Journées Équations aux dérivées partielles

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Time decay estimates for a perturbed wave equation

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We consider global time estimates for solutions $u(t, x)$ on $\mathbb{R} \times \mathbb{R}^n$, $n \geq 3$, to problems of the form

$$(\Box + V(x))u = 0,$$

(0.1)

$$u(0, x) = 0, \quad \partial_t u(0, x) = f(x).$$

In the unperturbed case ($V \equiv 0$), the map $f \to T_t f = u(t)$ is given by the operator corresponding to the Fourier multiplier $\dfrac{\sin t|\xi|}{|\xi|}$. It is well known that T_t is a bounded operator from $L^p(\mathbb{R}^n)$ to $L^p(\mathbb{R}^n)$ if and only if $\left|\dfrac{1}{p} - \dfrac{1}{2}\right| \leq \dfrac{1}{n-1}$ (with natural modifications at the endpoints for $n \leq 3$), while T_t is a bounded operator from $L^p(\mathbb{R}^n)$ to $L^{p'}(\mathbb{R}^n)$, $\dfrac{1}{p} + \dfrac{1}{p'} = 1$, if and only if $\dfrac{1}{p} - \dfrac{1}{2} \leq \dfrac{1}{n+1}$. See Marshall-Strauss-Wainger [3]. Moreover, the operator norms as a function of t may be determined by a scaling argument. In particular, for the values of p indicated above,

$$(0.2) \qquad \|T_t f\|_{p'} \leq C_p\, t^{\,1 - n(1/p - 1/p')} \|f\|_p.$$

The optimal decay as $t \to \infty$ occurs for the largest allowable value of $\dfrac{1}{p}$. For the remainder of this paper, we fix the following values:

$$(0.3) \qquad \frac{1}{p} = \frac{1}{2} + \frac{1}{n+1}, \quad \frac{1}{p'} = \frac{1}{2} - \frac{1}{n+1}, \quad d = n\left(\frac{1}{p} - \frac{1}{p'}\right) - 1 = \frac{n-1}{n+1}.$$

Then the solution to $\Box u = 0$, $u(0, x) = 0$, $\partial_t u(0, x) = f(x)$, satisfies

$$(0.4) \qquad \|u(t)\|_{p'} \leq C_p\, t^{-d} \|f\|_p.$$

Estimates of the form (0.2) may be used to study the long-time behavior (existence and scattering) of solutions to nonlinear wave equations, for instance the equation $\Box u + |u|^{p'/p} = 0$. See Strauss [5]. Analogous estimates for the

perturbed time-dependent Schrödinger equation have been considered recently by Journe-Soffer-Sogge [2].

We obtain estimates of the form (0.4) for solutions to the problem (0.1) under appropriate conditions on the potential. Of course, if V has the "wrong sign" and is sufficiently large, no decay estimate like (0.4) can hold. For instance, if $V \in C_0^\infty(\mathbb{R}^n)$, $V \le 0$, $V(0) < 0$, and M is sufficiently large, then $\exists$ $f \in C_0^\infty(\mathbb{R}^n)$ and $C, c > 0$ such that the solution to $(\square + MV) u = 0$, $u(0, x) = 0$, $\partial_t u(0, x) = f(x)$, satisfies $||u(t)||_{p'} \ge Ce^{ct}$. On the other hand, if the potential is nonnegative or small, then decay like that in the unperturbed case holds.

Theorem 1. *Let $V \in C_0^\infty(\mathbb{R}^n)$, and let $f \in L^p(\mathbb{R}^n)$, $n \ge 3$, have compact support. Let $u(t)$ be the solution to $(\square + V)u = 0$, $u(0, x) = 0$, $\partial_t u(0, x) = f(x)$. If either $||V||_{(n+1)/2}$ is sufficiently small or $V \ge 0$, then $||u(t)||_{p'} \le C_p t^{-d} ||f||_p$.*

The assumptions in the theorem as stated are not the best possible; for example, the compact support hypothesis on V can be replaced by sufficiently rapid decay (polynomial in $|x|^{-1}$) at ∞. The regularity assumption on V can be dropped entirely in the case of a small potential, and only limited regularity is needed in the nonnegative case. Similarly, finite propagation speed allows the assumption that f merely has sufficiently rapid decay at ∞.

