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Error estimate in the generalized Szegö theorem

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1. Let A be a positive selfadjoint elliptic pseudodifferential operator of order 1 on a smooth compact manifold M without boundary, dim M = $n \geq 2$. The spectrum of the operator A consists of infinite number of eigenvalues $\lambda_k \to +\infty$, $k \to \infty$. By $N(\lambda)$ we denote a counting function of the spectrum of operator A,

$$N(\lambda) = \# \{k : \lambda_k < \lambda\}$$

(we take into account the multiplicity of the eigenvalues). Let Π_λ be the spectral projectors of operator A corresponding to the intervals $(0,\lambda)$. We consider a family of operators

$$B_\lambda = \Pi_\lambda B \Pi_\lambda,$$

where B is a selfadjoint pseudodifferential operator of order zero. The rank of the operator B_λ is finite, so it has a finite number of eigenvalues $\mu_j(\lambda)$ lying in the interval

$$K = [-\|B\|, \|B\|] \subset \mathbb{R}^1.$$

The number of these eigenvalues is infinitely increasing when $\lambda \to +\infty$.

Let ρ_λ be a measure on K which is equal to the sum of the Dirac measures at the points $\mu_j(\lambda)$, i.e.

$$\rho_\lambda(f) = \sum_j f(\mu_j(\lambda)) = \operatorname{Tr} f\,(\Pi_\lambda B \Pi_\lambda)$$

for any function $f \in C(K)$. We shall study the asymptotic behaviour of ρ_λ when $\lambda \to +\infty$.

It is well known [1, theorem 29.1.7] that the measures $\lambda^{-n}\rho_\lambda$ converge weakly to the measure ρ_0 which is defined by the following formula

$$\rho_0(f) = (2\pi)^{-n} \int_{a_0(x,\xi)<1} f(b_0(x,\xi))\, dx\, d\xi,$$

where $f \in C(K)$ and a_0, b_0 are the principal symbols of the operators A and B. By other words, for any $f \in C(K)$

$$\rho_\lambda(f) = \rho_0(f)\,\lambda^n + o(\lambda^n). \tag{1}$$

This result is considered as a generalization of the classical Szegö theorem [2] on the contraction of a multiplication operator to the space of trigonometrical polynomials. It dues to Guillemin [3].

We prove that for sufficiently smooth function f the remainder in (1) is $o(\lambda^{n-1})$. Our main results are the following theorems.

<u>Theorem 1</u>. There exist an integer r and a positive constant C such that for any function $f \in C^r(K)$ the following inequality holds

$$|\rho_\lambda(f) - \rho_0(f)\,\lambda^n| \leqslant C(\lambda^{n-1}+1)\,\|f\|_{C^r(K)}. \tag{2}$$

<u>Theorem 2</u>. If B is a multiplication by sufficiently smooth function $b_0(x)$ then the estimate (2) is valid for $r = 2$.

2. Let $\varphi_j(x)$ be eigenfunctions of the operator A corresponding to the eigenvalues λ_j, and $(\varphi_j, \varphi_k) = \delta_j^k$. The proof of the generalized Szegö theorem is based on the following well known result (see [1, §29.1]).

<u>Theorem 3</u>. For any pseudodifferential operator H of order zero

$$\sum_{\lambda_j<\lambda} \overline{\varphi_j(x)}\, H\varphi_j(x) =$$

$$= (2\pi)^{-n} \int_{a_0(x,\xi)<1} h_0(x,\xi)\, d\xi\ \lambda^n + o(\lambda^{n-1})$$

uniformly with respect to $x \in M$, where h_0 is the principal symbol of the operator H.

