JOURNÉES ÉQUATIONS AUX DÉRIVÉES PARTIELLES

MITSURU IKAWA

On zeta function and scattering poles for several convex bodies

Journées Équations aux dérivées partielles (1994), p. 1-14
<http://www.numdam.org/item?id=JEDP_1994____A2_0>

ON ZETA FUNCTION AND SCATTERING POLES FOR SEVERAL CONVEX BODIES

MITSURU IKAWA

Department of Mathematics
Osaka University
Toyonaka 560, Japan

1. Introduction. Let $\mathcal{O}_j$, $j = 1, 2, \ldots, J$, be open bounded sets in $\mathbf{R}^3$ with smooth boundary Γ_j. We set

$$\mathcal{O} = \cup_{j=1}^{J} \mathcal{O}_j,$$

and assume the following:

(H.1) Each $\mathcal{O}_j$ is strictly convex, that is, the Gaussian curvature of Γ_j does not vanish.

(H.2) For each $\{l_1, l_2, l_3\} \in \{1, 2, \ldots, J\}^3$ such that $j_l \neq j_{l'}$ for $l \neq l'$,
$$(\text{convex hull of } \overline{\mathcal{O}_{j_1}} \text{ and } \overline{\mathcal{O}_{j_2}}) \cap \overline{\mathcal{O}_{j_3}} = \emptyset.$$

In this note, we consider the case of

$$J \geq 3. \tag{1.1}$$

We set

$$\Omega = \mathbf{R}^3 - \overline{\mathcal{O}},$$

and consider two dynamics in Ω. The one is the classical dynamics in Ω and the another is the quantum dynamics in Ω, and we are interested in relationships between these two dynamics. As the first step of study of relationships of two dynamics, we would like to take up the zeta function as the subject of the classical dynamics, and the scattering matrix as that of the quantum dynamics. Our interest as to these subjects is to know how the singularities of the zeta function relate to the poles of the scattering matrix, and vice versa.

Note that under the assumption (1.1), the classical dynamics in Ω, which is nothing but the geometric optics, becomes chaotic. This chaotic property makes the both dynamics difficult to treat.

Our result we shall talk about is as follows: In a small neighborhood of the axis of absolute convergence of the zeta function, the scattering matrix is holomorphic at points which are not so near to the poles of the zeta function.

2. Zeta Function for The Classical Dynamics

2.1. Definition of the zeta function

Let $A = [A(i,j)]_{i,j=1,2,\ldots,J}$ be a $J \times J$ matrix defined by

$$A(i,j) = \begin{cases} 1 & \text{for } i \neq j, \\ 0 & \text{for } i = j. \end{cases}$$

Set

$$\Sigma_A = \{\xi = (\ldots, \xi_{-n}, \xi_{-n+1}, \cdots, \xi_{-1}, \xi_0, \xi_1, \cdots, \xi_n, \ldots); \\ \xi_i \in \{1, 2, \ldots, J\} \text{ and } A(\xi_i, \xi_{i+1}) = 1 \text{ for all } i\}.$$

Then, for each element $\xi \in \Sigma_A$ there corresponds uniquely a ray $\mathcal{X}(\xi)$ of the geometric optics trapped by $\mathcal{O}$ in the future and in the past with the reflection order ξ. Namely, $\mathcal{X}(\xi)$ is the ray of geometric optics in Ω, which reflects on Γ_{ξ_0} and, following the advance of time in the future, reflects on $\Gamma_{\xi_1}, \Gamma_{\xi_2}, \cdots$ successively and following the time going back to the past, reflects on $\Gamma_{\xi_{-1}}, \Gamma_{\xi_{-2}}, \cdots$ successively. Denote the j-th reflection point of $\mathcal{X}(\xi)$ by $P_j(\xi)$. Then $\mathcal{X}(\xi)$ is an infinite broken ray connecting successively the points

$$\cdots, P_{-1}(\xi), P_0(\xi), P_1(\xi), \cdots.$$

We define a function $f(\xi)$ on Σ_A by

$$f(\xi) = |P_0(\xi) - P_1(\xi)|. \tag{2.1}$$

For $\xi \in \Sigma_A$ we can define a sequence of phase functions $\{\varphi_{\xi,j}(x)\}_{j=-\infty}^{\infty}$ satisfying for all j

$$\begin{cases} |\nabla\varphi_{\xi,j}(x)|^2 = 1 \quad \text{in a neighborhood of } P_j(\xi)P_{j+1}(\xi), \\ \nabla\varphi_{\xi,j}(P_j(\xi)) \quad \text{is parallel to } \overrightarrow{P_j(\xi)P_{j+1}(\xi)}, \\ \varphi_{\xi,j}(x) = \varphi_{\xi,j+1}(x) \quad \text{on } \Gamma_{\xi_{j+1}} \cap (\text{a neighborhood of } P_{j+1}(\xi)), \\ \text{the principal curvatures of } \mathcal{C}_{\xi,j}(P_j(\xi)) \text{ with respect to } \nabla\varphi_{\xi,j}(P_j(\xi)) \\ \qquad \text{are positive,} \end{cases} \tag{2.2}$$

where

$$\mathcal{C}_{\xi,j}(x) = \{y;\ \varphi_{\xi,j}(y) = \varphi_{\xi,j}(x)\}.$$

Note that the conditions (2.2) determine uniquely $\nabla\varphi_{\xi,j}(x)$ in a neighborhood of $P_j(\xi)P_{j+1}(\xi)$.

