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Control for Hyperbolic Equations

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This notes are concerned with control theory. We are interested by the following general problem : what are the states of a system that we can reach, starting from a given state, by acting on this system with a (control) function located in a given region of the space, during a certain amount of time.

Although these type of problems have been intensively studied for ODE's, the understanding of the PDE's case is still in progress.

Here, we shall mainly concentrate our study on the model case of the wave equation, with Dirichlet boundary conditions. (Of course, in practice, one can be interested by other type of equations, even nonlinear, or other boundary conditions, or other related problems. For a general view on control theory, the reader can look at the survey article by D.L. RUSSEL [20], and at the book of J-L. LIONS [18].) More precisely, our purpose is to show how one can use microlocal analysis to solve certain basic problems in control theory. To my knowledge, it is C. BARDOS and J. RAUCH who had remark that the multipliers techniques classicaly used in control problems, have to be replaced by microlocalization and propagation of singularities. If one notices that propagation of singularities is generally proved by multipliers techniques, one sees that the real break-through is the localisation in the cotangent bundle given by the microlocal study of the problem.

Historically, the main difficulty to achieve this program was certainly to obtain the result on propagation of singularities for boundary value problems. This has been done by R. MELROSE and J. SJÖSTRAND in [19]. (However, at this moment, the authors had in mind applications to scattering theory, no to control theory.)

These notes are organized as follow.

In I, we expose the H.U.M. method of J-L. LIONS (Hilbert Uniqueness Method), which gives a nice functionnal analysis setting for our problem.

In II, we recall some facts on microlocal analysis for boundary value problems.

In III, we give a proof of the lifting Lemma of C. BARDOS, G. LEBEAU, R. RAUCH **[1]** which is, with the theorem on propagation of singularities, the main ingredient for the study of exact controlability.

In IV, we treat the exact controlability problem.

In V, we give the best known estimates for the general case.

In VI, we give four examples to illustrate the preceeding techniques and results.

Finally, in VII we discuss the stabilization problem and in VIII, we just mention some results on the plate equation.

I. The H.U.M. method of J-L. Lions

Let (M, g) be a compact, analytic, riemanian manifold with boundary ∂M, $\Delta =$ div grad the Laplace operator on M, and $\Box = \partial_t^2 - \Delta$ the wave operator on the cylinder $X = \mathbb{R}_t \times M$.

Let E_0 be the Hilbert space $E_0 = H_0^1(M) \oplus L^2(M)$. For $u = (u_0, u_1) \in E_0$, let $w(t,x)$ be the solution of the evolution problem in X

$$(1) \qquad \Box w = 0\ , \quad w|_{\partial X} = 0\ , \quad w(0,x) = u_0(x)\ , \quad \frac{\partial w}{\partial t}(0,x) = u_1(x)\ .$$

Then $w(t,\cdot) \in C^0(\mathbb{R}_t; H_0^1(M)) \cap C^1(\mathbb{R}_t; L^2(M))$; we shall denote by $u(t) = \left(w(t,\cdot), \frac{\partial w}{\partial t}(t,\cdot)\right) \in E_0$ the Cauchy data of w at time t and will identify $u = u(0)$ with the solution w of (1). Recall that for any $u \in E_0$, we have

$$(2) \qquad \left.\frac{\partial w}{\partial n}\right|_{\partial X} \in L^2_{\mathrm{loc}}(\partial X)$$

where ∂_n is the unit exterior normal to the boundary of M, and more precisely $\int_I \int_{\partial M} \left|\frac{\partial w}{\partial n}\right|^2 \leq C(I)\, \|u\|^2_{E_0}$ for every bounded interval I of $\mathbb{R}_t$.

Let E_{-1} be the Hilbert space $E_{-1} = L^2(M) \oplus H^{-1}(M)$. For $u = (u_0, u_1) \in E_0$, $v = (v_0, v_1) \in E_{-1}$, let $\langle v, u\rangle$ be the duality

$$(3) \qquad \langle v, u\rangle = \int_M u_0 v_1 - u_1 v_0\ .$$

For $g \in L^2_{\mathrm{loc}}(\partial X)$, $v = (v_0, v_1) \in E_{-1}$, let $f(t,x)$ be the solution of the evolution problem in X

$$(4) \qquad \Box f = 0\ , \quad f|_{\partial X} = g\ , \quad f(0,x) = v_0\ , \quad \frac{\partial f}{\partial t}(0,x) = v_1\ .$$

Then $f \in C^0(\mathbb{R}_t, L^2(M)) \cap C^1(\mathbb{R}_t, H^{-1}(M))$; we shall denote by $v(t) = \left(f(t,\cdot), \frac{\partial f}{\partial t}(t,\cdot)\right) \in E_{-1}$ the Cauchy data of f at time t.

For every u solution of (1), and f solution of (4), we have

$$(5) \qquad \langle v(0), u(0)\rangle - \langle v(t), u(t)\rangle = \int_0^t \int_{\partial M} \partial_n w \cdot g\ .$$

Let Γ be an open subset of the boundary ∂M and T a positive number.

For any $u \in E_0$, we define $K(u)$ by

(6) $$K(u) = \frac{\partial w}{\partial n}\Big|_{]0,T[\times\Gamma}$$

where w is the solution of (1); then K is continuous from E_0 into $L^2(]0,T[\times\Gamma)$.

For any $g \in L^2(]0,T[\times\Gamma)$, let $h(t,x)$ be the solution of the evolution problem in X

(7) $$\Box h = 0\ ,\quad h|_{\partial X} = g\ 1_{]0,T[\times\Gamma}\ ,\quad h|_{t=T} = 0\ ,\quad \frac{\partial h}{\partial t}\Big|_{t=T} = 0\ .$$

Let $v(t) = \left(h(t,\cdot), \frac{\partial h}{\partial t}(t,\cdot)\right)$ be the Cauchy-data at time t of h. Define $S(g)$ by

(8) $$S(g) = v(0)\ .$$

Then S is continuous from $L^2(]0,T[\times\Gamma)$ into E_{-1}.

By (5), we have for every $u \in E_0$, $g \in L^2(]0,T[\times\Gamma)$ the identity

(9) $$\langle S(g), u\rangle = \int_0^T \int_\Gamma K(u) g\ .$$

DEFINITION. Let F be the range of S. By definition, F is the subspace of E_{-1} of the data controlable by Γ in time T.

The map S induces a bijection from $L^2(]0,T[\times\Gamma)/\mathrm{Ker}\, S \simeq (\mathrm{Ker}\, S)^\perp$ onto F and we put on F the Hilbert structure of $(\mathrm{Ker}\, S)^\perp$. Then the embedding $F \hookrightarrow E_{-1}$ is continuous.

Let ϕ be the isomorphism from E_0 onto the dual space $(E_{-1})'$ of E_{-1} defined by $\phi(u)(v) = \langle v, u\rangle$. From (9) one deduces that

(10) $$(\Im m K)^\perp = \mathrm{Ker}\, S$$

and if we define for $u \in E_0$, $|u|_G$ by

(11) $$|u|_G = \sup\left\{ |\langle v,u\rangle|\ ,\ v \in F,\ |v|_F \leq 1\right\}$$

we have

(12) $$|u|_G = \|K u\|_{L^2(]0,T[\times\Gamma)}\ .$$

Therefore, the isomorphism ϕ extends by continuity in an isometry from the space G, completion of E_0 for the semi-norm $|\ |_G$, which is isomorphic to $\overline{\Im m K} = (\mathrm{Ker}\, S)^\perp$, onto the dual space F' of F.

If v belongs to F, we shall say that v is controlable, and if $v = S(g)$, we call g a control for v; for a given v in F there exist an unique control g in $(\mathrm{Ker}\, S)^\perp$ for v; this one is optimal in the sense that his norm in $L^2(]0,T[\times\Gamma)$ is minimal.

Notice that v belongs to F and only if there exist a constant C such that

(13) $$\forall u \in E_0 \quad |\langle v, u\rangle| \leq C\,|u|_G = C\,\|K u\|_{L^2(]0,T[\times\Gamma)}$$

because if (13) is true, there exist $g \in L^2(]0,T[\times\Gamma)$ such that $\langle v,u\rangle = \int_0^T\int_\Gamma K(u)g$ for every $u \in E_0$ so the solution f of (4), with $f|_{\partial X} = g\ 1_{]0,T[\times\Gamma}$ $f(0,x) = v_0$, $\frac{\partial f}{\partial t}(0,x) = v_1$, $v = (v_0, v_1)$ satisfies $f(T,x) = \frac{\partial f}{\partial t}(T,x) = 0$ by (5), so $S(g) = v$.

Let $F^\perp$ be the orthogonal to F in E_{-1}. By (9), we have $F^\perp = \operatorname{Ker} K$, and so we have the equivalence

(14) $$F \text{ is a dense subspace of } E_{-1} \Leftrightarrow K \text{ is injective .}$$

Also by the closed graph theorem, (11) and (12) we have

(15) $$F = E_{-1} \Leftrightarrow \exists C,\quad \forall u \in E_0 \quad \|u\|^2_{E_0} \leq C\int_0^T\int_\Gamma |\partial_n w|^2 \ .$$

(Here w satisfies (1) with data u.)

When (15) is true, one say that we have exact controlability.

Here we have discuss the functional analysis setting when the control function acts on a part of the boundary of M. We shall now briefly discuss the interior control case.

Let ω be an open subset of M. For $g \in L^2(]0,T[\times\omega)$, let f be the solution of the evolution problem in X

(16) $$\Box f = g\ 1_{]0,T[\times\omega}\ , \quad f|_{\partial X} = 0\ , \quad f(T,x) = \frac{\partial f}{\partial t}(T,x) = 0\ .$$

Then we have $f \in C^0(\mathbb{R}_t, H_0^1(M)) \cap C^1(\mathbb{R}_t, L^2(M))$ and we define $S(g)$ by

(17) $$S(g) = \left(f(0,x), \frac{\partial f}{\partial t}(0,x)\right) \in E_0\ .$$

The operator S is now continuous from $L^2(]0,T[\times\omega)$ into E_0.