Analogous results hold for the solution to a perturbed Klein-Gordon equation $(\square + m^2 + V)u = 0$, $m > 0$. As in the unperturbed problem (see Marshall-Strauss-Wainger [3]), the optimal indices for decay in this case are

$$\frac{1}{p} = \frac{1}{2} + \frac{1}{n+2} \ , \ \frac{1}{p'} = \frac{1}{2} - \frac{1}{n+2} \ , \ d = \frac{n}{n+2} \ .$$

A description of the arguments used in proving Theorem 1 is provided below. Duhamel's formula and the known result (0.2) are used in Section 1 to reduce the result to an appropriate decay estimate local in space. In Section 2, it is shown that the decay properties of the kernel corresponding to the operator T_t away from the singular set of that kernel can be used to obtain the required local decay when the potential is small. For nonnegative potentials, it is established in Section 3 that the large-frequency part of the solution can be treated as in the small potential case, while the kernel for the small-frequency

piece can be analyzed more directly. Detailed proofs of these results will appear elsewhere.

1. Global Space estimate.

We assume from now on that V and f have support in $\{x: |x| \leq 1\}$. Let χ denote the characteristic function of this set. In order to establish estimates on the solution to (0.1) valid for all x, it is enough to obtain suitable time decay of an appropriate norm of χu. In fact, the following decay will suffice.

$$||\chi u(t)||_{p'} \leq \frac{C}{t} \, , \text{ and}$$

(1.1)

$$||\chi u(t)||_{p'} \leq \mu(t), \text{ with } \int_0^\infty \mu(t)dt < \infty.$$

Proposition 1.1. *Let* $V \in L^{(n+1)/2}(\mathbb{R}^n)$ *have support in* $\{x: |x| \leq 1\}$. *Let* $u(t)$ *be the solution to* $(\square + V)u = 0$, $u(0, x) = 0$, $\partial_t u(0, x) = f(x)$, *for* $f \in L^p(\mathbb{R}^n)$. *If* (1.1) *holds, then* $||u(t)||_{p'} \leq C_p t^{-d} ||f||_p$.

Proof. If v is the solution to $\square v = 0$, $v(0, x) = 0$, $\partial_t v(0, x) = f(x)$, we may write

$$(1.2) \qquad u(t) = v(t) - \int_0^t T_{t-s}(Vu(s)) \, ds \, ,$$

with T as in (0.2). From (0.4), v satisfies the desired estimate. From (0.3), we have $\frac{1}{p} = \frac{1}{p'} + \frac{2}{n+1}$. Thus (0.2) and Hölder's inequality yield

$$|| \int_0^t T_{t-s}(Vu(s)) \, ds \, ||_{p'} \leq C \int_0^t \frac{1}{(t-s)^d} ||Vu(s)||_p \, ds$$

$$\leq C ||V||_{(n+1)/2} \int_0^t \frac{1}{(t-s)^d} ||\chi u(s)||_{p'} \, ds \, .$$

From (1.1),

$$\int_0^t \frac{1}{(t-s)^d} ||\chi u(s)||_p \, ds \leq \int_0^{t/2} \frac{1}{(t-s)^d} \mu(s) \, ds + C \int_{t/2}^t \frac{1}{(t-s)^d} \frac{1}{s} ds$$

$$\leq \frac{C}{t^d} \int_0^\infty \mu(s) \, ds + \frac{C}{t} \int_{t/2}^t \frac{1}{(t-s)^d} \, ds \leq \frac{C}{t^d} \, . \qquad /////$$

Assuming the hypotheses of Theorem 1, we shall establish local decay as $t \to \infty$ which is better (for $n \geq 3$) than that required by (1.1): $||\chi u(t)||_{p'} \leq \dfrac{C}{t^{n-1}}$. (In odd dimensions, n - 1 may be replaced by any M arbitrarily large.) Since the norm estimate for small time is immediate from (0.4) and (1.2), Theorem 1 is then a consequence of Proposition 1.1.