Denote by $G_{\xi,j}(x)$ the Gaussian curvatue of $\mathcal{C}_{\xi,j}(x)$ at x. We define a function $g(\xi)$ by

$$g(\xi) = \log\sqrt{G_{\xi,0}(P_1(\xi))/G_{\xi,0}(P_0(\xi))}. \tag{2.3}$$

We see easily that

$$g(\xi) < 0 \quad \text{for all } \xi \in \Sigma_A. \tag{2.4}$$

Denote by σ_A the left shift operator in Σ_A, which is given by

$$(\sigma_A \xi)_i = \xi_{i+1} \quad \text{for all } i.$$

The zeta function attached to the boundary value problem with Dirichlet boundary condition is given by

$$\zeta(s) = \exp\left(\sum_{n=1}^{\infty} \frac{1}{n} \sum_{\sigma_A{}^n \xi = \xi} \exp\, S_n r(\xi, s)\right), \tag{2.5}$$

where

$$\begin{gathered} r(\xi, s) = -sf(\xi) + g(\xi) + \sqrt{-1}\,\pi, \\ S_n r(\xi, s) = r(\xi, s) + r(\sigma_A \xi, s) + \cdots + r(\sigma_A{}^{n-1}\xi, s). \end{gathered} \tag{2.6}$$

Evidently $\xi \in \Sigma_A$ satisfying $\sigma_A{}^n \xi = \xi$ is a periodic element with repect to σ_A of period n, and the corresponding ray $\mathcal{X}(\xi)$ is also a periodic ray in Ω with n reflection points. Denote by γ the corresponding periodic ray, that is, the ray starting from a point on Γ_{ξ_0} and reflecting on $\Gamma_{\xi_1}, \Gamma_{\xi_2}, \ldots$, successively, and after the reflection on $\Gamma_{\xi_{n-1}}$ returning to the starting point. Then, we have

$$S_n r(\xi, s) = e^{-sd_\gamma} (\lambda_{\gamma,1} \lambda_{\gamma,2})^{1/2} (-1)^n, \tag{2.7}$$

where d_γ denotes the length of γ and $\lambda_{\gamma,l}$ ($l = 1, 2$) the eigenvalues less than 1 of the Poincaré map of γ.

As to the convergence of $\zeta(s)$, we see immediately

$$\begin{gathered} {}^{\#}\{\xi \in \Sigma_A; \sigma_A{}^n \xi = \xi\} \le (N-1)^N, \\ S_n f(\xi) \ge n\, d_{\min}, \end{gathered}$$

where $d_{\min} = \min_{i \neq j} \operatorname{dis}(\mathcal{O}_i, \mathcal{O}_j)$. By taking account of (2.4), the right hand side of (2.5) converges absolutely in

$$\operatorname{Re} s > \frac{N-1}{d_{\min}}. \tag{2.8}$$

Remark that it holds that

$$-\frac{d}{ds} \log \zeta(s) = \sum_\gamma (-1)^{i_\gamma}\, T_\gamma\, e^{-sd_\gamma} (\lambda_{\gamma,1}\lambda_{\gamma,2})^{1/2}, \tag{2.9}$$

where the summation is taken over all the oriented periodic rays γ, and i_γ denotes the number of reflection points, and T_γ the prime period of γ.

Needless to say, $\zeta(s)$ converges absolutely in a larger domain than (2.8).

2.2. Ruelle operator and the abscissa of absolute convergence

For a function $k(\xi)$ defined on Σ_A, we set

$$\mathrm{var}_n\, k = \sup\{|k(\xi) - k(\xi')|;\ \xi_i = \xi'_i \ \text{ for all } |i| \leq n\},$$

and for $0 < \theta < 1$

$$||k||_\theta = \sup_n \frac{\mathrm{var}_n\, k}{\theta^n}.$$

We set

$$||k||_\infty = \sup_{\xi\in\Sigma_A} |k(\xi)|, \qquad \text{and} \qquad |||k|||_\theta = ||k||_\theta + ||k||_\infty,$$

and define the space $\mathcal{F}_\theta(\Sigma_A)$ by

$$\mathcal{F}_\theta(\Sigma_A) = \{k(\xi);\ |||k|||_\theta < \infty\}.$$

Now we introduce the spaces of one sided sequences

$$\Sigma_A^+ = \{\xi = (\xi_0, \xi_1, \cdots);\ A(\xi_i, \xi_{i+1}) = 1 \ \text{ for all } i \geq 0\},$$
$$\Sigma_A^- = \{\xi = (\cdots, \xi_{-2}, \xi_{-1});\ A(\xi_{i-1}, \xi_i) = 1 \ \text{ for all } i \leq -1\}.$$

Concerning the functions $f(\xi)$ and $g(\xi)$ introduced in the previous subsection it is easy to check

$$f,\, g \in \mathcal{F}_\theta(\Sigma_A)$$

for some $0 < \theta < 1$.