For $v \in E_{-1}$, let w be the solution of the evolution problem in X

(18) $$\Box w = 0\ , \quad w|_{\partial X} = 0\ , \quad w(0,x) = v_0\ , \quad \frac{\partial w}{\partial t}(0,x) = v_1\ .$$

We have $w \in C^0(\mathbb{R}_t, L^2(M)) \cap C^1(\mathbb{R}_t, H^{-1}(M))$ and we define $K(v)$ by

(19) $$K(v) = w\ 1_{]0,T[\times\omega}\ .$$

Then K is a continuous map from E_{-1} into $L^2(]0,T[\times\omega)$;

The formula (9) becomes

(20) $$\langle v, S(g)\rangle = \int_0^T\int_\omega K(v)g\ .$$

The space F of controlable data is still the range of S, with the Hilbert structure of $(\operatorname{Ker} S)^{\perp}$; it is now a subspace of E_0, and the $G-$norm on E_{-1} is defined by

$$(21) \qquad |v|_G = \sup\left\{ |\langle v,u\rangle|\,,\, u\in F,\ |u|_F \le 1\right\} = \|Kv\|_{L^2(]0,T[\times\omega)}\ .$$

The isomorphism ψ from E_{-1} to the dual space $(E_0)'$ of E_0 defined by $\psi(v)(u) = \langle v,u\rangle$ still extends to an isometry from $G(\simeq \overline{\Im m K})$ onto the dual space F' of F.

We still have

$$(22) \qquad u\in F \Leftrightarrow \exists C\ \forall v\in E_{-1}\ \ |\langle v,u\rangle| \le C\,|v|_G$$

(23) $\qquad F$ is a dense subspace of $E_0 \Leftrightarrow K$ is injective

$$(24) \qquad F=E_0 \Leftrightarrow \exists C,\ \forall v\in E_{-1}\ \ \|v\|^2_{E_{-1}} \le C\int_0^T\int_\omega |w|^2$$

(here w satisfies (18) with data v).

II. Microlocal study of boundary value problems

The results of this section are due to R. MELROSE and J. SJÖSTRAND. For more details, we refer to **[19]**, and **[6]**.

We shall denote by X the interior of the cylinder $\mathbb{R}_t\times M$, by ∂X the boundary $\partial X = \mathbb{R}_t\times\partial M$, and by $\overline{X}$ the closure of X, $\overline{X} = X\cup\partial X$. Let Y be a real neighborhood of $\overline{X}$, $\dot{T}^*Y$ his cotangent bundle (where $^{\bullet}$ means that we have removed the zero section), $\dot{T}^*\overline{X} = \dot{T}^*Y|_{\overline{X}}$, $\dot{T}^*X = \dot{T}^*Y|_X$, $\dot{T}^*_bX = \dot{T}^*X\cup\dot{T}^*\partial X$ and let $\dot{T}^*_{\partial X}$ be the conormal bundle to ∂X in Y. Let π be the canonical projection

$$(1) \qquad \dot{T}^*\overline{X}\setminus\dot{T}^*_{\partial X} \xrightarrow{\ \pi\ } \dot{T}^*X\cup\dot{T}^*\partial X = \dot{T}^*_bX\ .$$

We equipe $\dot{T}^*_bX$ with the topology defined by π. Let $\operatorname{car}(\Box)$ be the caracteristic variety of the wave operator

$$(2) \qquad \operatorname{car}(\Box) = \left\{(t,x,\tau,\xi)\in\dot{T}^*Y\ ;\ \tau^2 = |\xi|^2\right\}$$

and put

$$(3) \qquad \Sigma_b = \pi(\operatorname{car}(\Box))\ .$$

The cotangent bundle to the boundary, $\dot{T}^*\partial X$, is the disjoint union of the elliptic set $\mathcal{E}$, the hyperbolic set $\mathcal{H}$, and the glancing set $\mathcal{G}$, defined by

$$(4) \qquad \#\left\{\pi^{-1}(\rho)\cap\operatorname{car}(\Box)\right\} = \begin{cases} 0 & \text{if } \rho\in\mathcal{E}\\ 1 & \text{if } \rho\in\mathcal{G}\\ 2 & \text{if } \rho\in\mathcal{H}\ .\end{cases}$$

Let ρ_0 be a point of $\mathcal{G}$, β_0 the unique point in $\operatorname{car}(\Box)$ such that $\pi(\beta_0) = \rho_0$, and $\beta : s\mapsto\beta(s)$ the hamiltonian curve of $\Box$ such that $\beta_0 = \beta(0)$; then β is tangent to the boundary at β_0. If the order of contact of β with the boundary is exactly two, we have two cases : if

$\beta(s) \in \dot{T}^*X$ for $0 < |s|$ small, we shall write $\rho_0 \in \Sigma_b^{2,-}$, if $\beta(s) \notin \dot{T}^*\overline{X}$ for $0 < |s|$ small, we shall write $\rho_0 \in \Sigma_b^{2,+}$.

We put $\Sigma_b^0 = \Sigma_b \cap \dot{T}^*X$, $\Sigma_b^1 = \mathcal{H}$, $\Sigma_b^2 = \Sigma_b^{2,-} \cup \Sigma_b^{2,+}$, $\Sigma_b^{(3)} = \mathcal{G} \setminus \Sigma_b^2$; we have the disjoint union

$$\Sigma_b = \Sigma_b^0 \cup \Sigma_b^1 \cup \Sigma_b^2 \cup \Sigma_b^{(3)} . \tag{5}$$

Recall the following definition of a ray (see [**19**]).

DEFINITION. Let $p = |\xi|^2 - \tau^2$ be the principal symbol of $\Box$. A ray is a continuous curve $\gamma : I \to \Sigma_b$, where I is an open interval of $\mathbf{R}$ such that

(1) If $\gamma(s_0) \in \Sigma_b^0$, then γ is differentiable at s_0 and $\dot\gamma(s_0) = H_p(\gamma(s))$.

(2) If $\gamma(s_0) \in \Sigma_b^0 \cup \Sigma_b^{2,-}$, then $\gamma(s) \in \Sigma_b^0$ for $0 < |s - s_0|$ small.

(3) If $\gamma(s_0) \in \Sigma_b^{2,+}$, then $\gamma(s) \in \Sigma_b^{2,+}$ for $|s - s_0|$ small, γ is differentiable at s_0 (as a curve in $\Sigma_b^{2,+} \subset \dot{T}^*\partial X$) and $\dot\gamma(s_0) = H_q(\gamma(s_0))$, where $q(t,x;\tau,\xi) = |\xi|_b^2 - \tau^2$ where $|\xi|_b$ is the length of $\xi \in T_x^*\partial X$, for the metric induced by (M,g) on the boundary ∂M.

(4) If $\gamma(s_0) \in \Sigma_b^{(3)}$ and $\{\alpha^+(s), \alpha^-(s)\}$ are the (at most) two points in card $(\Box)$ such that $\pi(\alpha^\pm(s)) = \gamma(s)$, $(\alpha^+(s_0) = \alpha^-(s_0))$ then

$$\lim_{s \to s_0} \frac{\alpha^\pm(s) - \alpha^\pm(s_0)}{s - s_0} = H_p(\alpha^\pm(s_0)) .$$

Recall that near any point of the boundary ∂X, one can find a local chart (y,z), $z \in \mathbf{R}$, $y \in \mathbf{R}^m$, such that X is locally defined by $z > 0$, and $p = \zeta^2 + r(z,y,\eta)$ $(y,\eta) \in T^*\partial X$.

Then $\mathcal{G}$ is defined by $r(0,y,\eta) = 0$, $\Sigma_b^1 = \{r(0,y,\eta) < 0\}$,

$$\Sigma_b^{2,-} = \left\{ r(0,y,\eta) = 0 \, ; \frac{\partial r}{\partial z}(0,y,\eta) < 0 \right\} ,$$

$$\Sigma_b^{2,+} = \left\{ r(0,y,\eta) = 0 \, ; \frac{\partial r}{\partial z}(0,y,\eta) > 0 \right\}$$

$$\Sigma_b^{(3)} = \left\{ r(0,y,\eta) = 0 \, ; \frac{\partial r}{\partial z}(0,y,\eta) = 0 \right\} .$$

In these coordinates the principal symbol q of the induced wave equation on the boundary is $q = r(0,y,\eta)$. A ray contained in $\Sigma_b^{2,+}$ is called a gliding ray.

Because of the analyticity hypothesis on M, for any $\rho \in \Sigma_b$, there exist an unique ray $\gamma : \mathbf{R} \to \Sigma_b$ such that $\gamma(0) = \rho$ ([**19**]). We shall denote this ray by $\phi(s,\rho)$; the components $t(s)$, $\tau(s)$ on the ray satisfie $\tau(s) = \text{cte} = \tau \neq 0$ and $t(s) = t(0) - 2\tau s$.

If $u(t,x)$ is an extendible distribution on X which satisfies the wave equation $\Box u = 0$, its wave front set up to the boundary, $WF_b(u)$ is the subset of $\dot{T}_b^* X$ defined by

$$(6) \qquad \begin{cases} WF_b(u) \cap \dot{T}^* X = WF(u). \\ \text{For } \rho \in \dot{T}^* \partial X, \ \rho \notin WF_b(u) \text{ if there exist} \\ \text{a tangential pseudo-differential operator } A, \\ \text{elliptic at } \rho, \text{ such that } Au \in C^\infty(\overline{X}). \end{cases}$$

If (y, z) is a local chart near some point of the boundary, $z \in \mathbf{R}$, with X defined by $z > 0$, then for u solution of $\Box u = 0$ we have $u \in C^\infty(z \geq 0\, ; \mathcal{D}'_y)$. If $\underline{u}$ is the unique extension of u such that $\underline{u}|_{z<0} \equiv 0$ and $\Box \underline{u} = \varphi_1(y)\delta'_{z=0} + \varphi_0(y)\delta_{z=0}$, then we have

$$(7) \qquad WF_b(u) = \pi(WF(\underline{u}))$$

and so $WF_b(u)$ doesn't depend on the local chart in which we define the tangential pseudo-differential operators.

For u solution of $\Box u = 0$ in X, and $\rho \in \dot{T}_b^* X$, we shall write $u \in H^s_\rho$ if there exist a pseudo-differential operator A of degree s, elliptic at ρ (tangential if $\rho \in \dot{T}^* \partial X$) such that $Au \in L^2(X)$. This is equivalent to have $\underline{u} \in H^s_\beta$ for every $\beta \in \dot{T}^* \overline{X} \setminus \dot{T}^*_{\partial X}$ such that $\pi(\beta) = \rho$, where H^s_β is the usual microlocal Sobolev space at $\beta \in \dot{T}^* Y$.

The main result that we shall use is the following theorem on propagation of singularities due to R. MELROSE and J. SJÖSTRAND [19].

THEOREM. *Let $u(t, x)$ be an extendible distribution in X such that $\Box u = 0$ and $u|_{\partial X} = 0$. Then $WF_b(u)$ is contained in Σ_b, and if $\rho \in WF_b(u)$, then $\phi(s, \rho) \in WF_b(u)$ for every s.*

In the study of the stabilization, we shall also used a propagation result for the boundary condition $\left.\frac{\partial u}{\partial n} + \lambda(x) \frac{\partial u}{\partial t}\right|_{\partial X} = 0$, where $\lambda(x)$ is a smooth non negative function on ∂M. Then (cf. [19]), if u is an extendible distribution in X such that $\Box u = 0$, $\left.\frac{\partial u}{\partial n} + \lambda(x) \frac{\partial u}{\partial t}\right|_{\partial X} = 0$, we have $WF_b(u) \subset \Sigma_b$ and if $\rho \notin WF_b(u)$, we have $\rho' = \phi(s, \rho) \notin WF_b(u)$ if $t(\rho') \geq t(\rho)$.

