

JOURNÉES ÉQUATIONS AUX DÉRIVÉES PARTIELLES

DANIEL GRIESER

DAVID JERISON

Asymptotics of the first nodal line

Journées Équations aux dérivées partielles (1995), p. 1-8

http://www.numdam.org/item?id=JEDP_1995____A2_0

© Journées Équations aux dérivées partielles, 1995, tous droits réservés.

L'accès aux archives de la revue « Journées Équations aux dérivées partielles » (<http://www.math.sciences.univ-nantes.fr/edpa/>) implique l'accord avec les conditions générales d'utilisation (<http://www.numdam.org/conditions>). Toute utilisation commerciale ou impression systématique est constitutive d'une infraction pénale. Toute copie ou impression de ce fichier doit contenir la présente mention de copyright.

NUMDAM

Article numérisé dans le cadre du programme
Numérisation de documents anciens mathématiques
<http://www.numdam.org/>

ASYMPTOTICS OF THE FIRST NODAL LINE

DANIEL GRIESER AND DAVID JERISON

May 1995

INTRODUCTION

In this note, we announce the result that the first nodal line of a convex planar domain tends to a straight line as the eccentricity tends to infinity.

Let Ω denote a bounded convex domain in \mathbf{R}^2 . Denote by u a second Dirichlet eigenfunction. Then u satisfies

$$\begin{aligned}\Delta u &= -\lambda_2 u && \text{in } \Omega \\ u &= 0 && \text{on } \partial\Omega\end{aligned}$$

where $0 < \lambda_1 < \lambda_2 \leq \lambda_3 \leq \dots$ represent the eigenvalues in increasing order. The first nodal line Λ is the zero set of u .

$$\Lambda = \{z \in \Omega : u(z) = 0\}$$

In order to state a precise theorem, let us normalize the region to lie within an $N \times 1$ rectangle. Let P_x and P_y denote the orthogonal projection on the x and y axes, respectively. First rotate so that the projection $P_y\Omega$ is *smallest*. Then dilate and translate so that $P_y\Omega = (0, 1)$ and $P_x\Omega = (0, N)$. (The choice of orientation of the y axis is crucial for what follows, but the dilation and translation are merely for notational convenience.)

Theorem 1. *With the normalization above, there is an absolute constant C such that*

$$\text{length } P_x\Lambda \leq C/N$$

Furthermore, this estimate is sharp. The case of a long, thin, circular sector shows that $C \geq 1/2$.

1991 *Mathematics Subject Classification.* Primary 35J25, 35B65; Secondary 35J05.

Key words and phrases. nodal line, convex.

Work of the first author partially supported by NSF grant DMS-9306389. Work of the second author partially supported by NSF grant DMS-9401355.

OUTLINE OF THE PROOF

Step 1 $O(1)$ estimate. With the normalization of Theorem 1,

$$\Omega = \{(x, y) : f_1(x) < y < f_2(x), 0 < x < N\}$$

where f_1 is a convex function and f_2 is a concave function. Define the height function $h(x) = f_2(x) - f_1(x)$. Consider the ordinary differential operator \mathcal{L} defined by

$$\mathcal{L} = -\frac{d^2}{dx^2} + \frac{\pi^2}{h(x)^2}$$

Recall the following theorem, which implies a weaker version of Theorem 1, namely, the same estimate without the factor $1/N$.

Theorem 2 [J3]. *With the normalization of Theorem 1, let ϕ_2 be the second eigenfunction for \mathcal{L} with Dirichlet boundary conditions on $[0, N]$. Let x_0 be the unique zero of ϕ_2 in $(0, N)$. There is an absolute constant A such that*

$$P_x \Lambda \subset [x_0 - A, x_0 + A]$$

This theorem says in a very crude sense that u resembles the function

$$\phi_2(x) \sin \ell_x(y)$$

where

$$\ell_x(y) = \frac{\pi(y - f_1(x))}{h(x)}.$$

The function $\ell_x(y)$ is chosen to be the linear function in y that has the value 0 on the bottom, $(x, f_1(x))$, of Ω and π on the top, $(x, f_2(x))$, of Ω . Thus, $\sin \ell_x(y)$ is the lowest Dirichlet eigenfunction for $-(d/dy)^2$ on the interval $f_1(x) \leq y \leq f_2(x)$ of length $h(x)$. (The fact that we have rotated so that h is as small as possible plays a crucial role.)

In addition to this estimate, we will need another consequence of [J3], expressed in terms of a parameter L defined as follows.

Definition. *The length scale L of Ω is the length of the rectangle R contained in Ω with the lowest (first) Dirichlet eigenvalue.*

Up to order of magnitude, L is the largest number such that $h(x) > 1 - 1/L^2$ on an interval of length L . When Ω is a rectangle, $R = \Omega$ and $L = N$. When Ω is a triangle of length N , then $L \approx N^{1/3}$. In general, $N^{1/3} \lesssim L \leq N$. The example of a trapezoid shows that all intermediate sizes for L are possible.