For $r(\xi, s)$ defined by (2.6), we can construct $\tilde{r}(\xi, s)$ and $\chi(\xi, s)$ such that

$$\tilde{r}(\xi, s) \text{ depends only on } \ (\xi_0, \xi_1, \cdots),$$

that is, $\tilde{r}$ is a function in $\mathcal{F}_\theta(\Sigma_A^+)$, and

$$r(\xi, s) = \tilde{r}(\xi, s) - \chi(\xi, s) + \chi(\sigma_A \xi, s)$$

(for the construction of $\tilde{r}$ and χ, see, Bowen[1, page 11]).

We introduce the Ruelle operator, which is the operator in $\mathcal{F}_\theta(\Sigma_A^+)$ defined by

$$\mathcal{L}_s v(\xi) = \sum_{\sigma_A \eta = \xi} e^{\tilde{r}(\eta, s)}\, v(\eta) \quad \text{ for } v \in \mathcal{F}_\theta(\Sigma_A^+). \tag{2.10}$$

Related to $\mathcal{L}_s$, define $|\mathcal{L}_s|$ by

$$(|\mathcal{L}_s| v)(\xi) = \sum_{\sigma_A \eta = \xi} |e^{\tilde{r}(\eta, s)}|\, v(\eta) \quad \text{ for } v \in \mathcal{F}_\theta(\Sigma_A^+). \tag{2.11}$$

The Perron-Frobenius theorem shows that there is uniquely $h \in \mathbf{R}$ satisfying

$$(2.12) \qquad \begin{cases} \text{the spectrum of } |\mathcal{L}_{-h}| \text{ is contained in } \{\lambda \in \mathbf{C};\ |\lambda| \le 1\}, \\ |\mathcal{L}_{-h}| \text{ has 1 as an eigenvalue.} \end{cases}$$

Moreover, $|\mathcal{L}_{-h}|$ has the following decomposition:

$$|\mathcal{L}_{-h}| = 1\mathcal{P} + \mathcal{S},$$

where

$$\mathcal{P}\,v(\xi) = w(\xi)\int_{\Sigma_A^+} v(\eta)d\mu(\eta)$$
$$\text{the spectral radius of } \mathcal{S} < 1,$$

where $w(\xi)$ is an eigenvector associated to the eigenvalue 1, and $\mu(\xi)$ is a Gibbs measure satisfying

$$\int_{\Sigma_A^+} w(\eta)d\mu(\eta) = 1.$$

Remark that

(2.13) the abscissa of the absolute convergence of $\zeta(s)$ is $\mathrm{Re}\, s = -h$.

The Perron-Frobenius Theorem shows that the eigenvector $w(\xi)$ satisfies

$$(2.14) \qquad \inf_{\xi \in \Sigma_A^+} w(\xi) > 0.$$

We set

$$M = \max_{\xi,\eta \in \Sigma_A^+} \frac{w(\xi)}{w(\eta)}, \qquad m = \min_{\xi \in \Sigma_A^+} |e^{\tilde{r}(\xi,-h)}|.$$

As to the analytic continuation of $\zeta(s)$ beyond the abscissa of the absolute convergence, it is proved that $\zeta(s)$ is meromorphically continued into the domain

$$\mathrm{Re}\, s > -h - \frac{1}{2}\frac{1}{||f||_\infty}|\log\theta|$$

(see for example, [2], [8]).

3. Statement of Theorem

Denote by $\mathcal{S}(z)$ the scattering matrix for $\mathcal{O}$. It is well known that $\mathcal{S}(z)$ is holomorphic in $\{z \in \mathbf{C};\ \operatorname{Im} z \leq 0\}$ and meromorphic in the whole complex plane $\mathbf{C}$. In [3] we showed that, for $\mathcal{O}$ given by (1.1) with (H.1) and (H.2), under the assumption of $h > 0$, the number of poles of $\mathcal{S}(z)$ in $\{z \in \mathbf{C};\ \operatorname{Im} z \leq h - \varepsilon\}$ is finit for any $\varepsilon > 0$.

