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# BOUNDARY VALUES OF COHOMOLOGY CLASSES AS HYPERFUNCTIONS

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*Summary*

The lecture outlines the contents of the article Cordaro–Gindikin–Treves [1] which purports to formalize in the framework of hyperfunction theory the concept of the boundary value of a cohomology class (with coefficients in the sheaf of germs of holomorphic functions) propounded in the works [1], [2], [3] of Gindikin. In the article [1] of Cordaro–Gindikin–Treves the hyperfunctions are defined on a maximally real submanifold of complex space (and more generally on a hypo–analytic manifold). The formalization is facilitated by the treatement of hyperfunctions and of the boundary values of holomorphic functions in the recent monograph Cordaro–Treves [1]. In order to avoid technicalities that would obscure the overall picture, the present lecture will deal only with hyperfunctions in Euclidean space $\mathbb{R}^n$.

We begin by recalling some known facts about boundary values of a holomorphic function $f$ in a *wedge* with *edge* on $\mathbb{R}^n$. Consider an open set $\Omega \subset \mathbb{R}^n$ and an open cone $\Gamma \subset \mathbb{R}^n \backslash \{0\}$ (thus $x \in \Gamma$, $\lambda > 0 \Rightarrow \lambda x \in \Gamma$); it is convenient to assume that both $\Omega$ and $\Gamma$ are connected. The reader may think of a wedge as a tuboid $\Omega + \imath\Gamma$; or else, more precisely, as the cutoff of such a tuboid in the imaginary direction:

$$\mathscr{W}_\delta(\Omega,\Gamma) = \{ z \in \Omega + \imath\Gamma;\ |\mathscr{Im}\, z| < \delta(\mathscr{Re}\, z) \},$$

where $\delta(x)$ is a given continuous function in $\Omega$, $\delta > 0$; $\Omega$ regarded as a subset of $\mathbb{C}^n$ is called the edge of the wedge $\mathscr{W}_\delta(\Omega,\Gamma)$.

Suppose $f \in \mathcal{O}(\mathscr{W}_\delta(\Omega,\Gamma))$ and select at random a vector $\gamma \in \Gamma$, such, say, that $|\gamma| = 1$. Given any $\varphi \in C^\infty_{\text{comp}}(\Omega)$ we can form the integral

$$I_\gamma(t;f,\varphi) = \int_{\mathbb{R}^n} \varphi(x) f(x+\imath t\gamma)\mathrm{d}x,$$

provided $t > 0$ is sufficiently small ($t < \delta(x)$ if $x \in \text{supp}\ \varphi$). Furthermore suppose that, given any compact set $K \subset \Omega \cup \mathscr{W}_\delta(\Omega,\Gamma)$, there are constants $k \in \mathbb{Z}_+$, $C_K > 0$ such that

(1) $$|f(z)| \leq C_K |\mathscr{Im}\, z|^{-k},\ \forall\, z \in K \cap \mathscr{W}_\delta(\Omega,\Gamma).$$

It is then easily seen that $\lim_{t \to +0} I_\gamma(t;f,\varphi)$ exists: it suffices to note that $f(x+\imath t\gamma) = (\partial/\partial t)^{k+1} g(x+\imath t\gamma)$ with $g \in \mathcal{O}(\mathscr{W}_\delta(\Omega,\Gamma))$ and $t \to g(x+\imath t\gamma)$ continuous in the semiclosed interval $[0,\delta(x)[$. But $(\partial/\partial t)^{k+1} g(x+\imath t\gamma) = (\imath\gamma\cdot\partial/\partial x)^{k+1} g(x+\imath t\gamma)$, hence, as $t \to +0$,

$$I_\gamma(t;f,\varphi) = \int_{\mathbb{R}^n} [(-\imath\gamma\cdot\partial/\partial x)^{k+1}\varphi(x)] g(x+\imath t\gamma)\mathrm{d}x \to$$

$$\int_{\mathbb{R}^n} [(-\imath\gamma\cdot\partial/\partial x)^{k+1}\varphi(x)] g(x)\mathrm{d}x = \langle bv_\Omega f, \varphi \rangle.$$

Thus does the boundary value of $f$ define a distribution $bv_\Omega f$ in $\Omega$.

The constraint (1) creates unnecessary difficulties. If we were to limit our attention to distribution boundary values we would be forced to deal only with Dolbeault forms (see below) whose growth at the edge is tempered, in the sense of (1). For instance we would be asked to find solutions of this type to the Cauchy–Riemann equations in wedges $\mathcal{W}_\delta(\Omega,\Gamma)$, which is more technical than just finding solutions with unrestricted growth. It should be also said that some theorems, foremost the theorem of the Edge of the Wedge (see below), are more general when the condition (1) is absent. [For other results the version without (1) is weaker!] For our present purpose it is definitely advantageous to remove (1). To do this I shall recall the definition of hyperfunction boundary value.

A continuous linear functional $\mu\colon \mathcal{O}(\mathbb{C}^n) \to \mathbb{C}$ is called an *analytic functional* (in $\mathbb{C}^n$). The space of entire functions $\mathcal{O}(\mathbb{C}^n)$ is equipped with the topology of uniform convergence on compact sets; we shall denote by $\mathcal{O}'(\mathbb{C}^n)$ the space of analytic functionals. One says that $\mu \in \mathcal{O}'(\mathbb{C}^n)$ is *carried* by a compact set $K \subset \mathbb{C}^n$ if, given any $\epsilon > 0$, there is $C_\epsilon > 0$ such that

$$(2) \qquad |\langle\mu,h\rangle| \leq C_\epsilon \operatorname*{Max}_{K_\epsilon} |h|,$$

where $K_\epsilon = \{\, z \in \mathbb{C}^n;\ \mathrm{dist}(z,K) < \epsilon \,\}$. Below we write $\mu \in \mathcal{O}'(K)$ if an analytic functional $\mu$ in $\mathbb{C}^n$ is carried by a compact set $K \subset \mathbb{C}^n$. It is not true that, for any pair of compact subsets $K_1$ and $K_2$ of $\mathbb{C}^m$,

$$(3) \qquad \mathcal{O}'(K_1) \cap \mathcal{O}'(K_1) = \mathcal{O}'(K_1 \cap K_2).$$