The heuristic principle behind L is that ϕ_2 resembles $\sin(2\pi(x - x_0)/L)$, the second eigenfunction of the interval $[x_0 - L/2, x_0 + L/2]$ and u resembles $\sin(2\pi(x - x_0)/L) \sin \pi y$, the second eigenfunction of the rectangle of length L and width 1 with nodal line at $x = x_0$. This is true to within order of magnitude near the ‘‘central’’ portion of Ω , with an exponential tail in the thin regions of Ω . More precisely we have,

Proposition. *Let $u_+(x, y) = \max\{u(x, y), 0\}$ and $u_-(x, y) = \max\{-u(x, y), 0\}$. Then*

$$\begin{aligned} u_+(x_0 + A + 1, 1/2) &\approx u_-(x_0 - A - 1, 1/2) \approx \max |u|/L \\ u_+(x_0 + L/20, 1/2) &\approx u_-(x_0 - L/20, 1/2) \approx \max |u| \end{aligned}$$

The proposition follows from the methods of [J3]. (See especially Proposition A of [J3].)

Step 2. Denote by

$$e(x, y) = (h(x)/2)^{-1/2} \sin \ell_x(y)$$

the first Dirichlet eigenfunction on $I_x = \{y : f_1(x) \leq y \leq f_2(x)\}$, normalized in $L^2(I_x)$. Denote

$$\psi(x) = \int_{f_1(x)}^{f_2(x)} u(x, y) e(x, y) dy$$

Then

$$u(x, y) = \psi(x) e(x, y) + v(x, y)$$

where $\psi(x)$ is the “best” coefficient possible and $v(x, y)$ should be a small error term. Because of Theorem 2, there exists a number x_1 , such that $|x_0 - x_1| < A$ and $\psi(x_1) = 0$.

Lemma 1. *$\psi'(x) \approx 1/L$ on $|x - x_0| \leq L/20$. In particular, ψ is strictly increasing and x_1 is the only zero of ψ on that interval.*

Lemma 2. *$|v(x, y)| \lesssim S/L$ where*

$$S = \max_{|x-x_1| \leq L/20} (|f_1'(x)| + |f_2'(x)|) e^{-c|x-x_1|} + e^{-cL}$$

The number S represents the slope of the boundary near x_1 plus the slope at a further distance decreased by an exponential factor. In the range $|x - x_0| \leq L/20$, $|f_1'(x)| + |f_2'(x)| \leq C/L^3$. The ideas of the proofs of Lemmas 1 and 2 will be presented in the next section. For now let us complete the outline of the proof of Theorem 1.

Step 3. If $u(x, 1/2) = 0$, then

$$\frac{1}{L} |x - x_1| \approx |\psi(x)| = |v(x, 1/2)|/e(x, 1/2) \lesssim S/L$$

Therefore,

$$|x - x_1| \lesssim S$$

Moreover,

$$S \lesssim 1/L^3 \lesssim 1/N$$

This is the end of the proof for points of the nodal line in the middle of Ω . Near the boundary $\partial\Omega$, the denominator $e(x, y)$ is small, so additional ideas are needed. One uses maximum principle and Hopf type estimates of [J1, J2, J3] and extra estimates on the rate of vanishing of $v(x, y)$ at the boundary.

PROOFS OF LEMMAS 1 AND 2

The idea of the proof of Lemma 1 is as follows. One calculates that

$$\mathcal{L}\psi - \lambda_2\psi = -\psi'' + \left(\frac{\pi^2}{h(x)^2} - \lambda_2 \right) \psi = \sigma$$

where σ is small and

$$\left| \lambda_2 - \frac{\pi^2}{h(x)^2} \right| \leq \frac{100}{L^2}$$

in the range $|x - x_0| \leq L/20$. Next one deduces from the proposition above that with the normalization $\max |u| = 1$,

$$\psi(x_0 + L/20) - \psi(x_0 - L/20) \approx 1$$

Then comparison with constant coefficient ordinary differential equations gives $\psi'(x) \approx 1/L$ for $|x - x_0| \leq L/20$.

The idea of the proof of Lemma 2 is to follow the Carleman method of differential inequalities. In that method, one considers a *harmonic* function, say w , in a region, say Ω , which vanishes on a portion of the boundary. Then one considers the function

$$f(x) = \int_{f_1(x)}^{f_2(x)} w(x, y)^2 dy$$

Using the equation $\Delta w = 0$, the zero boundary values, and integration by parts, one can find a differential inequality for f of the form $f''(x) \geq a(x)f(x)$. This convexity property makes it possible to deduce rates of vanishing for w .