In this talk, we would like to show the following

Theorem. *Suppose that $\mathcal{O}$ given* (1.1) *satisfies* (H.1), (H.2) *and the condition*

$$\frac{M}{m}\sqrt{\theta} < 1. \tag{A}$$

Then we have $0 < \alpha < 1$ with the following property:

For any $\varepsilon > 0$, there is a positive constant C_ε such that, if z_0 is a point in $\mathcal{D}_{\alpha,\varepsilon}$ which is a domain given by

$$\mathcal{D}_{\alpha,\varepsilon} = \{z \in \mathbf{C};\ \operatorname{Im} z \leq h + |\operatorname{Re} z|^{-\alpha},\ |\operatorname{Re} z| \geq C_\varepsilon\},$$

and if the zeta function satisfies the estimate

$$|\zeta(iz_0)| \leq |z_0|^{1-\varepsilon}, \tag{3.1}$$

the scattering matrix $\mathcal{S}(z)$ is holomorphic at $z = z_0$.

REMARK 1. Note that α in the theorem will be chosen by the following way: First we choose $a > 0$ in such a way that

$$a|\log \theta| > 2, \quad a \log \frac{M}{m} < 1.$$

Next we choose α as

$$a \log \frac{M}{m} < \alpha < 1.$$

Therefore, according to the smallness of $\frac{M}{m}\sqrt{\theta}$, we can choose α small.

REMARK 2. As an example of $\mathcal{O}$ satisfying Condition (A), we have the following one: Let P_1, P_2, P_3 be points in $\mathbf{R}^3$ such that $\triangle P_1P_2P_3$ is a right triangle. For $\varepsilon > 0$ set

$$\mathcal{O}_\varepsilon = \cup_{j=1}^{3} \mathcal{O}_{j,\varepsilon}, \quad \mathcal{O}_{j,\varepsilon} = \{x \in \mathbf{R}^3;\ |x - P_j| < \varepsilon\}.$$

Then, $\mathcal{O}_\varepsilon$ satisfies Condition (A) when $\varepsilon > 0$ is small.

REMARK 3. If there is $z_0 \in \mathcal{D}_{\alpha,\varepsilon}$ such that

$$|\zeta(iz_0)| > |z_0|^{1-\varepsilon},$$

then, for all $z \in \mathcal{D}_{\alpha,\varepsilon}$ satisfying

$$|z_0|^{-1+\varepsilon} \leq |\operatorname{Re} z - \operatorname{Re} z_0| \leq (\log|z_0|)^{-1}$$

we have the estimate

$$|\zeta(z)| \leq |z|^{1-\varepsilon}.$$

REMARK 4. It is not yet known whether $\zeta(s)$ has a pole in general, and it is an interesting problem to know whether there exists a sequence of poles of $\zeta(s)$ converging to the line $\mathrm{Re}\, s = -h$.

4. Outline of The Proof of Theorem

4.1. Cosntruction of asymptotic solutions for oscillatory data

Consider the following boundary value problem with papameter $z \in \mathbf{C}$:

$$\begin{cases} (-\triangle - z^2)u = 0 & \text{in } \Omega, \\ u = f(x) & \text{on } \Gamma. \end{cases} \tag{4.1}$$

For $\mathrm{Im}\, z < 0$, the problem has a unique solution in $L^2(\Omega)$. Denote this solution as

$$u(x) = (R(z)f)(x).$$

Then, $R(z) \in \mathcal{L}(L^2(\Gamma), L^2(\Omega))$ and it depends analytically on $z \in \{z;\ \mathrm{Im}\, z < 0\}$. By uisng the regularity theory for $-\triangle$ we can regard $R(z)$ as a mapping from $C^\infty(\Gamma)$ into $C^\infty(\overline{\Omega})$. Then we have

$$R(z) \text{ is } \mathcal{L}(C^\infty(\Gamma), C^\infty(\overline{\Omega}))\text{-valued holomorphic function in } \mathrm{Im}\, z < 0.$$

It is known that this $R(z)$ can be prolonged meromorphically into the whole complex plane, and

the poles of $R(z)$ coincide with those of $\mathcal{S}(z)$.

Thus, the consideration of poles of $\mathcal{S}(z)$ is reduced to that of $R(z)$.

Our method to consider the analytic continuation of $R(z)$ into the upper half plane is an explicit construction of asymptotic solutions, which was done in [3].

For the oscillatory data given on Γ_1 of the form

$$f(x, z) = e^{-iz\varphi(x)}\, g(x) \tag{4.2}$$

we will construct an asymptotic solution of the boundary value problem

$$\begin{cases} (-\triangle - z^2)u = 0 & \text{in } \Omega, \\ u = f(x, z) & \text{on } \Gamma_1, \\ u = 0 & \text{on } \Gamma_2 \cup \Gamma_3 \cup \cdots \cup \Gamma_J \end{cases} \tag{4.3}$$

by the following way:

Let $\varphi_{(1)}(x)$ be a real valued function satisfying

$$\varphi_{(1)}(x) = \varphi(x) \qquad \text{on } \Gamma_1$$
$$|\nabla\varphi_{(1)}(x)| = 1$$

and construct $u_{(1)}(x,z)$ of the form

$$u_{(1)}(x,z) = e^{-iz\varphi_{(1)}(x)}\, g_{(1)}(x)$$

by the standard method. Next we construct $u_{(1,2)}, u_{(1,3)}, \cdots, u_{(1,J)}$ of the form

$$u_{(1,l)}(x,z) = e^{-iz\varphi_{(1,l)}(x)}\, g_{(1,l)}(x) \qquad (l = 2,3,\dots,J)$$

so that we have

$$u_{(1)}(x,z) + u_{(1,l)}(x,z) = 0 \quad \text{on } \Gamma_{1,l} \quad (l = 2,3,\dots,J),$$

where $\Gamma_{i,j}$ denotes the part of Γ_j seen from Γ_i. In order to repeat this procedure, we introduce some notations. For $n = 2,3,\dots,$ we set

$$I_n = \{\mathbf{i} = (i_1, i_2, \dots, i_n);\ i_j \in \{1,2,\dots,J\},\ i_1 = 1$$
$$\text{and } A(i_j, i_{j+1}) = 1 \quad \text{for all } j = 1,2,\dots,n-1\}.$$

Suppose that for all $\mathbf{i} \in I_n$

$$u_{\mathbf{i}}(x,z) = e^{-iz\varphi_{\mathbf{i}}(x)}\, g_{\mathbf{i}}(x) \tag{4.4}$$

is defined. For $\mathbf{j} \in I_{n+1}$, we define

$$u_{\mathbf{j}}(x,z) = e^{-iz\varphi_{\mathbf{j}}(x)}\, g_{\mathbf{j}}(x)$$

by the following way: denote $\mathbf{j} \in I_{n+1}$ as $\mathbf{j} = (\mathbf{i}, k)$, $\mathbf{i} \in I_n$. Let $u_{\mathbf{j}}(x,z)$ be an asymptotic solution of the form

$$u_{\mathbf{j}}(x,z) = e^{-iz\varphi_{\mathbf{j}}(x)}\, g_{\mathbf{j}}(x)$$

satisfying the boundary condition

$$u_{\mathbf{i}}(x,z) + u_{\mathbf{j}}(x,z) = 0 \quad \text{on } \Gamma_{i_n,k}.$$

By this procedure, we get a set of asymptotic solutions $\{u_{\mathbf{i}}(x,z)\}_{\mathbf{i}\in I}$, where $I = \cup_{n=1^\infty}\,{}_{I_n}$. Define $w(x,z)$ by

$$w(x,z) = \sum_{\mathbf{i}\in I} u_{\mathbf{i}}(x,z), \tag{4.5}$$

which is a first approximation of the solution to the problem (4.3).

In [3], the constant h is characterized as

$$h = \sup\{a; \sum_{\mathbf{i}\in I} |u_{\mathbf{i}}(x,z)| < \infty \quad \text{for all } \operatorname{Im} z < a\}. \tag{4.6}$$

Note that the constant h defined by (4.6) coincides with the one defined by (2.12).

Thus, our problem is mainly related to the analytic continuation of $w(x,z)$ defined (4.5) beyond the abscissa of absolute convergence.

4.2. Representation of $w(x,z)$ by the Ruelle operator

For $\xi \in \Sigma_A^+$, we have a sequence of points $\{Q_j(\xi)\}_{j=0}^{\infty}$ satisfying for all $j = 0, 1, 2 \dots$

$$Q_j(\xi) \in \Gamma_{\xi_j} \quad \text{and } Q_{j+1}(\xi) = Q_j(\xi) + |Q_{j+1}(\xi) - Q_j(\xi)|\, \nabla\varphi_{\mathbf{i}}(Q_j(\xi)),$$

where $\mathbf{i} = (\xi_0, \xi_1, \dots, \xi_j)$. Namely, $\{Q_j(\xi)\}_{j=0}^{\infty}$ is nothing but the sequence of reflection points of the ray which starts from a point on Γ_1 in the direction $\nabla\varphi_{(1)}$ and repeats the reflection on Γ_{ξ_1}, Γ_{ξ_2}, $\cdots$ successively. Set

$$f_j^+(\xi) = |Q_{j+1}(\xi) - Q_j(\xi)| \quad \text{and} \quad g_j^+(\xi) = \frac{1}{2}\log\frac{G_{\mathbf{i}}(Q_{j+1}(\xi))}{G_{\mathbf{i}}(Q_j(\xi))}$$

where $G_{\mathbf{i}}(x)$ denotes the Gaussian curvature of the surface $C_{\mathbf{i}}(x) = \{y;\ \varphi_{\mathbf{i}}(y) = \varphi_{\mathbf{i}}(x)\}$.

For each $j \in \{1, 2, \dots, J\}$, choose a sequence $\xi_{-n}^{(j)}$, $n = 1, 2, \dots$, such that $\xi_{-1}^{(j)} \neq j$ and $A(\xi_{-n-1}, \xi_{-n}) = 1$ for all $n = 1, 2, \dots$. For $\xi = (\xi_0, \xi_1, \cdots) \in \Sigma_A^+$ we corresponds an element $e(\xi)$ in Σ_A by

$$e(\xi) = (\cdots, \xi_{-n}^{(j)}, \xi_{-n+1}^{(j)}, \cdots, \xi_{-1}^{(j)}, \xi_0, \xi_1, \cdots)$$

when $\xi_0 = j$.