As a consequence of the preceding theorem, and the well posedness of the mixed problem for the wave equation in H^1, we have the following result on the propagation of the H^1 regularity.

THEOREM. *Let $u(t, x)$ be an extendible distribution in X such that $\Box u = 0$ and $u|_{\partial X} = 0$. If $\rho \in \Sigma_b$ and $u \in H^1_\rho$, then $u \in H^1_{\rho'}$ for every $\rho' = \phi(s, \rho)$. The same result holds for the boundary condition $\left.\frac{\partial u}{\partial n} + \lambda(x) \frac{\partial u}{\partial t}\right|_{\partial X} = 0$ if $t(\rho') \geq t(\rho)$.*

III. Lifting lemma

In the study of controlability and stabilization, we have to transfer information from the boundary to the interior; this is done by the lifting lemma.

DEFINITION. A point ρ in $\dot{T}^*\partial X$ is called non-diffractive if ρ belongs to $\mathcal{E}\cup\mathcal{H}$ or, if $\rho\in\mathcal{G}$ and if $\beta\in\operatorname{car}(\Box)$ is the unique point such that $\pi(\beta)=\rho$, the bicaracteristic curve $s\mapsto\gamma(s)$, $\gamma(0)=\beta$ of $\Box$ through β satisfies $\forall\varepsilon>0$, $\exists s\in\,]-\varepsilon,\varepsilon[$, $\gamma(s)\notin\dot{T}^*\overline{X}$.

LEMMA. *Let u be an extendible distribution in X solution of $\Box u=0$ and $\rho_0\in\dot{T}^*\partial X$ a non-diffractive point. Then if $u|_{\partial X}\in H^1_{\varphi_0}$ and $\left.\frac{\partial u}{\partial n}\right|_{\partial X}\in L^2_{\varphi_0}$, we have $u\in H^1_{\rho_0}$.*

PROOF : Recall that we have to prove $\underline{u}\in H^1_\beta$ for every $\beta\in\dot{T}^*\overline{X}\setminus\dot{T}^*_{\partial X}$ such that $\pi(\beta)=\rho_0$. By hypothesis we have $\Box\underline{u}\in H^{-1/2}_\beta$ for such β, so $\underline{u}\in H^{3/2}_\beta$ if $\beta\notin\operatorname{car}(\Box)$, and by propagation of singularities, using ρ_0 non-diffractive and $\underline{u}\equiv 0$ outside $\overline{X}$, $\underline{u}\in H^{1/2}_\beta$ if $\beta\in\operatorname{car}(\Box)$. If $\rho_0\in\mathcal{E}$, then every β such that $\pi(\beta)=\rho_0$ satisfies $\beta\notin\operatorname{car}(\Box)$ and the result follows. If $\rho_0\in\mathcal{H}$, let β_+,β_- be the two points of $\operatorname{card}(\Box)\cap\pi^{-1}(\rho_0)$; then the bicaracteristic curves of $\Box$ passing through β_+,β_- are transversal to the boundary, and by a classical construction in geometric optics, one can construct u_+,u_- solutions of $\Box u_\pm\in C^\infty(\omega)$ (where ω is a neighborhood of the base point of ρ_0), with $WF(u_\pm)|_{\partial X}$ closed to $\beta_\pm$, and $\rho_0\notin WF_b(u-(u_++u_-))$. Then the hypothesis on the traces of u is equivalent to $u_\pm\in H^1_{\beta_\pm}$, and so, if A is a tangential pseudo-differential operator of order 1, with support sufficiently closed to ρ_0, one has $Au_\pm\in L^2(X)$ and the result follows. Now, we take $\rho_0\in\mathcal{G}$, and a local chart (z,y), $z\in\mathbb{R}$ such that X is given by $z>0$, $\rho_0=(y=0,\eta_0)$, and $\Box=-\partial_z^2+R(z,y,D_y)+1^{\text{er}}$ order term. Let $\beta_0=(z=0,y=0;\zeta=0,\eta_0)$ be the unique point in $\pi^{-1}(\rho_0)\cap\operatorname{car}(\Box)$; we have $\underline{u}\in H^{1/2}_{\beta_0}$ and it remains to prove $\underline{u}\in H^1_{\beta_0}$. By hypothesis, we have $\Box\underline{u}=\varphi_0\,\delta_{z=0}+\varphi_1\,\delta'_{z=0}$, $\varphi_0\in L^2_{\rho_0}$, $\varphi_1\in H^1_{\rho_0}$. Let T_0 be the zero order o.p.d of symbol $\chi_1(\eta)\chi_2\left(\frac{\zeta}{|\eta|}\right)$ χ_1, χ_2 smooth, $\chi_1\equiv 0$ near $\eta=0$, $\chi_1\equiv 1$ for $|\eta|\geq 1$, $\chi_2(u)=0$ for $|u|\leq C_0$, $\chi_2(u)=1$ for $|u|\geq 2C_0$, with C_0 large enough so that $\operatorname{card}(\Box)\cap\operatorname{support}(T_0)=\phi$. Then, by the elliptic theory, if $A(z,y,D_y)$ is a tangential o.p.d of order d, with support closed to ρ_0, we have

$$AT_0\,\underline{u}|_{\pm x_n>0}\in C^k\ (\pm x_n\geq 0, H^{1-k-d}(\partial X))\ . \tag{1}$$

Let $C_1(z,y,D_z,D_y)$ be an o.p.d of order 1, elliptic at β_0, with support closed to β_0, and put $C_1^\varepsilon=C_1\ \theta(\varepsilon\langle|D|\rangle)$ $\langle|D|\rangle=(1+D_z^2+D_y^2)^{1/2}$, $\theta\in C_0^\infty$, $\theta\equiv 1$ near the origin. If c_1^ε is the symbol of C_1^ε, and p the principal symbol of $\Box$, let $\tilde{r}_1^\varepsilon$ be the solution of the equation

$\frac{1}{i}\{p,\widetilde{r}_1^\varepsilon\} = |c_1^\varepsilon|^2$ with support closed to the half bicaracteristic of p which goes outside $\overline{X}$, and take $r_1^\varepsilon = \psi \cdot \widetilde{r}_1^\varepsilon$, ψ homogeneous of degree zero, real, with support closed to support (c_1^ε), equal to 1 near support(c_1^ε), so that $\frac{1}{i}\{p,r_1^\varepsilon\} - |c_1^\varepsilon|^2$ has support in $z \leq -a$, for some $a > 0$. Then r_1^ε is bounded as a family of 1^{er} order symbols, and if R_1^ε is the o.p.d of symbol r_1^ε, one has

$$\square^* R_1^\varepsilon + (R_1^\varepsilon)^* \square = (C_1^\varepsilon)^* C_1^\varepsilon + F^\varepsilon + G^\varepsilon \tag{2}$$

where F^ε is a bounded family of o.p.d of order 1, with support near β_0, and G^ε is a bounded family of o.p.d of order 2, with support contained in $z \leq -a$. We have

$$\begin{aligned}(\underline{u}, (\square^* R_1^\varepsilon + (R_1^\varepsilon)^* \square)\underline{u}) &= 2\Re e(P\underline{u}, R_1^\varepsilon \underline{u}) \\ &= 2\Re e(\varphi_0, R_1^\varepsilon \underline{u}|_{z=0}) - 2\Re e(\varphi_1, \partial_z R_1^\varepsilon \underline{u}|_{z=0}).\end{aligned} \tag{3}$$

Therefore, by (2) and (3), and $\underline{u} \in H^{1/2}_{\beta_0}$, it remains to prove that uniformly in $\varepsilon > 0$, one has

$$R_1^\varepsilon \underline{u}|_{z=0} \in L^2(\partial X) \ , \quad \partial_z R_1^\varepsilon \underline{u}|_{z=0} \in H^{-1}(\partial X) \ . \tag{4}$$

Notice that because of $\underline{u} \in H^{1/2}_{\beta_0}$, and support (R_1^ε) is closed to β_0, we have uniformly in ε

$$R_1^\varepsilon \underline{u} \in C^k(z\,; H^{-1-k}) \ . \tag{5}$$

By (4) and (5), we just have to prove that, if R_1 is a first order o.p.d, with support sufficiently closed to β_0, one has

$$R_1 \underline{u}|_{z<0} \in C^k(z \leq 0\,; H^{-k}(\partial X)) \quad k = 0,1 \tag{6}$$

with bounds which depends only on a finite number of semi-norms of R_1.

By applying the Malgrange division theorem to the principal symbol of R_1 ans $\zeta^2 + r(z,y,\eta)$, one has

$$R_1 = \widetilde{A}_1 + \partial_z \widetilde{A}_0 + S_{-1}\square + R_0 \tag{7}$$

where $\widetilde{A}_i = E_0 A_i$, A_i tangential o.p.d of degree i, with support closed to ρ_0, E_0, S_{-1}, R_0 o.p.d of degree $0,-1,0$, supported near β_0, and $E_0 = \mathrm{Id}$ near support$(A_i) \cap$ card$(\square)$. (Apply the division theorem and multiply the result by E_0.) We have

$$R_1 \underline{u} = A_1 E_0 \underline{u} + \partial_n A_0 E_0 \underline{u} + v \tag{8}$$

with $v = [E_0, A_1]\underline{u} + \partial_n[E_0 + A_0]\underline{u} + S_{-1}(\varphi_0 \delta + \varphi_1 \delta') + R_0 \underline{u}$ so $v \in C^k(z\,; H^{-k}(\partial X))$. Now we have

$$A_1 E_0 \underline{u} = A_1 \underline{u} - A_1 T_0 \underline{u} + A_1(E_0 - 1)(1 - T_0)\underline{u} + A_1 E_0 T_0 \underline{u} \tag{9}$$

$$\begin{aligned}&\partial_n A_0 E_0 u = \\ &= \partial_n A_0 \underline{u} - \partial_n A_0 T_0 \underline{u} + \partial_n A_0(E_0 - 1)(1 - T_0)\underline{u} + \partial_n A_0 E_0 T_0 \underline{u}.\end{aligned} \tag{10}$$

One can suppose that support(T_0) and support(E_0) are disjoint so $E_0 T_0 \underline{u} \in C^\infty(Y)$; by (1) we have $A_i T_0 \underline{u}|_{z<0} \in C^k(z \leq 0, H^{1-k-i}(\partial X))$; obviously, one has $A_1 \underline{u}|_{z<0} \equiv \partial_n A_0 \underline{u}|_{z<0} \equiv 0$; finally, because $E_0 = \mathrm{Id}$

near support $(A_i) \cap \operatorname{car}(\square)$, if $\beta \in$ support $(A_i(E_0 - 1)(1 - T_0))$ one has $\beta \in$ support (A_i) and $\beta \notin \operatorname{car}(\square)$, so $\underline{u} \in H^{3/2}_\beta$ and therefore $A_i(E_0 - 1)(1 - T_0)\underline{u} \in C^k(z, H^{1-i-k}(\partial X))$. □

IV. Exact controlability

Here we keep the notations of the first paragraph.