To prove Lemma 2, one considers

$$g(x) = \int_{f_1(x)}^{f_2(x)} v(x, y)^2 dy,$$

and deduces a differential inequality of the form

$$g'' \geq 2 \left(\frac{(2\pi)^2}{h(x)^2} - \lambda_2 \right) g - \beta\sqrt{g} \geq g - \beta\sqrt{g}$$

The crucial point is that because we have subtracted the first eigenfunction in the y direction ($v = u - \psi(x)e(x, y)$), the coefficient on g involves $(2\pi)^2$ rather than π^2 . It follows that

$$g(x) \approx \frac{\cosh(x - x_1)}{\cosh(L/2)} + \beta \text{ dependence}$$

The first term is exponentially small and the second term is controlled by S , proving Lemma 2.

To illustrate the mechanism of the lemmas explicitly, we carry out a sine series computation in a special case. Note that the size and sign of $(k\pi)^2 - \lambda_2$ for $k = 1$ versus $k \geq 2$ is

at issue. We consider the special case in which $f_1(x) = 0$ and $f_2(x) = 1$ for $0 \leq x \leq N - 1$. Then $N - 1 \leq L \leq N$, so N and L are comparable. By comparison with rectangles of length N and $N - 1$ we find that

$$\pi^2 \left(1 + \frac{4}{N^2}\right) \leq \lambda_2 \leq \pi^2 \left(1 + \frac{4}{(N-1)^2}\right)$$

For $0 \leq x \leq N - 1$,

$$u(x, y) = \sum_{k=1}^{\infty} u_k(x) \sin(k\pi y)$$

where

$$u_k(x) = 2 \int_0^1 \sin(k\pi y) u(x, y) dy$$

Furthermore, the Fourier coefficient u_k satisfies

$$u_k''(x) + (\lambda_2 - (k\pi)^2)u_k = 0.$$

The function $\psi(x) = u_1(x)/\sqrt{2}$ and $\lambda_2 - \pi^2 \approx 1/N^2 \approx 1/L^2$. Thus

$$u_1(x) = -c_1 \sin \sqrt{\lambda_2 - \pi^2} x.$$

The coefficient satisfies $c_1 > 0$ because u is negative on the left half and positive on the right half of Ω . Normalize so that $\max u = 1$. By the proposition, u_{\pm} is large at $x_0 \pm L/20$, and hence c_1 is larger than a positive absolute constant. This yields Lemma 1, as well as the precise location of x_1 as a function of λ_2 .

On the other hand, the remaining terms of the series are small. For all $k > 1$, $\lambda_2 - k^2\pi^2 < -1$. Therefore,

$$u_k(x) = c_k \sinh \sqrt{(k\pi)^2 - \lambda_2} x$$

The unit bound on u implies

$$\sum_{k=1}^{\infty} u_k(x)^2 \leq 2$$

In particular,

$$\sum_{k=2}^{\infty} c_k^2 \sinh^2[\sqrt{(k\pi)^2 - \lambda_2}(N-1)] \leq 2$$

This implies that for $k \geq 2$,

$$|u_k(x)| \leq C e^{-kN} \quad \text{for} \quad |x - x_1| \leq N/10$$

Hence $v(x, y)$ is exponentially small, which proves Lemma 2 in the special case.

WHERE IS Λ ?

Recall that the nodal line may be in the exact middle ($x_1 = N/2$), as in the case where Ω is a rectangle, or it may be very near the fat end of the region, as in the case of a circular sector with vertex at the origin: $x_1 \approx N - cN^{1/3}$. Theorems 1 and 2 give numerical schemes for approximating the location of the nodal line as follows.

Recall that x_0 was defined above as the zero of the eigenfunction ϕ_2 . Since the second eigenvalue for \mathcal{L} on $[0, N]$ is the same as the first Dirichlet eigenvalue for the operator on the two intervals $[0, x_0]$ and $[x_0, N]$, Theorem 2 implies the following prescription.

ODE Eigenvalue Scheme. Choose x_0 to be the unique number such that the lowest Dirichlet eigenvalue for the operator \mathcal{L} on the intervals $[0, x_0]$ and $[x_0, N]$ are equal. Then

$$P_x \Lambda \subset [x_0 - A, x_0 + A]$$

The min-max principle implies that any curve dividing the region Ω into two halves with equal eigenvalues must intersect the nodal line. Theorem 1 implies that Λ is particularly close to a vertical straight line. This leads to the following prescription.