We define the function $\phi^+(\xi)$ for $\xi \in \Sigma_A^+$ such that $\xi_0 = 1$ by

$$\phi^+(\xi) = \sum_{n=0}^{\infty} \left\{-s\left(f({\sigma_A}^n e(\xi)) - f_n^+(\xi)\right) + g({\sigma_A}^n e(\xi)) - g_n^+(\xi)\right\}. \tag{4.7}$$

For each $k \in \{1, 2, \dots, J\}$, fix a point $x^{(k)} \in \Gamma_k$, and let ξ is an element in Σ_A such that $\xi_0 = k$. Then, there exists a unique broken ray arriving at $x^{(k)}$ after the reflection on $\Gamma_{\xi_{-j}}$ $(j = 1, 2, \dots)$. Denote the reflection points by $Q_{-j}^{(-)}(\xi)$, $j = 1, 2, \dots$. Also there is a sequence of phase functions $\psi_{\xi,-j}(x)$ defined in a neighborhood of $Q_{-j}^{(-)}(\xi)Q_{-j+1}^{(-)}(\xi)$ which satisfies

$$|\nabla\psi_{\xi,-j}(x)| = 1$$
$$\psi_{\xi,-j}(x) = \psi_{\xi,-j+1}(x) \qquad \text{on } \Gamma_{\xi_{-j}}.$$

We set

$$f_j^-(\xi) = |Q_{-j}^{(-)}(\xi) - Q_{-j+1}^{(-)}(\xi)| \quad \text{and} \quad g_j^-(\xi) = \frac{1}{2}\log \frac{G_{\xi,-j}(Q_{-j+1}^{(-)}(\xi))}{G_{\xi,-j}(Q_{-j}^{(-)}(\xi))}$$

where $G_{\xi,-j}(x)$ denotes the Gaussian curvature of the wave front $\{y;\ \psi_{\xi,-j}(y) = \psi_{\xi,-j}(x)\}$ at x. We define a function $\phi^-(\xi,s)$ for $\xi \in \Sigma_A$ by

$$\phi^-(\xi,s) = \sum_{n=1}^{\infty} \left\{-s\left(f(\sigma_A{}^{-n}\xi) - f_n^-(\xi)\right) + \left(g(\sigma_A{}^{-n}\xi) - g_n^-(\xi)\right)\right\}.$$

We introduce new operators $\mathcal{G}_s$ and $\mathcal{M}_{n,s}$ in $\mathcal{F}_\theta(\Sigma_A^+)$ by

$$\mathcal{G}_s v(\xi) = \sum_{\substack{\sigma_A\eta=\xi \\ \eta_1=1}} e^{\{\phi^+(\eta,s)+\chi(\eta,s)+\tilde{r}(\eta,s)\}}\, v(\eta) \tag{4.8}$$

and

$$(\mathcal{M}_{n,s} v)(\xi) = \sum_{\sigma_A\eta=\xi} e^{\{\phi^-(\sigma_A^n e(\eta),s)-\chi(\sigma_A^{n+1} e(\eta),s)+\tilde{r}(\eta,s)\}}\, v(\eta). \tag{4.9}$$

respectively.

Let $v_0(\xi)$ be the function given by

$$v_0(\xi) = \begin{cases} 1 & \text{if} \quad \xi_0 = 1, \\ 0 & \text{otherwise.} \end{cases}$$

Suppose that $\xi^{(k)} \in \Sigma_A^+$ satisfies $\xi_0^{(k)} = k$. Then, ther exists $C > 0$ such that

$$\begin{aligned} \Big|(\mathcal{L}_s{}^n\mathcal{M}_{n,s}\mathcal{G}_s v_0)(\xi^{(k)}) - \sum_{|\mathrm{i}|=n+2} u_{\mathrm{i}}(x^{(k)}, -is)\Big| \\ \le |s|\,(\theta + ca)^{n+2} \qquad \text{for all } \mathrm{Re}\, s > -h - a, \end{aligned} \tag{4.10}$$

where c is a positive constant.

The estimate (4.10) shows that the existence of $\sum_{n=0}^{\infty}(\mathcal{L}_s{}^n\mathcal{M}_{n,s}\mathcal{G}_s v_0)$ implies the existence of $w(x,-is)$, because we have

$$\Big|\sum_{n=0}^{\infty}(\mathcal{L}_s{}^n\mathcal{M}_{n,s}\mathcal{G}_s v_0) - w(x,-is)\Big| \le C.$$

Thus, the problem of analytic continuation of $w(x,s)$ is reduced to that of $\sum_{n=0}^{\infty}(\mathcal{L}_s{}^n\mathcal{M}_{n,s}\mathcal{G}_s v_0)$.