It is possible to have $K_1 \cap K_2 = \emptyset$ and yet $\mathcal{O}'(K_1) \cap \mathcal{O}'(K_1) \neq \{0\}$: for instance the Cauchy formula shows that the Dirac distribution in $\mathbb{C}$, $h \to h(0)$, is carried by the circle $\{ z;\ |z| = r \}$ whatever $r > 0$. However, and this is of the foremost importance for us, (3) is valid if $K_1$ and $K_2$ are subsets of *real space* $\mathbb{R}^n$. In this case we can talk of the *support* of $\mu$ (in $\mathbb{R}^n$), which we define to be the intersection of all the compact subsets of $\mathbb{R}^n$ which carry $\mu$. This property of real space follows from the fact that every compact subset K of $\mathbb{R}^n$ is polynomially convex, ie.,

$$K = \{ z \in \mathbb{C}^n;\ \forall\ h \in \mathcal{O}(\mathbb{C}^n),\ |h(z)| \leq \operatorname*{Max}_K |h| \}.$$

**Proof**: For $\zeta = \xi + \imath\eta \notin K$, take $h(z) = \exp\left[-\sum_{j=1}^{n} (z_j - \xi_j)^2\right]$. For all $x \in \mathbb{R}^n$, $0 < h(x) \leq 1$; and if $\xi \notin K$, $\operatorname*{Max}_K |h| < 1$. But $h(\zeta) = \exp(|\eta|^2)$. □

The polynomial convexity of the compact subsets of $\mathbb{R}^n$ has also the consequence that, if $K_j \subset\subset \mathbb{R}^n$ $(j = 1,2)$,

$$\mathcal{O}'(K_1 \cup K_2) = \mathcal{O}'(K_1) + \mathcal{O}'(K_2). \tag{4}$$

Finally we point out that if $K \subset\subset \mathbb{R}^n$, $\mathcal{O}'(K)$ carries a natural Fréchet space structure.

Properties (3) and (4) are all that is needed to develop the concept of a hyperfunction according to Martineau [1]. First let U be an open and bounded subset of $\mathbb{R}^n$. The hyperfunctions in U are the elements of the quotient linear space

$$\mathcal{B}(U) = \mathcal{O}'(\overline{U})/\mathcal{O}'(\partial U). \tag{5}$$

Here $\partial U$ is the boundary of U in $\mathbb{R}^n$. If $V \subset U$ is also open it follows from (4) that any $\mu \in \mathcal{O}'(\overline{U})$ can be decomposed as $\mu = \mu' + \mu''$, with $\mu' \in \mathcal{O}'(\overline{V})$, $\mu'' \in \mathcal{O}'(\overline{U\backslash V})$. If we also have $\mu = \nu' + \nu''$, $\nu' \in \mathcal{O}'(\overline{V})$, $\nu'' \in \mathcal{O}'(\overline{U\backslash V})$, then $\mu' - \nu' = \nu'' - \mu''$ is carried by $\overline{V}$ as well as by $\overline{U\backslash V}$, and therefore, accroding to (3), $\mu' - \nu' \in \mathcal{O}'(\partial V)$. In other words, the coset $[\mu']$ of $\mu'$ is unambiguously defined; it is taken to be the *restriction* of $[\mu]$ to V. This defines the restriction map $r_V^U$: $\mathcal{B}(U) \to \mathcal{B}(V)$, whence a presheaf, whence a sheaf, the sheaf $\mathscr{B}$ of hyperfunctions in $\mathbb{R}^n$. If now $\Omega$ is any open subset of $\mathbb{R}^n$, the continuous sections of the sheaf $\mathscr{B}$ over $\Omega$ are the hyperfunctions in $\Omega$. When $\Omega$ is bounded this is consistent with Definition (5). It is readily seen that the sheaf $\mathscr{B}$ is *flabby*, ie, every hyperfunction in $\Omega$ extends as a hyperfunction in $\mathbb{R}^n$ (whereas not every distribution in $\Omega \neq \mathbb{R}^n$ extends as a distribution in $\mathbb{R}^n$!). There is a natural linear injection $\mathcal{D}'(\Omega) \to \mathcal{B}(\Omega)$; however there is no natural Haus–dorff topology on the vector space $\mathcal{B}(\Omega)$ [due to the fact that $\mathcal{O}'(\partial U)$ is dense in $\mathcal{O}'(\overline{U})$].

Suppose now that the boundary of the open set $U \subset\subset \Omega$ is smooth, and let $f \in \mathcal{O}(\mathcal{W}_\delta(\Omega,\Gamma))$, $\gamma \in \Gamma \cap S^{n-1}$, as in the beginning. For any $h$ in $\mathcal{O}(\mathbb{C}^n)$ define

$$< \mu_{f,U}^{\gamma}(t),h > = \int_{U+\imath t\gamma} h(z)f(z)\mathrm{d}z$$

($\mathrm{d}z = \mathrm{d}z_1\wedge\cdots\wedge\mathrm{d}z_n$, $0 < t < \inf_U \delta$). It is obvious that $\mu_{f,U}^{\gamma}(t)$ can be regarded as an analytic functional carried by $\overline{U}+\imath t\gamma$. The following result is proved in Ye [1] (it is inspired by an argument of Hörmander [1]).

THEOREM 1.– *There is* $\mu_{f,U}^{\gamma} \in \mathcal{O}'(\overline{U})$ *such that the following is true:*

*To every open neighborhood* $\mathcal{N}$ *of* $\partial U$ *in* $\mathbb{C}^n$ *there is* $\varepsilon > 0$ *such that, if* $0 < t < \varepsilon$

*then* $\mu^{\gamma}_{f,\mathrm{U}} - \mu^{\gamma}_{f,\mathrm{U}}(t)$ *is carried by* $\mathcal{T}$.