PDE Eigenvalue Scheme. Choose x_2 so that the least eigenvalues for the Dirichlet problem for the Laplace operator on the two regions

$$\Omega \cap \{(x, y) : x < x_2\} \quad \text{and} \quad \Omega \cap \{(x, y) : x > x_2\}$$

are equal. Then Theorem 1 implies

$$P_x \Lambda \subset [x_2 - C/N, x_2 + C/N]$$

The first scheme requires knowledge of the lowest eigenvalue of an ordinary differential equation, which is in standard numerical packages. The second scheme requires knowledge of the lowest eigenvalue on a convex domain, which is not quite as standard. Toby Driscoll [D] has recently developed a very effective program for computing both eigenfunctions and eigenvalues on polygons. Preliminary experiments with triangles with $3 \leq N \leq 150$ indicate that A in the first scheme may be $1/100 + 1/N$. (This seems too good to be true, but perhaps $A = 1/10 + 1/N$ will work in general.) The bound C/N in Scheme 2 seems to be $1/N$ as predicted by the case of a sector. We must confess, however, that the rigorous proofs of these bounds give ridiculous values like $C = 10^{20}$.

CONJECTURES

The methods outlined here should also give information about the size of the first eigenfunction, improving by a factor of \sqrt{L} the bounds given in [J3].

Conjecture 1. *With the normalizations on Ω of Theorem 1, let u_1 denote the first eigenfunction for Ω such that $\max |u_1| = 1$. Then there is an absolute constant C and a suitable multiple of the first eigenfunction ϕ_1 for the operator \mathcal{L} on $[0, N]$ satisfies*

$$|u_1(x, y) - \phi_1(x) \sin \ell_x(y)| \leq C/L$$

Conjecture 1 is motivated by the elementary inequality

$$|\sin(x/L) - \sin(x/(L+1))| \leq C/L \quad \text{on } 0 \leq x \leq \pi L$$

The methods used to prove Theorem 1 also show the following.

Corollary. *Let (x, y) be a point of Λ satisfying $1/4 \leq y \leq 3/4$, that is, far from $\partial\Omega$. Let η be a unit vector tangent to Λ at (x, y) . Then*

$$|\eta \cdot e_1| \leq CS \leq C/L^3$$

Conjecture 2. *The corollary is valid up to the boundary.*

(Conjecture 2 implies Theorem 1.)

Finally let us speculate about the higher-dimensional case. We begin by explaining the significance of L in another way. Let e be a unit vector and define

$$\Omega(t, e) = \{x + se : x \in \Omega, 0 \leq s < t\}$$

Thus $\Omega(t, e)$ is Ω stretched by t in the direction e . Define

$$P(e) = -\frac{d}{dt} \lambda_1(\Omega(t, e))|_{t=0}$$

This is the first variation of the lowest eigenvalue. It is analogous to the projection body function in the theory of convex bodies. (See [J4].) In a convex domain normalized as above,

$$P(e_1) \approx \min_e P \approx 1/L^3 \quad \text{and} \\ P(e_2) \approx \max_e P \approx 1$$

Moreover, the direction e_1 is necessarily within $1/L^3$ of the values of e for which the exact minimum is attained.

In \mathbf{R}^n , $n \geq 3$ one can define the same function P on the unit sphere. In the spirit of quadratic forms, choose v_1 so that

$$P(v_1) = \min P$$

Choose v_2 perpendicular to v_1 such that

$$P(v_2) = \min_{v \perp v_1} P(v)$$

(One can continue inductively to form an orthonormal basis v_1, v_2, \dots, v_n .)

Conjecture 3. *There is a dimensional constant C such that if v is a unit vector tangent to Λ , then*

$$|v \cdot v_1| \leq CP(v_1)/P(v_2)$$

This conjecture is intended to give specific bounds on the way the nodal set tends to a plane as the eccentricity tends to infinity. It is an analogous conjecture concerning the shape of the second eigenfunction to conjectures in [J4] concerning the shape of the first eigenfunction. One could also formulate even more detailed and even more speculative conjectures relating all the numbers $P(v_k)$ to the location of Λ .

REFERENCES

- [D] Toby Driscoll, personal communication: The program will appear in his Ph.D. thesis, Cornell U..
- [J1] D. Jerison, *The first nodal line of a convex planar domain*, Internat. Math. Research Notes vol 1 (1991), 1–5, (in Duke Math. J. vol. 62).
- [J2] ———, *The first nodal set of a convex domain*, Essays in Fourier Analysis in honor of E. M. Stein (C. Fefferman et al., ed.), Princeton U. Press, 1995, pp. 225–249.
- [J3] ———, *The diameter of the first nodal line of a convex domain*, Annals of Math. vol 141 (1995), 1–33.
- [J4] ———, *Eigenfunctions and harmonic functions in convex and concave domains*, Proceedings of ICM, Zurich 1994 (to appear).

DEPARTMENT OF MATHEMATICS, MASSACHUSETTS INSTITUTE OF TECHNOLOGY, CAMBRIDGE, MA
02139-4307