DEFINITION. Let Γ be an open subset of the boundary ∂M and T a positive number. One says that (Γ, T) geometrically control (M, g) if for every ray $s \mapsto \phi(s, \rho)$ with $\tau(s) = -1/2$, $t(s) = s$, there exist $s \in]0, T[$ such that $\rho' = \phi(s, \rho)$ satisfies $\rho' \in \dot{T}^*\partial X|_{\Gamma \times]0,T[}$ and ρ' non-diffractive.

The following result is due to C. BARDOS-G. LEBEAU-J. RAUCH [1].

THEOREM. *Supposed that (Γ, T) geometrically control (M, g). Then we have exact controlability, that is $F = E_{-1}$.*

PROOF : Let H be the vector space

$$H = \left\{u \in L^2(]-2T, 2T[\times M),\ \square u = 0,\ u|_{\partial X} = 0,\ \partial_n u|_{]0,T[\times\Gamma} \in L^2\right\}$$

equipped with the norm

$$\|u\|_H = \|u\|_{L^2(]-2T,2T[\times M)} + \|\partial_n u\|_{L^2(]0,T[\times\Gamma)} \ .$$

Then H is a Banach space, and we have a natural injection $i : E_0 \hookrightarrow H$ by the identification of §.I.(1). We shall prove later that i is surjective. Then by the closed-graph theorem, there exist C_0 s.t

$$(1) \qquad \forall u \in E_0\ , \quad \|u\|_{E_0} \le C_0 \|u\|_H \ .$$

If there is no constant C s.t $\|u\|_{E_0} \le C\|Ku\|_{L^2(]0,T[\times\Gamma)}$ there is a sequence u_ν in E_0, $\|u_\nu\|_{E_0} = 1$, $\|Ku_\nu\|_{L^2} \to 0$. Then u_ν can be view as a bounded family in $H^1(]-2T, 2T[\times M)$ of solutions of $\square u_\nu = 0$, $u_\nu|_{\partial X} = 0$. By extracting a subsequence, one can suppose that $u_\nu \to u$ in $L^2(]-2T, 2T[\times M)$, and by (1) we have $u \in H$ and $\|u\|_H \ge \frac{1}{C_0}$, $\partial_n u|_{]0,T[\times\Gamma} \equiv 0$. Let N be the subspace of H

$$N = \left\{u \in H\ ;\ \partial_n u|_{]0,T[\times\Gamma} \equiv 0\right\} .$$

If $u \in N$, then $u \in H^1(]-2T, 2T[\times M)$ (because $E_0 \simeq H$) so $\frac{\partial u}{\partial t} = v \in L^2(]-2T, 2T[\times M)$ and $\partial_n v|_{]0,T[\times\Gamma} \equiv 0$, $\square v = 0$, $v|_{\partial X} = 0$, so N is stable by $\frac{\partial}{\partial t}$.

Also, by (1) the two norms $\|u\|_{L^2(]-2T,2T[\times M)}$ and $\|u\|_{H^1(]-2T,2T[\times M)}$ are equivalent on N, so N is finite dimensional. Let u be an eigenfunction of ∂_t on N. We have $u(t,x) = e^{\lambda t}g(x)$, $(\lambda^2 - \Delta)g = 0$, $g|_{\partial M} = 0$, $\partial_n g|_\Gamma = 0$ so by the Holmgren theorem (or the unicity theorem for second order real elliptic operators if one wants to work with smooth manifolds) we conclude that $g \equiv 0$ on every connected component of $\overline{M}$ which meets Γ. Or geometric control clearly imply that every connected component of $\overline{M}$ meets Γ, so $g \equiv 0$, and $N = \{0\}$. So there exist $C > 0$ s.t. $\|u\|_{E_0} \leq C \|Ku\|_{L^2(]0,T[\times\Gamma)}$, and $F = E_{-1}$ by §.I (15).

To prove that i is a surjective map, we just have to verify that H is a subspace of $H^1(]-2T, 2T[\times M)$ or equivalently that for $u \in H$ and $\rho \in \dot{T}_b^*X \cap \{t = 0\}$ we have $u \in H^1_\rho$. If $\rho \notin \Sigma_b$, one has $\rho \notin WF_b(u)$ because $\Box u = 0$ $u|_{\partial X} = 0$, so $u \in H^1_\rho$. If $\rho \in \Sigma_b$, let $\phi(s,\rho)$ be the ray through ρ. By the geometric control hypothesis, there exist $\rho' = \phi(s,\rho)$ a non diffractive point such that $\rho' \in \dot{T}^*\partial X|_{]0,T[\times\Gamma}$. We have $u|_{\partial X} = 0$, $\partial_n u|_{\Gamma\times]0,T[} \in L^2$ so $u|_{\partial X} \in H^1_{\rho'}$, $\partial_n u|_{\partial X} \in L^2_{\rho'}$, and by the lifting lemma one has $u \in H^1_{\rho'}$, and using the propagation theorem one conclude that $u \in H^1_\rho$. □

Notice that, if we made a small perturbation on $(M,g), \Gamma, T$ (for the C^∞ topology), we keep the geometric control property, so the geometric control property implies stable controlability.

Remark also that if there is a ray $s \mapsto \phi(s,\rho)$ $\tau(s) = -\frac{1}{2}$, $t(s) = s$ such that $\rho' = \phi(s,\rho)$ satisfies $\rho' \notin \bar{\Gamma} \times [0,T]$ for every $s \in [0,T]$, then every ray $s \mapsto \phi(s,\tilde{\rho})$ with $\tilde{\rho}$ closed enough to ρ satisfy the same property, so one can suppose that this ray γ contains some interior point, and construct a solution u of $\Box u = 0$, $u|_{\partial X}$ with $WF_b(u) \subset \gamma$, and $u \notin H^1_\rho$ at any $\rho \in \gamma$. Then $\varphi \in C_0^\infty(\mathbf{R})$, $\int \varphi = 1$ and $u_\varepsilon = \varphi_\varepsilon * u$, $\varphi_\varepsilon(t) = \frac{1}{\varepsilon}\varphi\left(\frac{t}{\varepsilon}\right)$, one has $\|u_\varepsilon\|_{E_0} \to \infty$ but $\|Ku_\varepsilon\|_{L^2(\Gamma\times]0,T[)}$ is bounded, so we loose exact controlability in this case.

In the C^∞ category, one can also remove the diffractive points $\rho' = \phi(s,\rho)$ with $\rho' \in \Gamma \times]0,T[$ by moving just a little the boundary near this points, so in this case, the geometric control property is equivalent to stable exact controlability.

We leave to the reader to give the related result for interior exact controlability (in that case, we don't need the lifting lemma).

Finally, let us remark that if $(M,g) = (\Omega, g_0)$ where Ω is a bounded open subset of $\mathbf{R}^n$, $\partial\Omega$ analytic (or smooth) and g_0 the flat metric, then $(\partial\Omega, T)$ satisfies the geometric control property for $T > \sup\{|x-y|, (x,y) \in \bar{\Omega} \times \bar{\Omega}\}$ but with $M = \{(x,y,z) \in \mathbf{R}^3, z \in [0,1], x^2+y^2 = 1\}$, and g the induced metric, then with $\Gamma = \partial M$ and every T, one has $F \neq E_{-1}$.

V. A priori estimates

In this part, we shall discuss what can be deduced from the analyticity hypothesis when the space F of controlable functions is dense in E_{-1}, in the general case where there exist rays which are not controlled by $]0,T[\times\Gamma$.

We suppose that we have

(1) $$T > 2\max\{\operatorname{dist}(x,\Gamma)\,;\, x \in M\}$$

so by the Holmgren uniqueness theorem, K is an injective map, and F is a dense subspace of E_{-1}. For any $v \in E_{-1}$, and $\varepsilon \in]0,1]$, let us define the cost function $C_v(\varepsilon)$

$$C_v(\varepsilon) = \inf\left\{ |v_1|_F \; ; v = v_1 + v_2 \, , \; |v_2|_{E_{-1}} \leq \varepsilon \right\} . \tag{2}$$

By the density of F in E_{-1}, $C_v(\varepsilon)$ is finite. If H is a Hilbert space with a compact injection $H \hookrightarrow E_{-1}$, then the function

$$C(\varepsilon ; H) = \sup\left\{ C_v(\varepsilon) \; ; v \in H \, , \; |v|_H \leq 1 \right\} \text{ is finite.} \tag{3}$$

By definition of the cost function, if $v \in E_{-1}$ and $\varepsilon \in]0,1]$ are given, one can find $g \in L^2(]0,T[\times \Gamma)$, with $|g| \leq C_v(\varepsilon)$ such that the solution f of the evolution problem in X

$$\Box f = 0 \; , \quad f|_{\partial X} = g \, 1_{]0,T[\times\Gamma} \; ; \quad f(0,x) = v_0 \; , \quad \frac{\partial f}{\partial t}(0,x) = v_1 \tag{4}$$

satisfies

$$\left\| f(T,\cdot), \frac{\partial f}{\partial t}(T,\cdot) \right\|_{E_{-1}} \leq \varepsilon \tag{5}$$

and if $g \in L^2(]0,T[\times \Gamma)$ is such that the solution of (4) satisfies (5), then $|g| \geq C_v(\varepsilon)$.

The function $C_v(\varepsilon)$ is decreasing and $v \in F$ if and only if $\lim_{\varepsilon \to 0} C_v(\varepsilon)$ is finite. (By the formula §.I (5), the map $g \mapsto \left(f(T,\cdot), \frac{\partial f}{\partial t}(T,\cdot) \right)$ from $L^2(]0,T[\times \Gamma)$ into E_{-1}, where f is the solution of (4) is continuous for the weak topology.)