If we change $\gamma' \in \Gamma \cap S^{n-1}$, $\gamma' \neq \gamma$, Stokes' theorem implies at once that $\mu^{\gamma}_{f,\mathrm{U}} - \mu^{\gamma'}_{f,\mathrm{U}} \in \mathcal{O}'(\partial \mathrm{U})$. Thus the coset of $\mu^{\gamma}_{f,\mathrm{U}}$ in $\mathcal{B}(\mathrm{U})$ is independent of $\gamma$. We shall call it the *hyperfunction boundary value* of $f$ in U and denote it by $bv_{\mathrm{U}}f$; if $\mathrm{V} \subset \mathrm{U}$ (with $\partial\mathrm{V}$ smooth) we have $bv_{\mathrm{V}}f = r^{\mathrm{U}}_{\mathrm{V}}(bv_{\mathrm{U}}f)$. We can let U expand to fill $\Omega$ and thus define $bv_{\Omega}f$.

THEOREM 2.— *The boundary value map*

$$\mathcal{O}(\mathcal{W}_{\delta}(\Omega,\Gamma)) \ni f \to bv_{\Omega}f \in \mathcal{B}(\Omega)$$

*is a linear injection.*

**Proof**: Let the open set $\mathrm{U} \subset\subset \Omega$ be connected and have a smooth boundary $\partial\mathrm{U}$. Suppose that $bv_{\mathrm{U}}f \equiv 0$; this means that $\mu^{\gamma}_{f,\mathrm{U}} \in \mathcal{O}'(\partial\mathrm{U})$ and therefore, given any compact neighborhood K of $\partial\mathrm{V}$ in $\mathbb{C}^n$, provided $t$ is sufficiently small,

$$\left| \int_{\mathrm{U}+\imath t\gamma} h(z) f(z) \mathrm{d}z \right| \leq C \sup_{\mathrm{K}} |h|. \tag{6}$$

Take $h(z) = (\nu/\pi)^{n/2}\exp\left[-\nu \sum_{j=1}^{n} (z_j - \xi_j - \imath t\gamma_j)^2\right]$ with $\xi \in \mathrm{U}$ away from K. As $\nu \to +\infty$ the left–hand side converges to $|f(\xi+\imath t\gamma)|$. The right–hand side is dominated by

$$C(\nu/\pi)^{n/2} \operatorname*{Max}_{\mathrm{K}} \exp(-\nu|z-\xi|^2 + \nu t^2).$$

If $t << \mathrm{dist}(\xi,\mathrm{K})$ it will converge to zero, implying that $f \equiv 0$ in an open subset of the connected wedge $\mathcal{W}_{\delta}(\mathrm{U},\Gamma)$. □

Of course the map (6) is *not* surjective: no hyperfunction which vanishes identically in an open subset of $\Omega$ can be the boundary value of a function $f \in \mathcal{O}(\mathcal{W}_\delta(\Omega,\Gamma))$ since $\mathcal{W}_\delta(\Omega,\Gamma)$ is connected. But suppose that a set of open and convex cones $\Gamma_j$ $(j = 1,\dots,r)$ has the following property:

$$\Gamma_1^o \cup \cdots \cup \Gamma_r^o = \mathbb{R}^n, \tag{7}$$

where $\Gamma_j^o = \{ \xi \in \mathbb{R}^n;\ \forall\ x \in \Gamma_j,\ \xi\cdot x \geq 0 \}$. Then, given any hyperfunction $u$ in $\mathbb{R}^n$, every point $x_0 \in \mathbb{R}^n$ has an open neighborhood U in which

$$u = \sum_{j=1}^{r} bv_U f_j,\quad f_j \in \mathcal{O}(\mathcal{W}_\delta(\Omega,\Gamma_j)). \tag{8}$$

Formula (8) is easily verified when $u$ is a distribution, which obviously can be assumed to have compact support. For the Fourier inversion formula and (7) allow us to write $u = \sum_{j=1}^{r} u_j$, with

$$u_j(x) = (2\pi)^{-n}\int_{A_j} e^{ix\cdot\xi}\hat{u}(\xi)d\xi$$

where $A_j \subset \Gamma_j^o$ is a Borel set. But for $y \in \Gamma_j$ the oscillatory integral

$$u_j(x+iy) = (2\pi)^{-n}\int_{A_j} e^{ix\cdot\xi-y\cdot\xi}\hat{u}(\xi)d\xi$$

defines a holomorphic function in $\mathbb{R}^n+i\Gamma_j$.

If we forget about Condition (7) we may introduce the following definition (Sato 1969):

DEFINITION 1.— *The hyperfunction u is said to be* **(microlocally) analytic** *at the point* $(x_0,\xi^0) \in \mathbb{R}^n\times(\mathbb{R}^n\setminus\{0\})$ *if the cones* $\Gamma_j$ $(j = 1,\dots,r)$ *can be chosen in the open half-space* $\{ x \in \mathbb{R}^n;\ x\cdot\xi_0 < 0 \}$ *in such a way that* (8) *is valid for some open neighborhood* U *of* $x_0$. *We shall refer to the complement of the set of points at which u is analytic as the* **analytic wave-front set** *of u and we shall denote it by* $\mathrm{WF}_a(u)$.

Other names for $\mathrm{WF}_a(u)$ are the *essential singular support* of u, the *essential spectrum* of *u*, the *microsupport* of *u*. The invariant interpretation of the space in which $(x,\xi)$ vary is of course phase space, ie, the cotangent bundle of $\mathbb{R}^n$ (from which the zero section has been deleted).

We are now going to introduce an alternate definition of hyperfunctions. In what follows $\mathcal{U}$ will denote an open subset of $\mathbb{C}^n$ and $C^\infty(\mathcal{U};\Lambda^{p,q})$ the space of differential forms

$$f = \sum_{|I|=p}\ \sum_{|J|=q} f_{I,J}\, dz_I\wedge d\bar{z}_J,\quad f_{I,J} \in C^\infty(\mathcal{U}).$$

We are using the multi-index notation: $I = \{i_1,\dots,i_p\}$ with $1 \leq i_1 <\cdots< i_p \leq n$, $dz_I = dz_{i_1}\wedge\cdots dz_{i_p}$; p is the *length* of I, and likewise for J and $d\bar{z}_J$. We have

$$\bar\partial f = \sum_{|I|=p}\ \sum_{|J|=q}\ \sum_{\ell=1}^{n} (\partial f_{I,J}/\partial\bar{z}_\ell)\, d\bar{z}_\ell\wedge dz_I\wedge d\bar{z}_J.$$

For each $p = 0,1,\dots,n$, we obtain the *Dolbeault complex*

$$\bar\partial\colon C^\infty(\mathcal{U};\Lambda^{p,q}) \to C^\infty(\mathcal{U};\Lambda^{p,q+1}),\ q = 0,1,\dots,n.$$

We shall denote by $H^{p,q}(\mathcal{U})$ its $q^{th}$ cohomology space. Below we shall make constant use of the fact that if $f \in C^\infty(\mathcal{U};\Lambda^{n,q})$ then $\bar\partial f = df$.