2. Local space estimate, small potentials.

If the kernel corresponding to the operator T_t is localized away from the surface of the light cone, it yields an operator bounded from $L^1(\mathbb{R}^n)$ to $L^\infty(\mathbb{R}^n)$ with decay like $t^{-(n-1)}$. This property allows us to obtain the desired local decay for solutions to (0.1) when the potential is suitably small by using (1.2); the key is that $(n-1) > 1$, so that the estimate may be integrated.

Lemma 2.1. *Let* $\widetilde{T}_t f(x) = \int \widetilde{K}_t(x,y) f(y) dy$, *with*

$$\widetilde{K}_t(x,y) = \chi(x) \int e^{i(x-y)\cdot\xi} \frac{\sin t|\xi|}{|\xi|} d\xi \, \chi(y) \, .$$

Then for $t \geq 4$, $||\widetilde{T}_t f||_\infty \leq Ct^{-(n-1)} ||f||_1$.

Proof. On the support of $\widetilde{K}_t$, $|x-y| \leq t/2$. Thus it suffices to show that, if

$$K_t(x) = \int e^{ix\cdot\xi} \frac{\sin t|\xi|}{|\xi|} d\xi$$

then $|K_t(x)| \leq Ct^{-(n-1)}$ for $|x| \leq t/2$. The usual Littlewood-Paley argument, involving a decomposition of the integral into sets where $t|\xi| \approx 2^j$, yields the desired estimate. $/////$

In the case of the time-dependent Schrödinger equation, the operator S_t corresponding to T_t is bounded from $L^1(\mathbb{R}^n)$ to $L^\infty(\mathbb{R}^n)$, and the decay rate is

$d = \dfrac{n}{2} > 1$. Consequently away from the surface of the light cone the kernel corresponding to T_t has properties better than that associated to S_t, but it is less well-behaved near the surface of the light cone.

Fix $u(t)$ a solution to $(\Box + V)u = 0$, $u(0, x) = 0$, $\partial_t u(0, x) = f(x)$, for $f \in L^p(\mathbb{R}^n)$ with support in $\{x \colon |x| \le 1\}$. In order to establish the time-decay of the local norm, we want to show that the following quantity is bounded.

(2.1)
$$F(t) = \sup_{\tau < t} \tau^{n-1} \| \chi \, u(\tau) \|_{p'} \, .$$

Proposition 2.2. *If V has support in $\{x \colon |x| \le 1\}$ and $\|V\|_{(n+1)/2}$ is sufficiently small, then $F(t) \le C < \infty$.*

Proof. For $\tau \le t$, we again write

$$\chi u(\tau) = \chi v(\tau) - \int_0^\tau \chi T_{\tau-s}(Vu(s)) \, ds \, ,$$

with v the solution of the unperturbed problem. The estimate on v is valid by Lemma 2.1, since $\chi v(t) = \widetilde{T}_t f$ if $t \ge 4$. For the second term, we may write

(2.2)
$$\int_0^\tau \chi T_{\tau-s}(Vu(s)) \, ds = \int_0^k \chi T_{\tau-s}(Vu(s)) \, ds + \int_k^{\tau-k} \chi T_{\tau-s}(Vu(s)) \, ds$$
$$+ \int_{\tau-k}^\tau \chi T_{\tau-s}(Vu(s)) \, ds$$

with $k \ge 4$ to be chosen later. (The case $\tau \le 2k$ is easily treated.) The first integral on the right hand side of (2.2) is bounded by a constant C_k. On the other hand, for $k \le \tau \le t$, since $s \le \tau - k$ implies $\chi T_{\tau-s}(Vu(s)) = \widetilde{T}_t(Vu(s))$, we have by Lemma 2.1,

$$\tau^{n-1} \int_k^{\tau-k} \| \chi T_{\tau-s}(Vu(s)) \|_{p'} \, ds \;\le\; \tau^{n-1} \int_k^{\tau-k} \| \chi T_{\tau-s}(Vu(s)) \|_\infty \, ds$$

$$\le\; C\tau^{n-1} \int_k^{\tau-k} \frac{1}{(\tau-s)^{n-1}} \| Vu(s) \|_1 \, ds \;\le\; C\tau^{n-1} \int_k^{\tau-k} \frac{1}{(\tau-s)^{n-1}} \| V \|_p \| u(s) \|_{p'} \, ds$$

$$\le\; C_V \, \tau^{n-1} \int_k^{\tau-k} \frac{1}{(\tau-s)^{n-1}} \frac{F(s)}{s^{n-1}} \, ds \;\le\; \frac{C_V}{k^{n-2}} F(t) \qquad \text{(since } n \ge 3\text{)}.$$