Let Δ_D be the realisation of the Laplace operator on $L^2(M)$, with Dirichlet boundary conditions, $\{e_k\}_{k \geq 1}$ an orthonormal basis in $L^2(M)$ of eigenfunctions, $-\Delta e_k = \lambda_k e_k \; e_k|_{\partial M} = 0$, $0 < \lambda_1 < \lambda_2 \leq \ldots$. For $v = (v_0, v_1) \in E_{-1}$, one has

$$\begin{cases} v_0 = \sum v_0^k e_k \; ; & v_0^k \in \ell^2 \\ v_1 = \sum v_1^k e_k \; ; & \lambda_k^{-1/2} v_1^k \in \ell^2 \; . \end{cases} \tag{6}$$

For $s \geq 0$, $\theta > 0$ we introduce the Sobolev spaces $E_{-1+s} = (-\Delta_D)^{-s/2} E_{-1}$, and the analytic spaces $E_{-1}^{\theta} = \exp(-\theta \sqrt{\Delta_D}) E_{-1}$. We have for $v \in E_{-1}^{\theta}$

$$\begin{cases} v_0 = \sum v_0^k e_k \; ; & e^{\theta \lambda_k^{1/2}} v_0^k \in \ell^2 \\ v_1 = \sum v_1^k e_k \; ; & \lambda_k^{-1/2} e^{\theta \lambda_k^{1/2}} v_1^k \in \ell^2 \; . \end{cases} \tag{7}$$

We put on E_{-1+s}, E_{-1}^{θ} their natural Hilbert structure.

By using a construction of geometric optic with complex phase function, one can prove the following estimate from below ([**17**]).

PROPOSITION. *Suppose that there is a ray* $s \mapsto \phi(s,\rho)$, $\tau(s) = -\frac{1}{2}$, $t(s) = s$, *such that* $\rho' = \phi(s,\rho)$ *satisfies for* $s \in [0,T]$, $\rho' \notin [0,T] \times \Gamma$, *and with only transversal points of reflection with the boundary. Then*

i) $\exists \theta_1 > 0$, $\alpha > 0$, $\varepsilon_0 > 0$, $D_0, D_1 > 0$ *such that*

$$C(\varepsilon, E^{\theta}_{-1}) \geq D_0 (D_1 \varepsilon)^{-\alpha/\theta} \tag{8}$$

for $\varepsilon \in]0, \varepsilon_0]$, $\theta \in]0, \theta_1]$.

ii) *For every* $s > 0$, *there exist* $\varepsilon_0 > 0$, $C > 0$ *such that*

$$C(\varepsilon, E_{-1+s}) \geq \exp(C\varepsilon^{-1/s}) \tag{9}$$

for $\varepsilon \in]0, \varepsilon_0]$.

Notice that in particular, under the hypothesis of the proposition, the space F of controlable functions doesn't contain the space E^{θ}_{-1} of analytic vectors, for θ small enough.

The main result here is that we have the same type of estimates from above [**17**].

THEOREM. *Suppose that* (1) *holds. Then, there exist* $\theta > 0$, $C > 0$, $\nu > 0$ *such that*

$$\forall \varepsilon \in]0,1] \quad C(\varepsilon, E^{\theta}_{-1}) \leq C\varepsilon^{-\nu} . \tag{10}$$

By a simple interpolation argument, the inequality (10) implies that we have the same type of inequalities (8), (9), with the opposite sign.

If H_0, H_1 are two Hilbert spaces, with a dense injection $H_0 \hookrightarrow H_1$, let us denote for $\delta \in [0,1]$ by $[H_0, H_1]_\delta$ the Hilbert interpolation space of index δ between H_0 and H_1. Then, for $\delta \in [0,1]$, put

$$\Theta(\delta) = \inf\left\{\theta > 0 \,;\, E^{\theta}_{-1} \subset [F, E_{-1}]_\delta\right\} . \tag{11}$$

We have $\Theta(\delta) \in [0, \infty]$, and the estimate (10) is equivalent to

$$\exists \delta_0 < 1 \quad \text{s.t} \quad \Theta(\delta_0) < \infty . \tag{12}$$

If $\Theta(\delta_1) < \infty$ and $\delta_2 \in]\delta_1, 1[$, then for $\theta > \Theta(\delta_1)$ one has $E^{\theta}_{-1} \hookrightarrow [F, E_{-1}]_{\delta_1}$, so for $\mu \in]0,1[$, $[E^{\theta}_{-1}, E_{-1}]_\mu \hookrightarrow [[F, E_{-1}]_{\delta_1}, E_{-1}]_\mu$. We have $[E^{\theta}_{-1}, E_{-1}]_\mu = E^{\theta(1-\mu)}_{-1}$ and $[[F, E_{-1}]_{\delta_1}, E_{-1}]_\mu = [F, E_{-1}]_{\delta_1 + \mu(1-\delta_1)}$; by choosing $\mu = \frac{\delta_2 - \delta_1}{1 - \delta_1}$, we obtain $\frac{1-\delta_2}{1-\delta_1}\theta \geq \Theta(\delta_2)$, and therefore the function $\frac{\Theta(\delta)}{1-\delta}$ is decreasing for $\delta \in]0,1[$. By (12), the limit

$$\lim_{\delta \to 1^-} \frac{\Theta(\delta)}{1-\delta} = \kappa \tag{13}$$

exists and belongs to $[0, \infty[$. Moreover, under the hypothesis of the proposition, one has $\kappa > 0$.

Finally, let us mention some open questions :

1) Are they examples where $\Theta(\delta) = \infty$ for some $\delta \in]0,1[$? On examples where eigenfunctions e_k does'nt belong to F ?

2) Does there exist some (complex) geometric interpretation for the constant κ ? (In some sense, the number κ measures how far we are from exact controlability.)

3) If (M,g), and $\Gamma \subset \partial M$ are fixed, $\kappa = \kappa(T)$ is a decreasing function of T. What can be said about $\lim\limits_{T\to\infty} \kappa(T)$?

VI. Examples

A. The first example that we shall discuss was communicated to us by J. Sjöstrand, about a question of Sikorav, from I.N.R.I.A. Let Ω be a bounded open set in $\mathbb{R}^d$, $d \geq 2$, with analytic boundary, $x_1, \dots, x_N$ a finite number of distinct points in Ω, and t positive. We suppose the following geometric assumption

$$(1) \quad \left\{ \begin{array}{l} \text{For every } x_j \text{ and } s_0 \in [0,T], \text{ there exist a ray} \\ s \mapsto \gamma(s) = \big(x(s), \xi(s),\ t(s) = s,\ \tau(s) = -\tfrac{1}{2}\big) \\ \text{such that } x(s_0) = x_j \text{ and } x(s) \neq x_k \text{ for every } k \\ \text{and } s \in [0,T] \setminus \{s_0\} \ . \end{array} \right.$$

Then, one has

$$(2) \quad \left\{ \begin{array}{l} \text{There exist a solution } u \text{ of } \Box u = 0,\ u|_{\partial X} = 0, \\ \text{such that } u(x_j, t) \equiv 0 \text{ for every } j,\ t \in]0,T[, \\ \text{and } u \not\equiv 0. \end{array} \right.$$

To prove this, let us introduce the Hilbert space E of Cauchy data (u_0, u_1) ıch that $u_j = \sum\limits_k u_j^k e_k$ where $\{e_k\}$ is an orthonormal basis of eigenfunctions of $-\Delta_D$ on Ω $-\Delta e_k = \lambda_k e_k$, $e_k|_{\partial\Omega} = 0$ and $\left\{ \lambda_k^{\frac{d-1}{4} - \frac{j}{2}} u_j^k \right\} \in \ell^2$. We identify $u \in E$ with the solution of the evolution problem $\Box u = 0$, $u|_{\mathbb{R}\times\Omega} = 0$, $u(0,x) = u_0$, $\frac{\partial u}{\partial t}(0,x) = u_1$; then $u \in H^{\frac{d-1}{2}}_{\mathrm{loc}\,(t)}(\mathbb{R} \times \Omega)$. Recall that any solution in $\mathbb{R}^d$ of $\Box u = 0$ is locally of the form

$$(3) \qquad u(t,x) = \int e^{it|\xi|} e^{ix\xi} \widehat{v}_+(\xi) \frac{d\xi}{(2\pi)^d} + \int e^{-it|\xi|} e^{ix\xi} \widehat{v}_-(\xi) \frac{d\xi}{(2\pi)^d}$$

with $v_\pm \in \mathcal{S}'(\mathbb{R}^d)$ and $u \in H^\sigma$ near $(x_0, 0)$ is equivalent to $v_\pm \in H^\sigma$ near x_0.

So, if H denotes the Hilbert space $H = \bigoplus\limits_j L^2(\{x_j\} \times]0,T[)$ the map from E to H

$$(4) \qquad u \mapsto R(u) = \bigoplus_j u(x_j, t)\, 1_{[0,T[}$$

is continuous. Moreover, if $g(t) \in L^2_{\mathrm{comp}}(\mathbb{R})$, and if we introduce $\widehat{v}_\pm(\xi) = \chi_1\left(\frac{\xi}{|\xi|}\right) \chi_0(|\xi|) \widehat{g}(\pm|\xi|)\, |\xi|^{-d+1}$ with χ_0 smooth, $\chi_0 \equiv 0$ near the origin,

$\chi_0(|\xi|) = 1$ for $|\xi| \geq 1$ and χ_1 a smooth non negative function on the sphere $\mathbb{S}^{d-1}$ one has, for some constant c_0, with u given by (3)

$$(5)\qquad \begin{cases} u(t,0) - c_0 \displaystyle\int \chi_1(\omega)\cdot g(t) \in C^\infty(\mathbb{R}) \\ v_\pm \in H^{\frac{d-1}{2}} \\ WF(v_\pm) \subset \{(x,\xi), \exists (t,\tau) \in WF(g), \\ \qquad \exists \omega \in \text{support}\,(\chi_1), \xi = |\tau|\omega, x = \alpha\omega, |\alpha| = 1\}. \end{cases}$$

Notice also that if $u = u_+ + u_-$ is the decomposition (3), one has $WF(u_\pm) \subset \pm\tau > 0$; Using the geometric hypothesis (1) the local construction given by (5) and the propagation of singularities, one construct a map $B : H \to E$ such that $\mathrm{Id} - R \circ B = K$, where K is bounded from H to $\underset{j}{\oplus} C^\infty(\{x_j\} \times [0,T])$, so is compact and the range of R is closed, and of finite codimension.

If R was injective, then $|||u||| = |Ru|$ will be a Hilbert norm on E (because $(E, |||\cdot|||) \simeq (\text{Range}\, R, |\ |_H)$ and by the closed graph theorem, there will exist $C > 0$ such that $\|u\|_E \leq C\,|Ru|_H$ for every u in E, and this is impossible because there exist a solution f of $\Box f = 0$, $f|_{\partial\Omega\times\mathbb{R}} = 0$, with $f \notin E$ and $WF_b(f) \subset \gamma$, where γ is a ray which does'nt intersect the $\{x_j\} \times [0,T]$, and $f_\varepsilon = f * \varphi_\varepsilon$, $\varphi_\varepsilon(t) = \frac{1}{\varepsilon}\varphi\left(\frac{t}{\varepsilon}\right)$, $\varphi \in C_0^\infty$, $\int \varphi = 1$, satisfies $|Rf_\varepsilon|_H$ bounded and $\|f_\varepsilon\|_E \to \infty$.