Consider now a compact set $K \subset \mathbb{R}^n$ and a form $f \in C^\infty(\mathbb{C}^n\setminus K;\Lambda^{n,n-1})$ such that $df = \bar\partial f = 0$. Let $\mathcal{D}$ be an open subset of $\mathbb{C}^n$ such that $K \subset \mathcal{D}$, whose boundary is a $C^\infty$

hypersurface $\Sigma$. Then $h \to \int_\Sigma hf$ defines an analytic functional $\mu_f$ carried by $\Sigma$. By Stokes' theorem $\Sigma$ can be contracted about K as much as we wish. It follows that $\mu_f$ is carried by K. On the other hand, since $\Sigma$ has no boundary, replacing $f$ by $f + dv$, $v \in C^\infty(\mathbb{C}^n\backslash K;\Lambda^{n,n-2})$, does not modify $\mu_f$, which means that $\mu_f$ is really associated to the Dolbeault cohomology class $[f]$ of $f$. Let us therefore write $\mu_{[f]}$. We obtain a linear map

$$H^{n,n-1}(\mathbb{C}^n\backslash K) \ni [f] \to \mu_{[f]} \in \mathcal{O}'(K). \tag{9}$$

First suppose n = 1. In this case $H^{1,0}(\mathbb{C}^n\backslash K)$ is the space of forms $\varphi(z)dz$ with $\varphi \in \mathcal{O}(\mathbb{C}\backslash K)$. If we recall that $\mathbb{C}\backslash K$ is connected (hence K is Runge) it is easy to construct a right–inverse of the map (9), namely $\mu \to \Gamma\mu\, dz$, where

$$\Gamma\mu = (2i\pi)^{-1}<\mu_w,(z-w)^{-1}>$$

is the *Cauchy transform* of $\mu$. Notice that $\Gamma$ maps $\mathcal{O}'(K)$ into the space $\mathcal{O}_0(\mathbb{C}\backslash K)$ of holomorphic functions in $\mathbb{C}\backslash K$ that vanish at infinity. Whatever $h \in \mathcal{O}(\mathbb{C})$,

$$< \mu,h > = \int_\Sigma h(z)\Gamma\mu(z)dz,$$

where $\Sigma$ is any smooth, closed curve in $\mathbb{C}\backslash K$ winding (once) around K. Laurent expansion gives a natural isomorphism $\mathcal{O}_0(\mathbb{C}\backslash K) \cong \mathcal{O}(\mathbb{C}\backslash K)/\mathcal{O}(\mathbb{C})$.

Now suppose n $\geq$ 2. We introduce the *Bochner–Martinelli current*,

$$E(z) = c_n \sum_{j=1}^{n} (-1)^{j-1} \frac{\bar{z}_j}{|z|^{2n}} dz_1\wedge\cdots\wedge dz_n\wedge d\bar{z}_1\wedge\cdots\wedge d\bar{z}_{j-1}\wedge d\bar{z}_{j+1}\wedge\cdots\wedge d\bar{z}_n.$$

We have

$$dE(z) = (-1)^n c_n \sum_{j=1}^{n} \frac{\partial}{\partial\bar{z}_j}\left(\frac{\bar{z}_j}{|z|^{2n}}\right) dz_1\wedge\cdots\wedge dz_n\wedge d\bar{z}_1\wedge\cdots\wedge d\bar{z}_n$$

and

$$\sum_{j=1}^{n}\frac{\partial}{\partial\bar z_j}\left(\frac{\bar z_j}{|z|^{2n}}\right)=-(n-1)\sum_{j=1}^{n}\frac{\partial}{\partial\bar z_j}\frac{\partial}{\partial z_j}\left(\frac{1}{|z|^{2n-2}}\right)=-\frac{n-1}{4}\Delta\left(\frac{1}{r^{2n-2}}\right),$$

and we select the constant $c_n$ in such a way that $(-1)^{n-1}\frac{n-1}{4}c_n\Delta(\frac{1}{r^{2n-2}})=\delta$. Thus

$$(10)\qquad dE=\delta dz_1\wedge\cdots\wedge dz_n\wedge d\bar z_1\wedge\cdots\wedge d\bar z_n.$$

From this and from Stokes' theorem it follows at once that, if $\Sigma$ is the smooth boundary of the domain $\mathcal{D}\ni w$, then, for any $h\in\mathcal{O}(\mathbb{C}^n)$,

$$h(w)=\int_\Sigma h(z)E(z-w).$$

Let now $K\subset\mathcal{D}\cap\mathbb{R}^n$ be compact and connected. We observe that, for any $z\in\Sigma$, $w\to\frac{\bar z_j-w_j}{|z-w|^{2n}}$ admits a holomorphic extension to an open (and connected) neighborhood $\mathcal{U}$ of K in $\mathbb{C}^n$: just extend $|z-w|^2$ as the function $\sum_{k=1}^{n}[(x_k-w_k)^2+y_k^2]$. This extend $E(z-w)$ as an (n,n–1)–form $\tilde E(z,w)$ in $z$–space whose coefficients are $C^\infty$ functions of $(z,w)$, holomorphic with respect to $w$, in $(\mathbb{C}^n\backslash\mathcal{U}')\times\mathcal{U}$. Here $\mathcal{U}'$ is a neighborhood of $\bar{\mathcal{U}}$ contained in $\mathcal{D}$; obviously $h(w)=\int_\Sigma h(z)\tilde E(z,w)$ in $\mathcal{U}$. If $\mu\in\mathcal{O}'(K)$ we may write

$$<\mu,h>=\int_\Sigma h(z)\tilde\mu(z).$$

Since we are free to contract the domain $\mathcal{D}$ about K as much as we wish provided we contract $\mathcal{U}$ correspondingly, we have

$$\tilde\mu=<\mu_w,\tilde E(z,w)>\in C^\infty(\mathbb{C}^n\backslash K;\Lambda^{n,n-1}).$$

In $\mathbb{C}^n\backslash\bar{\mathcal{U}}$ we have $\bar\partial\tilde\mu=<\mu_w,\bar\partial\tilde E(z,w)>=0$: to $\mu\in\mathcal{O}'(K)$ we have assigned a class $[\tilde\mu]$

$\in H^{n,n-1}(\mathbb{C}^n\backslash K)$. The very manner in which we have constructed the class $[\tilde{\mu}]$ shows that the map $\mu \to [\tilde{\mu}]$ is the inverse of the map (9), which thereby has been proved to be a bijection. [It is an easy exercise to derive Properties (3) and (4) from the isomorphisms we have established.]

