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FRANK MERLE

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Recent Progress on the Blow-up Problem for Zakharov Equations

By Frank Merle

Université de Cergy-Pontoise
Mathématiques, 8 Avenue du Parc, Le Campus,
95033 Cergy-Pontoise Cedex France

In this paper, we present recent progress for the blow-up problem for Zakharov equations.

More precisely, we consider Zakharov equations

$$\text{(I)}\qquad \begin{aligned} i\partial u/\partial t &= -\Delta u + nu \\ \partial n/\partial t &= -\nabla . v \\ c_0^{-2}\partial v/\partial t &= -\nabla n - \nabla |u|^2 \end{aligned}$$

with initial data $(u(-1),n(-1),v(-1)) = (u_0,n_0,v_0)$
where $u: R^2 \to C$, $n: R^2 \to R$, $v: R^2 \to R^2$,

and related equations which are

the cubic nonlinear Schrödinger equation

$$\text{(II)}\qquad i\partial u/\partial t = -\Delta u - |u|^2 u$$

with initial data $u(-1) = u_0$
where $u: R^2 \to C$,

and the Elliptic equation associated with (II)

$$\text{(III)}\qquad u = \Delta u + |u|^2 u$$

where $u: R^2 \to C$.

1) The local Cauchy theory for equations (I),(II).

We are interested to find a space H for equation (I) or (II) such that there is a unique solution of the equation on [0,T) and we have the following $T=+\infty$ or $T<+\infty$ and $|u(t)|_H \to +\infty$ as t goes to T.

i)Case of the nonlinear Schrödinger equation (II)
The case of the cubic nonlinear Schrödinger equation is now well-understood. A local (in time) Cauchy theory can be done in various natural space H^1, H^s, L^2 (see [GV],[K],[CaW],[Bo1]). Moreover, one can show that the blow-up time does not depend on the Cauchy space and in fact

we have at the blow-up a concentration phenomenon in L^2.

In [MT] (see also [W2], [GlM2]), it is proved the following. Let u(t) a blow-up solution (and T its blow-up time), there is then x(t) such that for all R>0, $\liminf_{t \to T} |u(t)|^2_{L^2(|x-x(t)|\leq R)} \geq a>0$ where a is an universal constant ($a=|Q|^2_{L^2}$ where Q will be defined in subsection 3).

In addition, we have the following conserved quantities for all t,

$|u(t)|_{L^2} = |u_0|_{L^2}$,
$E(u(t)) = E(u_0)$ where $E(u) = 1/2\int|\nabla u|^2 - 1/4\int|u|^4$.

ii)Case of Zakharov equations (I).
A local (in time) Cauchy theory can not be done up to now in the energy space $H_1=\{(u,n,v)\in H^1\times L^2\times L^2\}$ for a general initial data. The result is proved for the space $H_2=\{(u,n,v)\in H^2\times H^1\times H^1\}$ (see for exemple[OT2],[KePVg],[Bo2] and the references therein).

Moreover, one can show that we have at the blow-up time again the same concentration phenomenon in L^2. Indeed, let (u(t),n(t),v(t)) a blow-up solution (and T its blow-up time), there is then x(t) such that for all R>0, $\liminf_{t \to T} |u(t)|^2_{L^2(|x-x(t)|\leq R)} \geq |Q|^2_{L^2}$ where Q will be defined in subsection 3.

In addition, we have the following conserved quantities for all t,

$|u(t)|_{L^2} = |u_0|_{L^2}$,
$H(u(t),n(t),v(t)) = H(u_0,n_0,v_0)$ where $H(u) = \int|\nabla u|^2 + n|u|^2 + n^2/2+|v|^2/2c_0$.

iii) Blow-up problem
We are now interrested in the case $T<+\infty$, that is the case of a blow-up solution (or equivalently a singular solution) for equation (I) or (II). Most of the results can be extend in dimension $N\geq 1$in the case of a critical power for the nonlinear Schrödinger equation. Part of the results for the Zakharov equation can be extend to the dimension 3 (only dimensions 2,3 are relevant).

2) Elementary relations between equations (I)-(II)-(III)

i) Limit as c_0 goes to infinity.
We can easily see that as c_0 goes to infinity, the wave part of equation (I) give formally

$$\nabla(n + |u|^2) = 0,$$

or equivallently

$$n + |u|^2 = 0.$$

Thus equation (I) transform in equation (II) as c_0 goes to infinity.
If the initial data are compatible, this result of convergence has been rigourously proved by several authors ([AA2], [OT1], [KePVe]) when the limit solution u(t) (of equation (II)) is regular. Near the blow-up time, we do not have convergence results and in some sense we can not expect some. For example, in [GlM2], there is the case of a blow-up solution of equation (II) with initial data u_0 such that for all finite c_0 and all n_0,v_0 the solution of (I) (u,n,v)(t) is globally defined in time. Therefore, in some sense at the singularity, equation (I) when c_0 is large, can

not be consider as a perturbation of equation (II).