Notice however that if u satisfies $\Box u = 0$, $u|_{\partial X} = 0$ and $u(x_0, t) \equiv 0$ for a given $x_0 \in \Omega$, and every $t \in \mathbb{R}$, there exist an eigenfunction e of $(\Delta, \text{Dirichlet})$ such that $e(x_0) = 0$.

B. This second example is due to J. Rauch. Take $M = \mathbb{S}^2$, the unit sphere in $\mathbb{R}^3$, with the standard metric and $\omega = \{(x,y,z) \in \mathbb{R}^3; x^2 + y^2 + z^2 = 1, z > 0\}$. In this case, for $T > \pi$ we have exact controlability for (ω, T). This is just a limit case where the geometric control property is violated, with only one ray uncontrolled, the equatorial one (disregarding the orientation), which lives on the closure $\bar{\omega}$.

Recall that the action of the group of rotations induced an orthogonal decomposition of $L^2(\mathbb{S}^2)$

$$(6)\qquad L^2(\mathbb{S}^2) = \oplus E_k$$

where the E_k are the eigenspaces of the Laplace operator; the dimension of E_k is $2k+1$, and the associated eigenvalue $k^2 + k = \lambda_k$. The functions in E_k are exactly the restriction to the sphere of the harmonic polynomials in $\mathbb{R}^3$, homogeneous of degree k, and so for $f(x) \in E_k$ one has $f(-x) = (-1)^k f(x)$.

In §.I, we have identified $u = (u_0, u_1) \in E_{-1} = L^2 \oplus H^{-1}$ with the solution of the wave equation with data (u_0, u_1), so we have

$$(7)\qquad u(t,x) = \sum_k e^{it\sqrt{\lambda_k}} a_k^+ + e^{-it\sqrt{\lambda_k}} a_k^- = u^+ + u^-$$

and $\|u\|^2_{E_{-1}} = 2\sum |a_k^+|^2_{L^2} + |a_k^-|^2$, $a_k^\pm \in E_k$.

We have to prove that if $\varphi \in C_0^\infty(\mathbf{R})$, $\varphi \geq 0$, $\varphi(t) = 1$ for $t \in [0,\pi]$ one has, for some $C > 0$

$$(8) \qquad \forall u \in E_{-1}\ , \quad \|u\|^2_{E_{-1}} \leq C\int_{-\infty}^{\infty} \varphi(t) \int_\omega |u(t,x)|^2 \ .$$

First, we notice that it is sufficient to prove (8) for u_+ and u_- independantly, because, with $\|u\|^2_{E_{-2}} = 2\sum \frac{1}{(1+\lambda_k)}\left(|a_k^+|^2_{L^2} + |a_k^-|^2_{L^2}\right)$ one has

$$(9) \quad \int_{-\infty}^{+\infty} \varphi(t) \int_\omega \mathrm{Re}\,(u_+\bar{u}_-)$$

$$= \sum_{k,\ell} \mathrm{Re}\left(\int_{\mathbf{S}^2} a_k^+\,\overline{a_{\bar{p}}}\cdot\widehat{\varphi}(\sqrt{\lambda_k}+\sqrt{\lambda_\ell})\right) \leq C_0\,\|u\|^2_{E_{-2}}$$

with C_0 independant of u, so we will obtain (8) with an extra term $C\,\|u\|^2_{E_{-2}}$ on the right, and using the compacity of the injection $E_{-1} \hookrightarrow E_{-2}$, and the uniqueness argument $(T > \pi)$, the result will follow.

Now, we just remark that we have $\sqrt{\lambda_k} = \sqrt{k^2+k} = k + \frac{1}{2} + 0\left(\frac{1}{\sqrt{\lambda_k}}\right)$, and using $a_k^+(-x) = (-1)^k\ a_k^+(x)$, one has

$$(10) \qquad u_+(t+\pi,x) = i u_+(t,-x) + r_+(t,x)$$

with $\|r_+\|_{E_{-1}} \leq C_1\,\|u_+\|_{E_{-2}}$, C_1 independant of u_+.

Now, take $\psi \in C_0^\infty(\mathbf{R})$, $\psi \geq 0$, ψ supported near $t = 0$ such that $\psi(t) + \psi(t-\pi) \leq \varphi(t)$ and $\psi(0) > 0$. We have, using (10)

$$(11) \quad \int \varphi(t) \int_\omega |u_+|^2 \geq \int \psi(t) \int_\omega |u_+(t,x)|^2 + |u_+(t,-x) - i r_+(t,x)|^2$$

$$\geq \frac{1}{2}\int \psi(t) \int_{\mathbf{S}^2} |u_+|^2 - 2\int \psi(t) \int_\omega |r_+|^2$$

and we have $\int \psi(t) \int_{\mathbf{S}^2} |u_+|^2 \geq C_2\,|u_+|^2_{E_{-1}}$, $\int \psi(t) \int_\omega |r_+|^2 \leq C_3\,|u_+|^2_{E_{-2}}$, for $C_{2,3} > 0$ independant of u, and the result follows by the compactness and unicity argument.

C. This example was communicated to us by C. Bardos. We are here interested by the finite dimensional control problem. We keep the notations of the first paragraph.

Let $\varphi_1,\ldots,\varphi_N$ be a finite set of linearly independant elements of $(E_1)'$; we want to know if for every data $v(0) = (v_0,v_1) \in E_{-1}$, there exist a control function $g \in L^2(]0,T[\times\Gamma)$ such that the solution of the evolution problem

$$(12) \qquad \Box f = 0,\ f|_{\partial X} = g\,1_{]0,T[\times\Gamma},\ f(0,x) = v_0,\ \frac{\partial f}{\partial t}(0,x) = v_1$$

satisfies, with $v(T) = \left(f(T,x), \frac{\partial f}{\partial t}(T,x)\right) \in E_{-1}$,

$$\varphi_i(v(T)) = c_i \tag{13}$$

where the c_i are given numbers. If $u_i \in E_0$ is such that $\langle v, u_i\rangle = \varphi_i(v)$, using §.I (5) one sees that this problem is equivalent to solve

$$\int_0^T \int_\Gamma \partial_n h_i \cdot g = \langle v(0), h_i(0)\rangle - c_i \tag{14}$$

where h_i satisfies $\Box h_i = 0$, $h_i|_{\partial X} = 0$, $\left(h_i(T,\cdot), \frac{\partial h_i}{\partial t}(T,\cdot)\right) = u_i$. So if H is the finite dimensional space spaned by the h_i, this problem is always solvable if and only if the map $K_H : H \to L^2(]0,T[\times \Gamma)$, $h \mapsto K(h) = \frac{\partial h}{\partial n}\Big|_{]0,T[\times\Gamma}$ is injective. Let us give some examples (we suppose M connected).

a) If K is injective, K_H is, a fortiori, injective.

b) If H is spaned by analytic vectors, K_H is injective as soon as we have $T > 0$, and $\Gamma \neq \phi$. In particular, it is the case when H is spaned by eigenfunctions of $(-\Delta$, Dirichlet$)$.

c) More generally, K_H is injective if every $h \in H$ is microlocally analytic except on rays with no diffractive points, and intersecting $]0,T[\times \Gamma$. (Because in this case, on can apply the theorem of propagation of analytic singularities of J. Sjöstrand to conclude that if $h \in H$, and $K_H(h) = 0$, then h is analytic up to the boundary, so $h \equiv 0$.)

Of course, the finit dimensional argument injective $\Leftrightarrow$ surjective gives no information on the norm ρ_H of the map (with $c_i \equiv 0$) $v(0) \mapsto \inf \|g\|_{L^2}$.

If we take $H = H_N$ the space spaned by the $N-$ first eigenfunctions, and $T > 2\max\{\mathrm{dist}(x,\Gamma); x \in M\}$, let $\alpha_N = \rho_{H_N}$; then α_N is an increasing function of N and from the result of §.V, one deduces

$$\exists C_0 > 0, \quad \alpha_N \le C_0 \exp(C_0\sqrt{\lambda_N}) \tag{15}$$

[by §.V (10), one has $C(\varepsilon, E^\theta_{-1}) \le C\varepsilon^{-\nu}$; Let Π_N be the orthogonal projection of E_{-1} onto the space generated by the N first eigenfunctions; by definition $\alpha_N = \sup_{|v|_{E_{-1}} \le 1} \{\inf(|y|_F ; \Pi_N(v-y) = 0)\}$. Take $v \in E_{-1}$, $|v|_{E_{-1}} \le 1$; then $\Pi_N(v) = w \in E^\theta_{-1}$ and $|w|_{E^\theta_{-1}} \le e^{\theta\sqrt{\lambda_N}}$, so $w = y + z$ with $|z|_{E_{-1}} \le \varepsilon e^{\theta\sqrt{\lambda_N}}$, and $|y|_F \le C\varepsilon^{-\nu} e^{\theta\sqrt{\lambda_N}}$, and there exist $y' \in F$ such that $\Pi_N(z - y') = 0$, $|y'|_F \le \alpha_N \varepsilon e^{\theta\sqrt{\lambda_N}}$ and therefore, $\Pi_N(v - (y + y')) = 0$ imply $\alpha_N \le e^{\theta\sqrt{\lambda_N}}[C\varepsilon^{-\nu} + \varepsilon\alpha_N]$ and the result follows by taking $\varepsilon = \frac{1}{2}e^{\theta\sqrt{\lambda_N}}$].