4.3. Summation of $\mathcal{L}_s{}^n \mathcal{M}_{n,s}$

Note that ϕ^- satisfies

$$|\phi^-(\sigma_A{}^n\, e(\xi), s) - \phi^-(\sigma_A{}^{n-1}\, e(\sigma_A\xi), s)| \le c|s|\,\theta^{n-1}, \tag{4.11}$$

which implies that

$$||\mathcal{L}_s{}^n \mathcal{M}_{n,s} - \mathcal{L}_s{}^{n-1}\mathcal{M}_{n-1,s}\mathcal{L}_s||_\infty \le c|s|\,\theta^{n-1}. \tag{4.12}$$

Then we see that

$$\begin{aligned}\mathcal{R}_s =& \mathcal{M}_{0,s} + (\mathcal{L}_s\mathcal{M}_{1,s} - \mathcal{M}_{0,s}\mathcal{L}_s) + \left(\mathcal{L}_s{}^2\mathcal{M}_{2,s} - \mathcal{L}_s{}^1\mathcal{M}_{1,s}\mathcal{L}_s\right) \\ &+ \cdots + \left(\mathcal{L}_s{}^n\mathcal{M}_{n,s} - \mathcal{L}_s{}^{n-1}\mathcal{M}_{n-1,s}\mathcal{L}_s\right) + \cdots\end{aligned}$$

converges absolutely for all $\operatorname{Re} s > -h - a$. On the other hand, we have

$$\begin{aligned}\mathcal{L}_s{}^n\, \mathcal{M}_{n,s} = &\left(\mathcal{L}_s{}^n\mathcal{M}_{n,s} - \mathcal{L}_s{}^{n-1}\mathcal{M}_{n-1,s}\mathcal{L}_s\right) \\ &+ \left(\mathcal{L}_s{}^{n-1}\mathcal{M}_{n-1,s} - \mathcal{L}_s{}^{n-2}\mathcal{M}_{n-2,s}\mathcal{L}_s\right)\mathcal{L}_s \\ &+ \cdots + (\mathcal{L}_s\mathcal{M}_{1,s} - \mathcal{M}_{0,s}\mathcal{L}_s)\,\mathcal{L}_s{}^{n-1} + \mathcal{M}_{0,s}\mathcal{L}_s{}^n.\end{aligned}$$

Suppose that $\operatorname{Re} s > -h$. Then, since we have $||\mathcal{L}_s||_\infty < 1$, it follows from the above relation that

$$\sum_{n=0}^{\infty} \mathcal{L}_s{}^n\, \mathcal{M}_{n,s} = \mathcal{R}_s \sum_{n=0}^{\infty} \mathcal{L}_s{}^n. \tag{4.13}$$

Moreover, suppose that $\mathcal{L}_s$ can be decomposed as

$$\mathcal{L}_s = \lambda_s \mathcal{P}_s + \mathcal{Q}_s, \tag{4.14}$$

where

$$\text{the spectral radius of } \mathcal{Q}_s \le 1 - \varepsilon_0 \quad (\varepsilon_0 > 0) \quad \text{for all } \operatorname{Re} s > -h - a. \tag{4.15}$$

Then, we have

$$\sum_{n=0}^{\infty} \mathcal{L}_s{}^n = \frac{1}{1-\lambda_s}\mathcal{P}_s + \sum_{n=0}^{\infty} \mathcal{Q}_s{}^n. \tag{4.16}$$

This formula shows that even for s_0 with $\operatorname{Re} s_0 < -h$, if s_0 is connected by a curve where $\mathcal{L}_s$ is decomposable as (4.14) with property (4.15), the left had side of (4.13) can be continued analytically up to s_0, where it is also of the form

$$\frac{1}{1-\lambda_s}\mathcal{R}_s\mathcal{P}_s + \mathcal{R}_s \sum_{n=0}^{\infty} \mathcal{Q}_s{}^n.$$

This fact implies that

$$w(x^{(k)}, z) \text{ can be continued analytically up to } \quad -is_0.$$

Since

$$(1-\lambda_s)\zeta(s) \text{ is uniformly bounded for all } \mathrm{Re}\, s > -h-a,$$

we get

$$|w(x^{(k)}, -is)| \le C\,|\zeta(s)|. \tag{4.17}$$

Therefore, if $|\zeta(s_0)| \le |s_0|^{1-\varepsilon}$, the standard argument for asymptotic solutions gives us the existence of $R(z)$ at $z=-is_0$.

Thus, in the next subsection, we shall consider the decomposition of $\mathcal{L}_s$.

