact, the argument is quite simple. A direct computation show that

$$(t^2-r^2)\Delta u = t\partial_t S u - r\partial_r S u - t\partial_t u + r\partial_r u - r^2(\Delta u - \partial_r^2 u) - t^2 \Box u,$$

and now observe that $\Delta u - \partial_r^2 u = \frac{2}{r}\partial_r + \frac{1}{r^2}\Omega^2 u$. This can also be applied to derivatives of u. (More work is needed in order to bound first derivatives.)

Next, by squaring and integrating (8) one gains control of $\|\alpha(t-r)\partial\nabla\Gamma^a u(t,x)\|_{L^2}$ in terms of $E_k^{1/2}(u(t))$ and $t\|\Box \Gamma^a u\|_{L^2}$, using the argument of Gårding's inequality. Only now do we use the equation. The nonlinear terms are interpolated between L^∞ and L^2, similar to what we do in the next paragraph to complete the derivation of energy inequality.

We now return to the energy inequality. Recall that we needed a bound for the terms

$$\|\partial\nabla\Gamma^b u \nabla\Gamma^c u\|_{L^2}.$$

Introduce the weight $\alpha(r)\alpha(t-r)$ and use the inequality $1 \leq \alpha(r)\alpha(t-r)/(1+t)$ to get the upper bound

$$(1+t)^{-1}\|\alpha(r)\alpha(t-r)\partial\nabla\Gamma^b u\nabla\Gamma^c u\|_{L^2}.$$

¿From here, the strategy is to interpolate between L^∞ and L^2, taking $\alpha(r)$ times the smaller derivative in L^∞. The factor $\alpha(t-r)$ goes with the term with two derivatives. Then apply the estimates (5), (6), (7).

We conclude with a few words about the null condition and global existence. Nonlinearities which satisfy the null condition have additional decay along the light cone. This comes from the decomposition

$$\nabla = \frac{x}{r}\partial_r - \frac{x}{r^2}\wedge\Omega,$$

in terms of radial and angular derivatives. Thus, if we replace the derivatives in $D^{ijk}_{\ell mn}\partial_\ell(\partial_m u^j \partial_n v^k)$ by this decomposition, the leading term with only radial derivatives vanishes precisely when $D^{ijk}_{\ell mn}x^\ell x^m x^n = 0$. This gives an enhanced decay rate of t^{-2} along the cone. On the other hand, we get the the same enhancement from the degenerate factor $\alpha(t-r)$ which can be introduced into *both* the L^∞- and the L^2-norms. The price of this is that we need (5), (6) for $\nabla\Gamma u$, now. This requires us to introduce the smoothing operator $|D|^{-1}$. This works out in the end thanks to the fact that the equation is in divergence form, however it forces us to use a nonlocal version of the energy, which explains the strange assumption about the initial data g.

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