In the opposite direction, if the hypothesis of the proposition of §.V is fulfilled, one can show [**17**]

$$\exists C_1 > 0, \quad \forall N, \quad \alpha_N \ge C_1 \exp C_1\sqrt{\lambda_N}. \tag{16}$$

D. In this last example, we shall briefly indicate what is the structure of the space F of controlable data, for the interior control problem with

$$(17)\qquad \begin{cases} M = \mathbf{S}^2 \text{ unit sphere in } \mathbf{R}^3 \\ \omega = \{(x,y,z) \in M\,;\, \sqrt{1-r^2} < z\}, 0 < r < 1\,;\; T > 2\pi . \end{cases}$$

Let $\mathcal{B}(M)$ be the space of Sato hyperfunctions on M and for $f \in \mathcal{B}(M)$

$$(18)\qquad K(f) = \frac{1}{4\pi}\int_{\mathbf{S}^2} \frac{f(y)}{(z-y)^{3/2}}\, dy\ .$$

Then K is an isomorphism from $\mathcal{B}(M)$ onto the space of holomorphic functions on the open strictly pseudo-convex subset $\Omega_0 = \left\{z \in C,\ |z| < \frac{1}{\sqrt{2}}\right\}$ of the complex isotropic cone, $C = \left\{z \in \mathbb{C}^3,\ z^2 = 0\right\}$. If for $d \in \mathbf{N}$ we denote by E_d the space of harmonic polynomials homogeneous of degree d and if we choose an orthonormal basis $e_d^j(x)$ of E_d, $1 \le j \le 2d+1$, for the Hilbert structure on E_d induce by $L^2(\mathbf{S}^2)$, such that $e_d^j(x)$ are orthogonal for the Hilbert structure on E_d induce by $L^2(\Omega_0)$ (for the measure on C induce by the Lebesgue measure on $\mathbb{C}^3$) one has for $f = \sum f_{j,d}\, e_d^j(x)|_{|x|=1}$, $K(f) = \sum f_{j,d}\, e_d^j(z)|_{z \in C}$, and the equivalence

$$(19)\qquad (x,\xi)_{|\xi|=1} \in SS(f) \Leftrightarrow \frac{x - i\xi}{2} \in \text{supp sing}\, K(f)|_{\partial\Omega_0}\ .$$

Take $t_0 > T$, and let S be the map from $\Big\{g(t,x) \in \mathcal{B}(\mathbf{R}\times\mathbf{S}^2)\,;\ \text{support}\,(g) \subset [0,T] \times \bar{\omega}\Big\} = \mathcal{D}$ into $\mathcal{B}(M)^2$ defined by : if $u(t,x)$ is the solution of $(\partial_t^2 - \Delta)u = g$; support $(u) \subset (t \ge 0)$, $S(g) = \big(u(t_0,x),\ \frac{\partial u}{\partial t}(t_0,x)\big)$; Let I the map from $\mathcal{D}$ into $\mathcal{O}(\Omega_0)^2$, $I = K \circ S$. Then if $g(t,x) = \sum g_d^j(t)\, e_d^j(x)$ one has, with $\lambda_d = d^2 + d$

(20)

$$I(g) = \sum \int_{-\infty}^{\infty} \left(\frac{\sin{(t_0 - s)}\sqrt{\lambda_d}}{\sqrt{\lambda_d}},\ \cos{(t_0 - s)}\sqrt{\lambda_d}\right) g_d^j(s)\, ds \cdot e_d^j(z)$$
$$= \int_{-\infty}^{\infty} ds \int_{\mathbf{S}^2} \Big(A_0(s, 2z\cdot x),\ A_1(s, 2z\cdot x)\Big)\, g(s,x)$$

with $A_j(s,z) = \sum_{d=0}^{\infty} A_j^d(s)\, z^d$, $A_0^d(s) = \lambda_d^0 \frac{\sin{(t_0-s)}\sqrt{\lambda_d}}{\sqrt{\lambda_d}}$,
$A_1^d(s) = \lambda_d^0 \cos{(t_0 - s)}\sqrt{\lambda_d}$, with $\sum \lambda_d^0 z^d = \frac{1}{4\pi}(1-z)^{-3/2}$, so $A_j(s,z)$ is defined for $s \in \mathbb{R}$, $z \in \mathbb{C}$, $|z| < 1$, is holomorphic in z, and is holomorphic near (s,z), $|z| = 1$ if $e^{\pm i(t_0 - s)}\ z \ne 1$. By (20), if $\Omega = \left\{z \in C\,;\ \sup_{x \in \bar{\omega}} |z \cdot x| < \frac{1}{2}\right\}$ one has

$$(21)\qquad I(g) \in \mathcal{O}(\Omega)^2\ .$$

In particular, one has $K(F) \subset \mathcal{O}(\Omega)^2$, which gives a precise microlocal information on F. The boundary of Ω is not smooth, but has a natural stratification

$$\partial\Omega = \partial\Omega_h \cup \partial\Omega_g \cup \partial\Omega_e \cup \partial\Omega_c \tag{22}$$

with $\dim\partial\Omega_h = 3$, $\dim\partial\Omega_e = 3$, $\dim\partial\Omega_g = 2$, $\dim\partial\Omega_c = 1$, $\overline{\partial\Omega_h} = \partial\Omega_h \cup \partial\Omega_g = \partial\Omega \cap \partial\Omega_0$, $\overline{\partial\Omega_c} = \partial\Omega_g \cup \partial\Omega_e \cup \partial\Omega_c$, $\partial\Omega_g$, $\partial\Omega_c$ are closed. We have $z = u + iv \in \partial\Omega_h$, (resp. $\partial\Omega_g$) if and only if $z = u + iv$, $u^2 = v^2 = \frac{1}{4}$, $u \cdot v = 0$, and the plane, in $\mathbf{R}^3$, $\Pi(u,v)$ spaned by (u,v) intersect $\partial\omega$ in two different points (resp. $\Pi(u,v)$ is tangent to $\partial\omega$); we have also $\Omega_c = \partial\Omega \cap (z_3 = 0) = \left\{z \in C;\, z_3 = 0,\ |z| = \frac{1}{r\sqrt{2}}\right\}$; $\partial\Omega$ has a conical singularity at $\partial\Omega_c$, and $\partial\Omega_h$, $\partial\Omega_e$ are strictly pseudo-convex.

Notice that Ω (and $\partial\Omega$) are invariant under the $\mathrm{U}(1)$ action $z \mapsto z\,e^{it}$ (which corresponds to the geodesic flow on the sphere), and so the spaces E_d are mutually orthogonal in $L^2(\Omega)$; Therefore, for $x \in \mathcal{O}(\Omega_0) \cap L^2(\Omega_0)$ $x = \sum x_d$, $x_d \in E_d$, one has $x \in \mathcal{O}(\Omega) \cap L^2(\Omega)$ if and only if $\sum \|x_d\|^2_{L^2(\Omega)} < \infty$, and the space $\mathcal{O}(\Omega)^2$ is invariant under the free evolution group of the wave equation (one has $x = \sum x_d \in \mathcal{O}(\Omega)$ if and only if $\sum \|x_d\|^2_{L^2(\Omega)}\, e^{-\varepsilon d} < \infty$ for every $\varepsilon > 0$).

By a detailed analysis of the map I, which involves the construction of analytic parametrix for the diffraction near $\partial\Omega_g$, one can show that the range of I is equal to $\mathcal{O}(\Omega)^2$. [Notice however that if we replace the disc ω by the union of two disjoint small disc ω_1, ω_2, we will have range$(I) = \mathcal{O}(\Omega_1)^2 + \mathcal{O}(\Omega_2)^2$ which is not of the form $\mathcal{O}(\Omega)^2$ because of the cohomological obstruction in the Cousin-problem.]

VII. Stabilization

A) Neuman Stabilization.

In this part, we take a non negative smooth function $\lambda(x) \in C^\infty(\partial M; [0,\infty[)$, and we are interested in the asymptotic behavior in time of solutions of the evolution problem in $M \times [0,\infty[$

$$\begin{cases} \Box u = 0 \text{ in } M \times]0,\infty[\ ; \\ \frac{\partial u}{\partial n} + \lambda \frac{\partial u}{\partial t} = 0 \text{ on } \partial M \times]0,\infty[\\ u|_{t=0} = u_0 \in H^1 \ , \ \frac{\partial u}{\partial t}\Big|_{t=0} = u_1 \in L^2 \ . \end{cases} \tag{1}$$

Notice that the constant functions are trivial solutions of (1). This mixte problem is well posed in $t \geq 0$, and we have $u(t,x) \in C^0(\mathbb{R}_+, H^1) \cap C^1(\mathbb{R}_+, L^2)$.

For every $t > 0$, one has $\sqrt{\lambda}\frac{\partial u}{\partial t}\Big|_{\lambda>0} \in L^2(]0,t[\times (\partial M \cap \lambda > 0))$ and, if we denote by $E(t)$ the energy of the solution at time t,

$E(t) = \frac{1}{2}\int_M |\nabla_x u|^2 + |\frac{\partial u}{\partial t}|^2,$

$$(2) \qquad E(0) - E(t) = \int_0^t \int_{\partial M \cap \lambda > 0} \lambda \left|\frac{\partial u}{\partial t}\right|^2 .$$

Then we have the following result on uniform exponential decay for the energy (i.e. the norm of the solution modulo the constant functions).

THEOREM. *Suppose that there exist $T > 0$ such that $(\lambda > 0, T)$ has the geometric control property.*

There exist $C > 0$ such that for every data $(u_0, u_1) \in H^1 \times L^2$ one has

$$(3) \qquad E(t) \leq \frac{1}{C} e^{-ct} \cdot E(0) .$$

PROOF : Let us take $t_2 > t_1 > T$ and $\varepsilon \in]0, t_2 - t_1[$. Let $H = H^1 \oplus L^2$, and G, F^0, F^1 the Banach spaces

$$(4) \quad G = \Big\{ u \in L^2(]0, t_2[\times M); \Box u = 0, \frac{\partial u}{\partial n} + \lambda \frac{\partial u}{\partial t}\Big|_\partial = 0, \sqrt{\lambda}\, \frac{\partial u}{\partial t}\Big|_{\lambda > 0} \in L^2(]0, t_2[\times (\lambda > 0)\Big\}$$

equipped with the norm

$$\|u\|_G = \|u\|_{L^2(]0,t_2[\times M)} + \left\|\sqrt{\lambda}\, \frac{\partial u}{\partial t}\right\|_{L^2(]0,t_2[\times(\lambda>0))}$$

$$(5) \quad F^s = \Big\{ u \in H^s(]t_1, t_1 + \varepsilon[\times M), \Box u = 0, \frac{\partial u}{\partial n} + \lambda \frac{\partial u}{\partial t}\Big|_\partial = 0 \Big\}$$

equipped with the norm

$$\|u\|_{F^s} = \|u\|_{H^s(]t_1,t_1+\varepsilon[\times M)} \cdot$$

Let i be the inclusion of H into G, which is continuous by (2), and r the restriction map from G to F^0. If $u \in G$ and $\rho \in \dot{T}^*\partial X|_{]0,t_2[\times(\lambda>0)}$ one has $\frac{\partial u}{\partial t}\Big|_\partial \in L^2_\rho$, so $\frac{\partial u}{\partial n} \in L^2_\rho$ by the boundary condition; also, u satisfy $\Box u = 0$, $\frac{\partial u}{\partial n} + \lambda \frac{\partial u}{\partial t}\Big|_\partial$ so $WF_b(u) \subset \Sigma_b$ and $\frac{\partial u}{\partial t}\Big|_\partial \in L^2_\rho$ implies $u|_\partial \in H^1_\rho$, because we have $(\tau = 0) \subset \mathcal{E}$.