The theorem 3 (with $H = I$) immediately implies that for any bounded function $h(x)$ and corresponding multiplication operator $\{h\}$

$$|\mathrm{Tr}\, \Pi_\lambda \{h\}\, \Pi_\lambda - (2\pi)^{-n} \int_{a_0(x,\xi)<1} h(x)\, dx\, d\xi\, \lambda^n \,| \leqslant$$

$$\leqslant C\, \lambda^{n-1} \sup_x |h(x)|\,,$$

where $\lambda \geqslant 1$, and the constant C does not depend on h. In particular,

$$N(\lambda) = (2\pi)^{-n} \int_{a<1} dx d\xi\, \lambda^n + o(\lambda^{n-1}).$$

If f is a smooth function then $f(B)$ is a pseudodifferential operator and its principal symbol is $f(b_0)$. Therefore according to the theorem 3, for $f \in C^\infty(K)$ we have

$$\mathrm{Tr}\, \Pi_\lambda\, f(B)\, \Pi_\lambda = \rho_0(f)\, \lambda^n + o\,(\lambda^{n-1})\,,$$

where remainder somehow depends on f. It is easy to see from the proof of the theorem 3 [1, §29.1] that this remainder term is estimated for $\lambda \geqslant 1$ by

$$C\, \lambda^{n-1} \|f\|_{C^r(K)}$$

where the constant C and the integer r are independent of f.

Remark 4. We suppose that this estimate holds for $r = 2$. If it is true then the theorem 1 is valid for $r = 2$ as well.

3. Now we shall prove the following abstract theorem.

Theorem 5. Let A be a positive selfadjoint operator and B be a bounded selfadjoint operator in a Hilbert space. Suppose that spectrum of the operator A consists of eigenvalues, and let Π_λ be the spectral projectors corresponding to the intervals $([0,\lambda])$, $N(\lambda)$ be the counting eigenvalues function, and

$$N_\varepsilon(\lambda) = \sup_{\mu \leqslant \lambda} [N(\mu) - N(\mu-\varepsilon)].$$

Assume that the comutator $\widetilde{B} = [A,B]$ is a bounded operator. Then for any $\varepsilon > 0$ and for any function $f \in C^2(K)$ the following inequality holds

$$|\mathrm{Tr}\, \Pi_\lambda f(B) \Pi_\lambda - \mathrm{Tr}\, f(\Pi_\lambda B \Pi_\lambda)| \tag{3}$$
$$\leqslant (2\|B\|^2 + C_\varepsilon \|\widetilde{B}\|^2)\, N_\varepsilon(\lambda) \max_K |f''|,$$

where $K = [-\|B\|, \|B\|]$, and the constant C_ε depends on ε only.

On account of (3) the theorems 1 and 2 follow from the results mentioned in the section 2.

We deduce (3) from the following well known Berezin's inequality.

<u>Theorem 6</u>. Let B be a bounded self adjoint operator in a Hilbert space, $K = [-\|B\|, \|B\|]$, and Π be a selfadjoint projector, rank $\Pi < \infty$. Then for any convex function $\psi \in C(K)$

$$\mathrm{Tr}\, \Pi\, \psi(B)\, \Pi \geqslant \mathrm{Tr}\, \psi(\Pi B \Pi).$$

<u>Corollary 7</u>. Let $\varphi \in C^2(K)$ is a strictly convex function.Then for any $f \in C^2(K)$

$$|\mathrm{Tr}\, \Pi f(B)\Pi - \mathrm{Tr}\, f(\Pi B\Pi)| \leqslant \tag{4}$$
$$\leqslant \left(\max \left|\frac{f''}{\varphi''}\right|\right) (\mathrm{Tr}\, \Pi \varphi(B)\Pi - \mathrm{Tr}\, \varphi(\Pi B \Pi)|.$$

In particular (if $\varphi(t) = t^2$),

$$|\mathrm{Tr}\, \Pi f(B) \Pi - \mathrm{Tr}\, f(\Pi B \Pi)| \leqslant \frac{1}{2} (\max_k |f''|)\, \|(I-\Pi) B \Pi\|_2^2, \tag{5}$$

where $\|\,.\,\|_2$ is the Hilbert-Schmidt norm.