4.4. Decomposition of the Ruelle operator

Choose $a>0$ so that

$$a|\log\theta| > 2.$$

Then, we have

$$\theta^{a\log k} = k^{a\log\theta} < k^{-2-\varepsilon} \quad (\varepsilon > 0). \tag{4.18}$$

Set

$$n = a\log k \tag{4.19}$$

and

$$\Sigma^+_{A,(n)} = \{\xi_{(n)} = (\xi_0, \xi_2, \dots, \xi_n);\ A(\xi_i, \xi_{i+1}) = 1 \quad \text{for } i = 0, 1, \dots, n-1\}.$$

For each $\xi_{(n)}$ take an extension $\tilde{\xi} \in \Sigma^+_A$ and set

$$\tilde{r}(\xi_{(n)}, s) = \tilde{r}(\tilde{\xi}, s).$$

Define $L_{s,(n)}$ an operator in $\Sigma^+_{A,(n)}$ by

$$(L_{s,(n)}v)(\xi_{(n)}) = \sum_{\sigma_A\eta_{(n)} = \xi_{(n)}} e^{\tilde{r}(\eta_{(n)},s)}\, v(\eta_{(n)}),$$

where $\sigma_A\eta_{(n)} = \xi_{(n)}$ signifies that $\eta_{i+1} = \xi_i$ for $n = 0, 1, \dots, n-1$.

We fix n and consider $L_{s,(n)}$ for $k \le |\mathrm{Im}\, s| \le k+1$. We have from (4.18) that

$$||\mathcal{L}_s - L_{s,(n)}||_\infty \le k^{-1-\varepsilon}. \tag{4.20}$$

Denote by $\lambda_{\max,s}$ the largest eigenvalue of $|\mathcal{L}_s|$ and by $w(\xi)$ the associate eigenvector such that $w(\xi) > 0$.

Lemma 4.1. *Let $\alpha > 0$ be the constant in Theorem. Then there exists a positive constant β such that the estimates*

$$||L_{s,(n)}v - \lambda\, v|| \le k^{-\alpha}||v||, \quad v \neq 0 \tag{4.21}$$

and

$$|\lambda| \ge \lambda_{\max,s} - k^{-\alpha} \tag{4.22}$$

implies that

$$\left|\frac{v(\xi_{(n)})}{w(\tilde{\xi})} - 1\right| \le k^{-\beta}. \tag{4.23}$$

Proposition 4.2. *Suppose that (4.21) and (4.22) hold. Then, there is a unitary matrix Θ such that*

$$||\,|\mathcal{L}_s| - \lambda^{-1}\Theta^{-1}L_{s,(n)}\Theta||_\infty < k^{-\beta}. \tag{4.24}$$

Recall that $|\mathcal{L}_s|$ has the following decomposition for all $\mathrm{Re}\, s > -h - a$:

$$\begin{gathered} |\mathcal{L}_s| = \lambda_{\max,s}\mathcal{P}_s + \mathcal{Q}_s, \\ \text{the spectral radius of } \mathcal{Q}_s < 1 - \varepsilon_0 \quad (\varepsilon_0 > 0). \end{gathered} \tag{4.25}$$

If $L_{s,(n)}$ has an eigenvalue λ such that

$$|\lambda| > \lambda_{\max,s} - k^{-\alpha},$$

Proposition shows that

$$||\,|\mathcal{L}_s| - \lambda^{-1}\Theta^{-1}L_{s,(n)}\Theta||_\infty < k^{-\beta}.$$

Then, the decomposition (4.25) implies that

$$\begin{gathered} \lambda^{-1}\Theta^{-1}L_{s,(n)}\Theta = P'_s + Q'_s, \\ \text{the spectral radius of } Q'_s < 1 - \varepsilon_0 + k^{-\beta}. \end{gathered}$$

from which it follows that

$$\begin{gathered} L_{s,(n)} = \lambda\, P_s + Q_s, \\ \text{the spectral radius of } Q_s < 1 - \varepsilon_0 + k^{-\beta}. \end{gathered}$$

By combining the above decomposition and the estimate (4.20), we have the desired decomposability of $\mathcal{L}_s$.

References

1. R. Bowen, *Equilibrium states and the ergodic theory of Anosov differomorphism*, S. L. M., 470, Springer-Verlag, Berlin, 1975.
2. N. Haydn, *Meromorphic extension of the zeta function for Axiom A flows*, Ergod. Th. & Dynam. Sys. **10** (1990), 347–360.
3. M. Ikawa, *Decay of solutions of the wave equation in the exterior of several convex bodies*, Ann. Inst. Fourier **38** (1988), 113–146.
4. M. Ikawa, *On the existence of poles of the scattering matrix for several convex bodies*, Proc. Japan Acad. **64** (1988), 91–93.
5. M. Ikawa, *Singular perturbation of symbolic flows and poles of the zeta functions*, Osaka J. Math. **27** (1990), 281–300.
6. P. D. Lax and R. S. Phillips, *Scattering theory. Revised edition*, Academic Press, New York, 1989.
7. W. Parry and M. Pollicott, *An analogue of the prime number theorem for closed orbits of Axiom A flows*, Ann. Math. **118** (1983), 537-591.
8. M. Pollicott, *Meromorphic extension of generalized zeta function*, Invent. Math. **85** (1986), 147–164.