Let now $\rho \in \Sigma_b \cap (t = t_1)$, and γ_- be the half ray with end point at ρ, contained in $t \leq t_1$. By hypothesis, there exist $\rho' \in \gamma_-$, $\rho' \in \dot{T}^*\partial X|_{]0,t_2[\times(\lambda>0)}$, ρ' non diffractive. We have shown that, for $u \in G$, $u|_\partial \in H^1_{\rho'}$, $\frac{\partial u}{\partial n} \in L^2_{\rho'}$, so by the lifting lemma, one has $u \in H^1_{\rho'}$, and by propagation (the time increases from ρ' to ρ) $u \in H^1_\rho$. Therefore the range of r is contained in F^1, and by the closed graph theorem, r is continuous from G to F^1. By (2), $E(t)$ is a decreasing function for every $u \in H$, so we conclude that there exist $C > 0$ such that

$$(6) \qquad \forall u \in H \quad E(t_2) \leq C \|i(u)\|_G^2$$

and, using (2) this imply

$$(7) \qquad \forall u \in H \qquad \|u\|_H^2 \le (1+C)\,\|i(u)\|_G^2 \ .$$

As in the proof of the exact controlability, if the inequality (8) were false

$$(8) \qquad \exists C' \ , \ \ \forall u \in H \ , \ \ E(0) \le C' \int_0^{t_2} \int_{\partial M \cap \lambda > 0} \lambda \left| \frac{\partial u}{\partial t} \right|^2$$

one construct a sequence u_ν, $E_{u_\nu}(0) = 1$, $\int_M u_\nu(0,x) = 0$, $\int_0^{t_2} \int_{\partial M \cap (\lambda < 0)} \lambda \left| \frac{\partial u_\nu}{\partial t} \right|^2 \to 0$ and by (7) and the compacity of the injection from H into $L^2(]0,t_2[\times M)$, one can supposed $u_\nu \to u \in H$, and we are reduced to prove that the space $N_{t_2} = \left\{ u \in H \,;\, \frac{\partial u}{\partial t}\Big|_{]0,t_2[\times(\lambda>0)} \equiv 0 \right\}$ are reduced to constants (because if $u_\nu \to u$ in H, $\frac{\partial u_\nu}{\partial t}\Big|_\partial \to \frac{\partial u}{\partial t}\Big|_\partial$ in $\mathcal{D}'$). By (7), these spaces are finite dimensional and decreasing in t_2 ; so for α small enough and $s \in]t_2 - \alpha, t_2[$ the spaces N_s are independant of s, and therefore stable by $\frac{\partial}{\partial t}$ (for $u \in N_s$, $\frac{\partial u}{\partial t} \in G$, so $\frac{\partial u}{\partial t} \in H$). But if $v(x)e^{\mu t} \in N_s$, one has $(\mu^2 - \Delta)v = 0$, $\partial_n v + \mu\lambda(x) v|_\partial = 0$ and $\mu v(x)|_{\partial M \cap (\lambda > 0)} = 0$, so for $\mu \ne 0$ one has $v = \partial_n v = 0$ on $\partial M \cap (\lambda > 0)$ and $v \equiv 0$ by unicity. If $\mu = 0$, then $\Delta v = 0$, $\partial_n v|_\partial = 0$ so $v = $ cte.

So (8) is true and by (2) one has

$$(9) \qquad E(t_2) \le E(0)\left(1 - \frac{1}{C'}\right)$$

and the theorem follows from the semi-group property.

B) Dirichlet stabilization.

The situation here is different from the Neuman case. We take a continuous non negative function $\lambda(x) \in C^\infty(\partial M; [0,\infty[)$ such that the boundary is a disjoint union $\partial M = \Gamma^- \cup S \cup \Gamma^+$, with $\Gamma^\pm$ open, S a smooth hypersurface, $\lambda|_{\Gamma^-} \equiv 0$, $\lambda|_{\Gamma^+} > 0$, $\lambda|_{\Gamma^+}$ smooth, and near any point $x_0 \in S$, with S defined by $\rho = 0$, $d\rho \ne 0$, and $\Gamma^+ = \{\rho > 0\}$ we suppose

$$(1) \qquad \sqrt{\lambda} = a(x)\,\rho_+^{\alpha/2} \ , \quad a \in C^\infty \ , \quad a > 0 \ , \quad \rho_+ = \sup(\rho, 0) \ .$$

Here, $\alpha \in]0,\infty[$ is independant of $x_0 \in S$. We look at the evolution problem

$$(2) \qquad \begin{cases} \Box u = 0 \ \text{in} \ M \times]0,\infty[\ ; \ \frac{\partial u}{\partial t} + \lambda(x) \frac{\partial u}{\partial n}\Big|_{\partial M \times]0,\infty[} \equiv 0 \\ u|_{t=0} = u_0 \ , \quad \frac{\partial u}{\partial t}\Big|_{t=0} = u_1 \end{cases}$$

with $(u_0, u_1) \in H = H^1(M) \oplus L^2(M)$. This mixed problem is well posed, the solution $u(t,x)$ satisfies $u \in C^0(\mathbb{R}_+, H^1) \cap C^1(\mathbb{R}_+, L^2)$

$\sqrt{\lambda}\frac{\partial u}{\partial n}\Big|_{\partial M\times]0,T[} \in L^2(\partial M \times]0,T[)$ and

(3) $$E(0) - E(T) = \int_0^T \int_{\partial M} \left|\sqrt{\lambda}\ \frac{\partial u}{\partial n}\right|^2$$

where $E(t) = \int_M |\nabla_x u(x,t)|^2 + \left|\frac{\partial u}{\partial t}(x,t)\right|^2$.

We denote by H_0 the closed subspace of H, $H_0 = \{(u_0,u_1) \in H\,;\, u_0|_{\Gamma_-} \equiv 0\}$. If $(u_0,u_1) \in H_0$, then the solution u of (2) satisfy $(u,u_t') \in C^0(\mathbf{R}_+, H_0)$, and $\sqrt{E(0)}$ is a norm on H_0.

Then we have the following result (see **[15]** for the proof).

THEOREM.

1) *Suppose that* $\alpha \in\,]0,1]$ *and that there exist* T *such that* (Γ_+, T) *has the geometric control property. Then there exist* $C > 0$ *such that for every data* $(u_0,u_1) \in H_0$*, one has*

(4) $$E(t) \le \frac{1}{C} e^{-Ct} E(0)\ .$$

2) *Suppose that* $\alpha \in\,]1,\infty[$. *Then for every* $\varepsilon > 0$, $T > 0$, *there exist data* $(u_0,u_1) \in H_0$, *such that* $E(0) = 1$ *and* $E(T) \ge 1 - \varepsilon$ *; in particular* (4) *is false.*

So the stabilization of the Dirichlet boundary condition gives an example of unstable stabilization. The reason for the failure of exponential decay when $\alpha \in\,]1,\infty[$ is that, for solutions of (2), one has only $WF_b(u) \subset \Sigma_b \cup (\dot{T}^*\partial X \cap (\tau = 0))$, and singularities can live in the elliptic part of the boundary; when $\alpha \in\,]0,1]$, the Hardy inequality allows to treat this difficulty. Remark also that $\sqrt{\lambda}\frac{\partial u}{\partial n}\Big|_\partial \in L^2$ imply $\frac{\partial u}{\partial t}\Big|_\partial \in L^2$ so $u|_\partial \in H^1_\rho$ for $\rho \in \mathcal{H} \cup \mathcal{G}$ and in the proof, one can use the argument of propagation of singularities with the Dirichlet condition, instead of the condition $\frac{\partial}{\partial t} + \lambda\frac{\partial}{\partial n}$.

VIII. Plate equation

In this part, we shall briefly discuss what results can be obtained for the plate equation $\partial_t^2 + \Delta^2$, using the same type of microlocal analysis. This equation is a model for the vibration of fine elastic plates; it is not of hyperbolic type, but of Schrödinger type: $\partial_t^2 + \Delta^2 = (\partial_t + i\Delta)(\partial_t - i\Delta)$, so the idea here is to decompose in pieces the energy space $E = \oplus E_k$, so that in E_k the frequency is of order 2^k, like in the Littlewood-Paley theory, using the eigenfunctions of the underlying problem, so E_k is stable by the time evolution, and to make for each k the scaling in time $t = 2^{-k}s$, so on E_k the equation becomes $\partial_s^2 + h_k^2\Delta^2$, $h_k = 2^{-k}$, which reduces the problem to the semi-classical analysis of an operator of principal type (see **[16]**). Then one can prove:

THEOREM. *Take $\Gamma \subset \partial M$, such that for some $T > 0$, (Γ, T) has the geometric control property. Then for every $T_0 > 0$, and data $v_0 \in H_0^1(\Omega)$, $v_1 \in H^{-1}(\Omega)$ there exist a control function $g \in L^2(]0, T_0[\times \Gamma)$ such that the solution of the evolution problem*

$$(1) \qquad \begin{cases} (\partial_t^2 + \Delta^2)v = 0\ , & v|_{t=0} = v_0\ , \qquad \left.\frac{\partial v}{\partial t}\right|_{t=0} = v_1 \\ v|_\partial = 0\ , & \Delta v|_\partial = g\, 1_{]0,T_0[\times\Gamma} \end{cases}$$

satisfies $v \equiv 0$ for $t \geq T_0$.

Using the same type of techniques, T. HARGÉ [4] has obtained the same result with a control function acting on $\partial_n v|_\partial$ [although in that case, one has to suppose that the space F generated by the eigenfunctions of $(\Delta^2,\ u|_\partial = 0,\ \partial_n u|_\partial = 0)$ such that $\Delta u|_\Gamma = 0$ which is of finite dimension by the proof, is trivial; for example, one can suppose that the boundary ∂M is connected]. For other results in this direction, see [8], [9], [10], [11], [12].

One interesting thing in the study of control theory for Schrödinger-type equation is that the infinite speed of propagation (more precisely, the speed is proportionnal to the frequency) makes the situation better that in the hyperbolic case.

For example, in [7] (see also [5]) it is shown that one has the interior control property for the plate equation in a rectangular domain of the plane by acting on an arbitrary non-void subset. More recently, N. BURQ [2] has shown the same type of phenomena when $M = \Omega$, open in $\mathbf{R}^3$, $\partial\Omega = \partial\Omega_1 \cup \partial K_1 \cup \ldots \cup \partial K_N$ where $K_1, \ldots, K_N$ are disjoint, strictly convex, bounded subset of Ω_1, with convex-hull $(K_1 \cup \ldots \cup K_N) \Subset \Omega_1$, satisfying the Ikawa hypothesis, and a control function acting on $\partial\Omega_1$ only.

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