ii) Periodic solutions of (I),(II)

By direct calculation, we can check that if w(x) is a solution of equation (III) then

- $u(t,x) = e^{it}\, w(x)$ is a periodic solution of equation (II)
- $(u(t,x),n(t,x),v(t,x)) = (e^{it}\, w(x),- |w(x)|^2,0)$ is also a periodic solution of equation (I).

iii)Conformally self-similar blowing-up solution

For this power in two dimension, the nonlinear Schrödinger equation has one more invariance : if u(t,x) is a solution of equation (II) then

$$1/t\ddot{u}(1/t,x/t)\exp(i|x|^2/4t)$$

is also a solution of equation (II).
In particular, if w(x) is a real solution of the equation (III), then

$$1/t\ w(x/t)\exp(-i/t + i|x|^2/4t)$$

is also a solution of equation (II) which blow-up at T = 0. We then obtain explicit blow-up solutions of equation (II).

Unfortunatly, such invariance does not exist for the Zakharov equation. In particular, there is no direct way to obtain explicit blow-up solutions of Zakharov equations.

3) On minimal solutions of (III)

In this section, we recall briefly some results on the elliptic equation (III). From [BeL],[St] it is now classical that equation (III) have infinitly many solutions in H^1 (up to the invariance of the equation).
Let us defined the unique positive radially symmetric solution of equation (III) (see [Kw] for uniqueness). We have in fact that the solution w=0 is isolated in the set of solution in L^2. More precisely,

i)Assume that w(x) is a nonzero solution of equation (III) then $|w|_{L^2} \geq |Q|_{L^2}$.

ii) Moreover, we have the following carracterisation of the minimal solution (or ground state) of equation (III). Assume that w is a nonzero solution of equation (III) and $|w|_{L^2} = |Q|_{L^2}$ then up to the invariance of the equation w = Q (that is there exist x',ω,θ such that $w(x) = e^{i\theta}\omega Q(\omega(x-x'))$).

4) Equation (II)

The problem of singularity for equation (II) has been studied in the last 20 years, and we give here part of results obtained.

i) No blow-up for small data

In [W1], it has been proved that for $u \in H^1$, we have the following

$$1/4\int|u|^4 \leq 1/2\int|\nabla u|^2 \{\int|u|^2 /\int Q^2 \}.$$

It follows from this identity that if

$$|u_0|_{L^2} < |Q|_{L^2}$$

then there is non blow-up phenomon and the solution is globally defined in time.

ii)blow-up for large data

For this equation there are two way to obtain blow-up solutions.

- explicit blow-up solution.

From the conformal invariance of the equation if w(x) is a real solution of the equation (III), then

$$1/t\, w(x/t) \exp(-i/t + i|x|^2/4t)$$

is also a solution of equation (II) which blow-up at $T = 0$.

In particular $\quad S(t,x) = 1/t\, Q(x/t) \exp(-i/t + i|x|^2/4t)$

is a blow-up solution such that $|u_0|_{L^2} = |Q|_{L^2}$.

-Viriel identity.

From [SoSyZ], [Gla], we have the following property of the solution of equation (II).

Assume that $|x|u_0 \in L^2$ then for all time t, $|x|u_0 \in L^2$ and

$$d^2t/dt^2 \{ \int |x|^2 |u(t,x)|^2 \, dx \} = 16\, E(u_0).$$

From this viriel identity, we have that

if $E(u_0) < 0$ then the solution blow-up in finite time ($T<+\infty$).

iii)Minimal blow-up solutions

Since if $|u_0|_{L^2} < |Q|_{L^2}$ then there is non blow-up, and there is blow-up solution in the case where $|u_0|_{L^2} = |Q|_{L^2}$, one can ask is it possible to carracterize all minimal blow-up solutions in L^2 (that is solution which blows-up and such that $|u_0|_{L^2} = |Q|_{L^2}$).

In [M1] (see also [M4] for a another approach of the proof), the following is proved.

Assume that u(t) is a blow-up solution with minimal mass (and u(t) is an H^1 solution of equation (II)), that is $|u_0|_{L^2} = |Q|_{L^2}$. Then up to the invariance of the equation, we have

$$u(t,x) = S(t,x) = 1/t\, Q(x/t) \exp(-i/t + i|x|^2/4t)$$

(that is there exist x', x'',ω,θ such that $u(t,x) = e^{i\theta}\omega/t\, Q((x-x')\omega/t - x'') \exp(-i\omega^2/t + i|x-x'|^2/4t)$).