Proof. Applying the Berezin's inequality to the convex functions

$$\psi_{\pm} = \left(\max_{K} \left|\frac{f''}{\varphi''}\right| \right) \varphi \pm f$$

we obtain exactly (4).

In view of (5), to prove the theorem 5 it is sufficient to estimate $\|(I-\Pi_\lambda)B\Pi_\lambda\|_2^2$ by $(2\|B\|^2 + C_\varepsilon\|\tilde{B}\|^2)\, N_\varepsilon(\lambda)$. Note that

$\|(I-\Pi_\lambda)B\Pi_\lambda\|_2^2 \leqslant 2\,(\|(I-\Pi_\lambda)B\Pi_{\lambda-\varepsilon}\|_2^2 + \|(I-\Pi_\lambda)B(\Pi_\lambda - \Pi_{\lambda-\varepsilon})\|_2^2)$,

and $\|(I-\Pi_\lambda, B(\Pi_\lambda-\Pi_{\lambda-\varepsilon})\|_2^2 \leqslant \|B\|^2 N_\varepsilon(\lambda)$. So it remains to estimate $\|(I-\Pi_\lambda)\, B\,\Pi_{\lambda-\varepsilon}\|_2^2$ only . According to the definition

$$\|(I-\Pi_\lambda)\, B\, \Pi_{\lambda-\varepsilon}\|_2^2 = \sum_{\lambda_k \geqslant \lambda} \sum_{\lambda_j < \lambda-\varepsilon} |(B\,\varphi_j, \varphi_k)|^2,$$

where φ_j are the eigenfunctions of the operator corresponding to the eigenvalues λ_j. Since $(B\,\varphi_j, \varphi_k) = (\lambda_k-\lambda_j)^{-1}(\hat{B}\,\varphi_j, \varphi_k)$, we obtain that

$$\|(I-\Pi_\lambda)\, B\, \Pi_{\lambda-\varepsilon}\|_2^2 = \sum_{\lambda_k \geqslant \lambda} \sum_{\lambda_j < \lambda-\varepsilon} |(B\varphi_j, \varphi_k)|^2,$$

$$\leqslant \sum_{k} \sum_{\lambda_j < \lambda-\varepsilon} (\lambda-\lambda_j)^{-2}\, |(\tilde{B}\,\varphi_j, \varphi_k)|^2 \leqslant$$

$$\leqslant \|\tilde{B}\|^2 \sum_{\lambda_j < \lambda-\varepsilon} (\lambda-\lambda_j)^{-2} = \|B\|^2 \int_0^{\lambda-\varepsilon} (\lambda-\mu)^{-2}\, dN(\mu)$$

$$\leqslant \|\tilde{B}\|^2\, N_{\varepsilon/2}(\lambda) \sum_{k=0}^{k^*} (\lambda - k\varepsilon/2)^{-2}$$

where $(\lambda-\varepsilon/2) \geqslant k^*\varepsilon/2 > (\lambda-\varepsilon)$. The sum in the right hand side is estimated by some constant C_ε not depending on λ. Therefore

$$\|(I-\Pi_\lambda)\, B\, \Pi_{\lambda-\varepsilon}\|_2^2 \leqslant C_\varepsilon\, \|B\|^2 N_{\varepsilon/2}(\lambda).$$

It completes the proof of the theorem 5 and of the theorems 1 and 2.

Remark 8. Under some additional assumptions one can obtain a two-term asymptotic formula for $\mathrm{Tr}\,\Pi_\lambda f(B)\,\Pi_\lambda$. However, even under these assumptions the difference

$Tr\, \Pi_\lambda f\,(B)\, \Pi_\lambda - Tr\, f\,(\Pi_\lambda\, B\, \Pi_\lambda)$

can really have the order $0\,(\lambda^{n-1})$. So the second term in (1) (if it exists) can be different one.

Remark 9. The theorem 5 can be applied in various different problems as well. For example, it allows to improve some results from [4].

References

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