We can sligthly re–interpret what was just said to confirm the analogy with the picture in Theorems 1 & 2: Here the cone $\Gamma$ is taken to be $\mathbb{R}^n\backslash\{0\}$; suppose $K = \overline{U}$, with U open in $\mathbb{R}^n$. The preceding argument shows that, if $f \in C^\infty(\mathbb{C}^n\backslash K;\Lambda^{n,n-1})$ is closed, then the analytic functionals $\mu_{f,U}(t)$ defined by

$$< \mu_{f,U}(t),h > = \int_{U+\imath t S^{n-1}} hf$$

"converge" (in the sense of Theorem 1) to $\mu_{f,U} \in \mathcal{O}'(\overline{U})$ whose coset modulo $\mathcal{O}'(\partial U)$ only depends on the class $[f] \in H^{n,n-1}(\mathbb{C}^n\backslash K)$. That coset can be denoted $bv_U[f]$ and called the boundary value of $[f]$ in U. The following statement has already been essentially proved:

THEOREM 3.– *Let* U *be an open and bounded subset of* $\mathbb{R}^n$. *The linear map*

(11) $$H^{n,n-1}(\mathbb{C}^n\backslash\overline{U}) \ni [f] \to bv_U[f] \in \mathcal{B}(U).$$

*is bijective.*

[When $n = 1$ the cohomology space at the left in (11) must be interpreted as the quotient space $\mathcal{O}(\mathbb{C}\backslash\overline{U})/\mathcal{O}(\mathbb{C}) \cong \mathcal{O}_0(\mathbb{C}\backslash\overline{U})$.]

In summary we have seen that we can define boundary maps from the cohomology spaces $H^{n,q}(\Omega+i\Gamma)$ into $\mathcal{B}(\Omega)$ when $q = 0$, in which case the cone $\Gamma$ can be taken to be convex; and when $q = n-1$, in which case $\Gamma = \mathbb{R}^n\setminus\{0\}$. When $q = 0$ we obtain a "small" subspace of $\mathcal{B}(\Omega)$, whereas when $q = n-1$, we obtain all of $\mathcal{B}(\Omega)$. In both cases the boundary value map is injective.

We shall now tackle the cases $1 \leq q \leq n-2$ (henceforth $n \geq 3$). The definition of the boundary value map is a natural extension of the ones in the cases $q = 0, n-1$. As before let $\Gamma \subset \mathbb{R}^n\setminus\{0\}$ be an open and connected cone and let $c$ be a Lipschitz q–cycle in $\Gamma$, ie., the image of a Lipschitz map $j_c\colon S^q \to \Gamma$. We are going to integrate a $C^\infty$ differential form $\omega$ of degree q in $\Gamma$ over c: this means that we integrate over $S^q$ the pullback $j_c^*\omega$. Let then $U \subset \Omega \subset \mathbb{R}^n$ be as before and consider a closed Dolbeault form $f \in C^\infty(\mathcal{W}_\delta(\Omega,\Gamma);\Lambda^{n,q})$. We can define the analytic functionals

$$\mathcal{O}(\mathbb{C}^n) \ni h \to < \mu^c_{f,U}(t),h > = \int_{U+itc} hf.$$

The analogue of Theorem 1 is valid here. As a matter of fact, the statement for q arbitrary can be deduced from that when $q = 0$ by taking

$$\mu^c_{f,U} = \int_c \mu^\gamma_{f,U} d\gamma,$$

where $d\gamma$ stands for an appropriate measure on c. At any rate, by reasoning directly, we get an analytic functional $\mu^c_{f,U} \in \mathcal{O}'(\overline{U})$ such that $\mu^c_{f,U} - \mu^c_{f,U}(t)$ is carried by ever "smaller" neighborhoods of $\partial U$ provided $t > 0$ is sufficiently small. As before the coset modulo $\mathcal{O}'(\partial U)$ of $\mu^c_{f,U}$ depends only on the class $[f] \in H^{n,q}(\mathcal{W}_\delta(\Omega,\Gamma))$; it also depends only on the *homology class* of the q–cycle c in $\Gamma$. We get a bilinear map

$$H_q(\Gamma)\times H^{n,q}(\mathcal{W}_\delta(\Omega,\Gamma)) \ni ([c],[f]) \to bv_U^{[c]}[f] \in \mathcal{B}(U).$$

Below we keep the cycle c fixed; letting $U \nearrow \Omega$ we define the boundary value map

(12) $$\mathrm{H}^{n,q}(\mathcal{W}_\delta(\Omega,\Gamma)) \ni [f] \to bv_\Omega^{\mathfrak{c}}[f] \in \mathcal{B}(\Omega).$$

We cannot expect this map to be surjective, but we must demand that it be injective, for otherwise the information extracted from boundary values evaporates. This forces us to shift the focus from the cone $\Gamma$ to the cycle $\mathfrak{c}$ and to microlocalize, in the sense that the open set U is allowed to contract about a central point $O$ and the open cone $\Gamma$ to contract about $\mathfrak{c}$ (it is convenient to take $\mathfrak{c} \subset S^{n-1}$; at any rate microlocalization ignores the radial dilations in $\Gamma$).