5) Equation (I)

Until recently, there were no results on existence of solutions which blow-up for Zakharov equations. Indeed the two ingredients; the conformal invariance and the viriel identity which give blow-up results for the limit equation as c_0 goes to infinity do not hold. We can note that there were numerical evidence of singular behavior of solution of equation (I) in [LPSSW] and [PSSW].

i) No blow-up for small data

One can show (see [AA1],[SS]) as for the Schrödinger equation, that if $|u_0|_{L^2} < |Q|_{L^2}$ then ther is non blow-up phenomon and the solution is globally defined in time.

ii)blow-up for large data

As for equation (II), we are able to construct explicit blow-up solution and give obstructions to regular behavior.

- explicit blow-up solutions.

We do not have anymore the conformal invariance to obtain explicit blow-up solutions. We use in fact a bifurcation argument at "infinity" (using the structure of the nonlinear Schrödinger equation) to obtain explicit blow-up solution.

In [GlM1], a family of blow-up solutions in the energy space of the form

$$u(t,x) = \omega/t\, P(\omega x/t) \exp(-\omega^2 i/t + i|x|^2/4t)$$

$$n(t,x) = \{\omega/t\}^2 N(\omega x/t)$$

where $P(x) = P(|x|)$ and $N(x) = N(|x|)$
and

$$P = \Delta P + NP$$
$$(c_0\omega)^{-2}\{r^2N_{rr} + 6rN_r + 6N\} - \Delta N = \Delta P^2$$

is investigated.
More precisely, it is proved using this kind of construction, that there are blow-up solutions such that $|u_0|_{L^2} = |Q|_{L^2} + \varepsilon$, for *all* $\varepsilon > 0$.

We can note that the solutions constructed are numerically stable (see [LPSSW]). The problem now is the following, we have construct blow-up solutions but we do not existence of many (or a large set) of singular solution. For this purpose, we use a different approach.

-viriel identity.

In [M2], it is derived a perturbed viriel identity for the Zakharov equation. More precisly, for a regular solution with decay at infinity we have

$$d^2t/dt^2 \{1/4 \int |x|^2|u(t,x)|^2 dx + c_0^{-2} \int_0^t \int (x.v(t,x))n(t,x)\, dx\, dt\} = 2H(u_0,n_0,v_0) - c_0^{-2}\int |v(t,x)|^2 dx.$$

From this pertubed viriel identity, we have in [M2] that

if $H(u_0,n_0,v_0) < 0$ and the initial data are radially symmetric then the solution blow-up in finite time ($T<+\infty$) or in infinite time in H_1 (with a concentration of $u(t)$ in L^2 as t goes to infinity).

We suspect that in the case where $H(u_0,n_0,v_0) < 0$ then the solution away blows-up in finite time. This result give in particular the existence of a large class of singular solutions.

iii)Minimal blow-up solutions

Since if $|u_0|_{L^2} < |Q|_{L^2}$ then there is non blow-up, and there are blow-up solution such that in the case where $|u_0|_{L^2} = |Q|_{L^2} + \varepsilon$, for all $\varepsilon > 0$. one can ask, as for the nonlinear Schrödinger equation about minimal blow-up solutions (that is solution which blows-up and such that $|u_0|_{L^2} = |Q|_{L^2}$).

In [GlM2], we in fact proved that there is no minimal blow-up solution:

if $|u_0|_{L^2} = |Q|_{L^2}$ then there is non blow-up.

Therefore, the situation is different from the one of the nonlinear Schrödinger equation.

iv)Instability and stability results of blow-up behavior

Let us first recall some results for the nonlinear Schrödinger equation. We have explicit blow-up solution such that the blow-up rate in H^1 is of the type $1/(T-t)$. In particular, the one which the minimal blow-up solution has this rate of blow-up. From a physical point of view, we can expect that this rate is stable. It is not the case. Indeed,in [LPSS] for example it is observed numerically blow-up rate of the type $\mathrm{Log}|\mathrm{Log}|T-t|| / (T-t)^{1/2}$.

In [M3], it show for the Zakharov equation (with c_0 finite but eventually very large), that the blow-up rate is stonger than $1/(T-t)$. More precisely, let $(u,n,v)(t)$ a blow-up solution and T its blow-up time, we have for t near

$$|u(t)|_{H^1} \geq c/(T-t).$$

This shows that in fact the blow-up rate of the type $\mathrm{Log}|\mathrm{Log}|T-t|| / (T-t)^{1/2}$ is unstable with respect to perturbations of the equation (with a term involving a wave equation). In contrary, the one with blow-up rate $1/(T-t)$ seem numerically stable. This in particular shows the physical interest of the minimal blow-up solution of the nonlinear Schrödinger equation : the solution of the form

$$S(t,x) = 1/t\, Q(x/t) \exp(-i/t + i|x|^2/4t).$$

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