We are unable to prove directly the injectivity of (12) even in good circumstances (e. g., when $\mathfrak{c}$ is a q–sphere). But we recall that $\mathrm{H}^{n,q}(\mathcal{U})$ is but one of the "realizations" of the cohomology space $\mathrm{H}^q(\mathcal{U},\mathcal{O}^{(n)})$ with coefficients in the sheaf $\mathcal{O}^{(n)}$ of germs of n–forms $h\mathrm{d}z$ ($\mathrm{d}z = \mathrm{d}z_1\wedge\cdots\wedge\mathrm{d}z_n$, $h$ holomorphic). Another realization is the Čech space $\mathfrak{H}^q(\mathcal{U},\mathcal{O}^{(n)})$, and the switch from Dolbeault to Čech enables us to prove what we want, but only under a special assumption about the cycle $\mathfrak{c}$:

(13) *the cone $\Gamma_\mathfrak{c}$ generated by $\mathfrak{c}$ in $\mathbb{R}^n\backslash\{0\}$ is equal to the boundary of its convex hull $\hat{\Gamma}_\mathfrak{c}$.*

By the boundary of $\hat{\Gamma}_\mathfrak{c}$ we mean its boundary in $\mathrm{L}\backslash\{0\}$ where L is the smallest linear subspace of $\mathbb{R}^n$ that contains $\hat{\Gamma}_\mathfrak{c}$. Actually $\Gamma_\mathfrak{c}$ is generated by its intersection with a (q+1)–dimensional affine subspace A of $\mathbb{R}^n$; there is no loss of generality in assuming that $\mathfrak{c}$ is equal to that intersection. In this case $\mathfrak{c}$ is a Lipschitz hypersurface in A and (13) amounts to saying that $\mathfrak{c}$ is the boundary in A of a relatively open, bounded and *convex* subset of A.

We shall skectch the proof of the injectivity of the map (12) in a simple situation, when q = 1 and $\mathfrak{c}$ is, say, a *square abcd* in a plane A. Call $\Gamma_{ab}$ an open and convex cone contained in $\Gamma$ whose intersection with A contains the side $[a,b]$, and similarly for $bc$, $cd$ and $da$. By selecting the cones $\Gamma_{ab}$, $\Gamma_{bc}$, $\Gamma_{cd}$ and $\Gamma_{da}$ sufficiently "thin" we ensure that the intersections $\Gamma_a = \Gamma_{ab} \cap \Gamma_{da}$ $(\ni a)$, $\Gamma_b = \Gamma_{ab} \cap \Gamma_{bc}$ $(\ni b)$, $\Gamma_c = \Gamma_{bc} \cap \Gamma_{cd}$ $(\ni c)$ and $\Gamma_d = \Gamma_{cd} \cap \Gamma_{da}$ $(\ni d)$ are mutually far apart. Let $\Omega$ be an open subset of $\mathbb{R}^n$ and let $U \subset\subset \Omega$ be an open ball centered at a point $O$. If $f \in C^\infty(\Omega;\Lambda^{n,1})$ and if $\bar\partial f \equiv 0$ we can find $u_{ab} \in C^\infty(U+\imath\Gamma_{ab};\, \Lambda^{n,0})$ such that $\bar\partial u_{ab} = f$ in $U+\imath\Gamma_{ab}$, and similarly for $bc$, $cd$ and $da$ in the place of $ab$. In $U+\imath\Gamma_a$ we have $u_{ab} - u_{da} = g_a dz$, $g_a \in \mathcal{O}(U+\imath\Gamma_a)$. Whatever $h \in \mathcal{O}(\mathbb{C}^n)$, we have

$$\int_{U+\imath t\mathfrak{c}} hf =$$
$$\int_{U+\imath t[a,b]} hf + \int_{U+\imath t[b,c]} hf + \int_{U+\imath t[c,d]} hf + \int_{U+\imath t[d,a]} hf =$$
$$\int_{U+\imath ta} h(u_{da}-u_{ab}) + \int_{U+\imath tc} h(u_{bc}-u_{cd}) + \int_{U+\imath tb} h(u_{ab}-u_{bc}) + \int_{U+\imath td} h(u_{cd}-u_{da}) =$$
$$\int_{U+\imath ta} hg_a dz + \int_{U+\imath tc} hg_b dz + \int_{U+\imath tb} hg_c dz + \int_{U+\imath td} hg_d dz.$$

By letting $t > 0$ go to zero we conclude at once that

$$bv_U^{\mathfrak{c}}[f] = bv_U\, g_a + bv_U\, g_b + bv_U\, g_c + bv_U\, g_d. \tag{14}$$

Suppose now that $bv_U^{\mathfrak{c}}[f] \equiv 0$. At this point we apply the theorem of the Edge of the Wedge, actually a refined version of it. The standard version states that there are six functions $F_{ab} \in \mathcal{O}(U'+\imath\Gamma'_{ab})$, $F_{ac} \in \mathcal{O}(U'+\imath\Gamma'_{ac})$, $F_{ad} \in \mathcal{O}(U'+\imath\Gamma'_{ad})$, $F_{bc} \in \mathcal{O}(U'+\imath\Gamma'_{bc})$, $F_{bd} \in \mathcal{O}(U'+\imath\Gamma'_{bd})$, $F_{cd} \in \mathcal{O}(U'+\imath\Gamma'_{cd})$, with $U' \subset U$ an open ball centered at $O$ and $\Gamma'_{pq}$ an open and convex cone in $\mathbb{R}^n\backslash\{0\}$ containing the segment $[p,q]$, such that

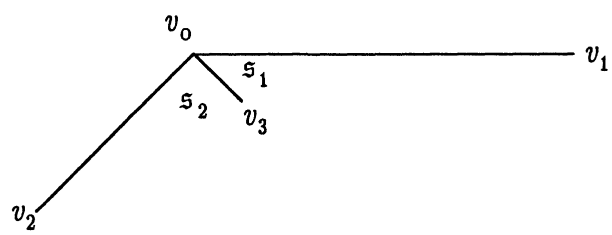
$$g_a = F_{ab} + F_{ac} + F_{ad} \text{ in } U'+\imath(\Gamma'_{ab}\cap\Gamma'_{ac}\cap\Gamma'_{ad}\cap\Gamma_a),$$
$$g_b = -F_{ab} + F_{bc} + F_{bd} \text{ in } U'+\imath(\Gamma'_{ab}\cap\Gamma'_{bc}\cap\Gamma'_{bd}\cap\Gamma_b),$$
$$g_c = -F_{ac} - F_{bc} + F_{cd} \text{ in } U'+\imath(\Gamma'_{ac}\cap\Gamma'_{bc}\cap\Gamma'_{cd}\cap\Gamma_c),$$
$$g_d = -F_{ad} - F_{bd} - F_{cd} \text{ in } U'+\imath(\Gamma'_{ac}\cap\Gamma'_{bc}\cap\Gamma'_{cd}\cap\Gamma_d).$$

The refined version, proved by fully exploiting the convexity of the square, allows us to assume that the functions corresponding to the diagonals, $F_{ac}$ and $F_{bd}$, vanish identically. The meaning of this is that the quadruplet $\{g_a,g_b,g_c,g_d\}$, a priori a Čech cocycle for the covering $\{U'+\imath\Gamma'_{ab},U'+\imath\Gamma'_{bc},U'+\imath\Gamma'_{cd},U'+\imath\Gamma'_{ab}\}$ of $U'+\imath\Gamma_c$, is in fact a Čech coboundary. But then the natural isomorphism between the Dolbeault and Čech cohomology tells us that $f = \bar\partial u$ in $U''+\imath\Gamma''$, with $U'' \subset U$ an open ball centered at $O$, $\Gamma'' \subset \Gamma$ an open cone containing $\mathfrak{c}$ and $u \in C^\infty(U''+\imath\Gamma'';\Lambda^{n,0})$. [A more careful reading of the equations might avoid any shrinking of U or of Γ, but the shrinking is at any rate unavoidable when dealing with boundary values on curved edges.] The method we have just outlined can be generalized to any "convex" q–dimensional polyhedron $\mathfrak{c}$, and thence to any Lipschitz q–cycle by a careful poly–hedral approximation, leading to the proof of

THEOREM 4.— *Assume* $n \geq 3$ *and* $1 \leq q \leq n-1$. *Let* $\mathfrak{c}$ *be a Lipschitz* q*–cycle satisfying* (13) *contained in an open cone* $\Gamma \subset \mathbb{R}^n\backslash\{0\}$. *Let* U *be an open neighborhood in* $\mathbb{R}^n$ *of a point O and* $\delta$ *a number* $> 0$. *There are an open neighborhood* $U' \subset U$ *of O, an open cone* $\Gamma'$, $\mathfrak{c} \subset \Gamma' \subset \Gamma$, *and a number* $\delta'$, $0 < \delta' < \delta$, *such that, for any* $[f] \in H^q(\mathcal{W}_\delta(U,\Gamma),\mathcal{O}dz)$, *if* $bv_U^{\mathfrak{c}}[f] \equiv 0$ *then the restriction of* $[f]$ *to* $\mathcal{W}_{\delta'}(U',\Gamma')$ *vanishes.*

The correct way of stating Theorem 4, especially its version for curved wedges, is microlocal, in the following sense. The concept of germ of a wedge $\mathcal{W}_\delta(U,\Gamma)$ is obvious: let the open subset U of $\mathbb{R}^n$ contract about the point $O$, the cone $\Gamma \subset \mathbb{R}^n\backslash\{0\}$ about the q–cycle $\mathfrak{c}$ and let $\delta \to +0$. Denote such a germ by $\mathcal{W}(\mathbb{R}^n,O,\mathfrak{c})$. We may then talk of the *germ of a cohomology class* $[f] \in H^q(\mathcal{W}(\mathbb{R}^n,O,\mathfrak{c}),\mathcal{O}dz)$ in the germ of wedge $\mathcal{W}(\mathbb{R}^n,O,\mathfrak{c})$; and of its *boundary value* along $\Gamma_\mathfrak{c}$, $bv_O^\mathfrak{c}[f]$, which is the germ at $O$ of a hyperfunction in $\mathbb{R}^n$. We shall debote by $\mathcal{B}_O(\mathbb{R}^n,\mathfrak{c})$ the space of germs of hyperfunc– tions at $O$ of the form $bv_O^\mathfrak{c}[f]$, $[f] \in H^q(\mathcal{W}(\mathbb{R}^n,O,\mathfrak{c}),\mathcal{O}dz)$. [The degree q is the dimen– sion of the cycle $\mathfrak{c}$.]

Before proceeding we discuss an example which shows that Hypothesis (13) is unavoidable. Let $\Pi$ be a two–plane in $\mathbb{R}^3$, not passing through the origin, and consider four points of $\Pi$, $v_0$, $v_1$, $v_2$, $v_3$ such that the union of the triangles $\mathfrak{s}_j$ with respective vertices $v_0$, $v_j$, $v_3$ (j = 1,2) is not convex:




Due to the lack of convexity of $\mathfrak{s}_1 \cup \mathfrak{s}_2$ there exists a germ of a holomorphic function $h$ in the germ of wedge $\mathcal{W}(\mathbb{R}^3,O,[v_0,v_3])$ which cannot be represented as the difference of two such germs in the wedges $\mathcal{W}(\mathbb{R}^3,O,\mathfrak{s}_j)$ (j = 1,2; we are using the terminology of germ of a wedge in a case where the directrix is a chain and not a cycle, but what is meant should be clear). Let then $\mathfrak{c}$ denote the boundary of the (nonconvex) polygone $\mathfrak{s}_1 \cup \mathfrak{s}_2$. For each i = 0,1,2,3 we define the germ of a holomorphic function $g_i$ in the germ of wedge $\mathscr{W}(\mathbb{R}^3, O, \{v_i\})$ as follows: $g_o = - g_3 = h$ (restricted to the relevant wedge); $g_1 = g_2 = O$. This defines a Čech one–cocycle $\check{g}$ and the boundary value of its cohomology class vanishes. Suppose $\check{g}$ were a coboundary. It would mean that to each pair (i,j) $\neq$ (O,3), $O \leq i < j \leq 3$, there is a germ of holomorphic function $F_{ij}$ in the germ of wedge $\mathscr{W}(\mathbb{R}^3, O, [v_i, v_j])$ such that the following is true

$$(15) \qquad F_{01} - F_{02} = h,\ F_{13} - F_{01} = O,\ F_{02} - F_{23} = O,\ F_{23} - F_{13} = -h,$$

in the germs of wedges $\mathscr{W}(\mathbb{R}^3, O, \{v_j\})$, j = 0,1,2,3, respectively. Thanks to the micro–local Bochner tube theorem we derive from the two middle equations (15) that, for i = 1,2, there is the germ of a holomorphic function $F_i$ in $\mathscr{W}(\mathbb{R}^3, O, \mathfrak{s}_i)$ such that

$$F_i = F_{oi} \text{ in } \mathscr{W}(\mathbb{R}^3, O, [v_o, v_i]),\ F_i = F_{i3} \text{ in } \mathscr{W}(\mathbb{R}^3, O, [v_i, v_3]).$$

But then the first (as well as the fourth) equation (15) implies $h = F_1 - F_2$ in $\mathscr{W}(\mathbb{R}^3, O, [v_o, v_3])$, which contradicts our choice of $h$.

Back to the square *abcd* we note that the equation (14) has further implica–tions, beyond the injectivity of the boundary value map. Let us not assume that $bv_U^{\mathfrak{c}}[f] \equiv 0$ but, instead, that the four vertices $a$, $b$, $c$, $d$ all lie in the closed half space $H_- = \{ y \in \mathbb{R}^n;\ y \cdot \xi_o \leq 0 \}$, which means that $\mathfrak{c} \subset H_-$. Since the intersection of each cone $\Gamma_a$, $\Gamma_b$, $\Gamma_c$, $\Gamma_d$ with the interior of $H_-$ is nonempty it follows that $(O, \xi_o) \notin WF_a(bv_U^{\mathfrak{c}}[f])$. In other words, $WF_a(bv_U^{\mathfrak{c}}[f])|_O$ does not intersect the "antipolar" $-\Gamma_{\mathfrak{c}}^o = \{ \xi \in \mathbb{R}^n \backslash \{0\};\ \forall\ y \in \mathfrak{c},\ \xi \cdot y \leq 0 \}$. And as a matter of fact, if $u \in \mathcal{B}(\mathbb{R}^n)$ is such that

$WF_a(u)|_O \cap (-\Gamma_c^o) = \emptyset$ then $u = bv_U^c[f]$ for some $[f] \in H^{n,1}(\mathscr{W}_\delta(U,\Gamma))$, if the open neighborhood U of $O$ is sufficiently small ($\Gamma$: an open cone in $\mathbb{R}^n \backslash \{0\}$, $\Gamma \supset c$). This can be proved in full generality:

THEOREM 5.– *Let* $c$ *be a Lipschitz* q*–cycle satisfying Condition* (13) *and let* $O$ *be an arbitrary point of* $\mathbb{R}^n$. *For the germ at* $O$ *of a hyperfunction* $u$ *in* $\mathbb{R}^n$ *to belong to* $B_O(\mathbb{R}^n, c)$ *it is necessary and sufficient that* $WF_a(u)|_O \cap (-\Gamma_c^o) = \emptyset$.

The following consequence of Theorem 5 is noteworthy:

COROLLARY 1.– *Let* $c$, $c'$ *be two Lipschitz cycles in* $\mathbb{R}^n \backslash \{0\}$, *both satisfying Condition* (13). *If* $\hat{\Gamma}_{c'} \subset \hat{\Gamma}_c$ *then* $B_O(\mathbb{R}^n, c') \subset B_O(\mathbb{R}^n, c)$.

The cycles $c$ and $c'$ need not have the same dimension, but note that $\hat{\Gamma}_{c'} \subset \hat{\Gamma}_c$ ($\Leftrightarrow \hat{\Gamma}_c^o \subset \hat{\Gamma}_{c'}^o$) entails $\dim c' \leq \dim c$.

Let us give a simple illustration of Theorem 5. Consider an arbitrary (hyper–function) solution $u$ of the wave–equation

$$\partial^2 u/\partial x_{n+1}^2 - \sum_{k=1}^{n} \partial^2 u/\partial x_k^2 = 0$$

in $\mathbb{R}^{n+1}$ ($n \geq 2$). By a celebrated theorem of Sato we know that $WF_a u \subset \{ \xi \in \mathbb{R}^n;\ \xi_{n+1}^2 = \xi_1^2 + \cdots + \xi_n^2 \}$. For any $\theta \in \mathbb{R}^n$, $|\theta| < 1$, call $c_\theta$ be the intersection of the unit sphere $S^n$ with the hyperplane $H_\theta = \{ \xi \in \mathbb{R}^n;\ \xi_{n+1} = \theta_1\xi_1 + \cdots + \theta_n\xi_n \}$. Observe that $H_\theta$ intersects the light cone solely at the origin; and that $\dim c_\theta = n-1$. It follows that, in the neighborhood U of an arbitrary point of $\mathbb{R}^{n+1}$, $u = bv_U^{c_\theta}[f]$ with $f \in H^{n+1,n-1}(\mathcal{W}_\delta(U,\Gamma_\theta))$ where $\Gamma_\theta \subset \mathbb{R}^{n+1}\backslash\{0\}$ is an open cone containing $H_\theta\backslash\{0\}$. As we show in a forthcoming article all these cohomology classes can be glued together, to yield an (n+1,n–1)–class in the whole region $\mathbb{R}^{n+1}+\imath\Gamma$, where now $\Gamma$ is the entire complement of the solid light cone: $\Gamma = \{ y \in \mathbb{R}^{n+1};\ y_{n+1}^2 < y_1^2 +\cdots+ y_n^2 \}$. It would be interesting to describe explicitely the cohomology classes corresponding to the classical solutions of the wave equation, say, to the solution $u$ such that

$$u\Big|_{x_{n+1}=0} = \delta(x_1,\dots,x_n),\ \partial u/\partial x_{n+1}\Big|_{x_{n+1}=0} = 0.$$

Considerations similar to those above apply to more general differential operators with $C^\omega$ coefficients.

The article Cordaro–Gindikin–Treves [1] proves the results above, and more, when the edges of the wedges lie on a totally real $C^\infty$ submanifold of $\mathbb{R}^n$. The difficulty in extending the results is that we cannot exploit the convexity of the tuboids $U+\imath\Gamma$ when U and $\Gamma$ are convex subsets of Euclidean space. The difficulty is resolved by a local approximate convexification (see Appendix, *loc. cit.*). This procedure leaves no other choice than to microlocalize.

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