We may choose k large enough that this term is bounded by $\frac{1}{4}F(t)$.

For the remaining integral in (2.2), we use the smallness of the potential.

$$\tau^{n-1} \int_{\tau-k}^{\tau} \|\chi T_{\tau-s}(Vu(s))\|_{p'}\, ds \;\leq\; C\tau^{n-1} \int_{\tau-k}^{\tau} \frac{1}{(\tau-s)^d}\|Vu(s)\|_p\, ds$$

$$\leq\; C\,\|V\|_{(n+1)/2}F(t) \int_{\tau-k}^{\tau} \frac{1}{(\tau-s)^d}\, ds \;=\; C_k\,\|V\|_{(n+1)/2}F(t)\ .$$

If $\|V\|_{(n+1)/2}$ is sufficiently small, this term is bounded by $\frac{1}{4}F(t)$. We conclude that $F(t) \leq C + \frac{1}{2}F(t)$, with C independent of t.   /////

The argument given above is unchanged if the potential V depends on t, as long as the size of its norm is uniformly small in time (and its support is bounded).

3. Nonnegative potentials.

In the case of large nonnegative potentials V in Theorem 1, a further decomposition of the solution u is convenient. Sufficiently large frequencies are handled by the methods of section 2, including the use of Duhamel's formula. Small frequencies are analyzed by an examination of the action of the kernel relating f to u. The frequency decomposition is time-independent. The explicit time dependence of the kernel is handled by writing it in the form given by the spectral representation associated with $-\Delta + V$.

Fix a cutoff function $\alpha \in C_0^\infty(\mathbb{R}^+)$, $\alpha \geq 0$, $\alpha \equiv 1$ for $\rho \leq 1$, $\alpha \equiv 0$ for $\rho \leq 0$, and let $\alpha_M(\rho) = \alpha(\rho/M)$. For potentials $V \geq 0$ as in Theorem 1, the functional calculus for unbounded self-adjoint operators on $L^2(\mathbb{R}^n)$ allows us to define the operators

$$(3.1) \qquad A_M = \alpha_M\left((-\Delta + V)^{1/2}\right)\ ,\ B_M = I - A_M\ .$$

If for u the solution of (0.1) we write $u = A_M u + B_M u$, then $B_M u$ satisfies

$$(3.2) \qquad (\Box + V)B_M u = 0,\ B_M u(0) = 0,\ \partial_t B_M u(0) = B_M f.$$

(It is easily verified that B_M extends to a bounded operator on $L^p(\mathbb{R}^n)$.)

Proposition 1.1 can be applied to $B_M u$ in order to reduce the desired estimate for this high frequency part of u to a spatially local estimate. The proof of the analogue of Proposition 2.2 goes through essentially unchanged, except for the last term in (2.2) to be handled. (This term was the only one in the proof of Proposition 2.2 which required V to be small.) Thus, for k fixed and $\tau \leq t$, we consider

$$\tau^{n-1} \int_{\tau-k}^{\tau} ||\chi T_{\tau-s}(VB_M u(s))||_{p'}\, ds \, .$$

Since we know that $||T_{\tau-s}(VB_M u(s))||_{p'} \leq C_V(\tau - s)^{-d}||u\,(s)||_p$ with a bound independent of M, an estimate of the form $||T_{\tau-s}(VB_M u(s))||_2 \leq \delta(M)||u\,(s)||_2$ with $\delta(M) \to 0$ as $M \to \infty$ would allow us to conclude that, for M sufficiently large,

$$\tau^{n-1} \int_{\tau-k}^{\tau} ||\chi T_{\tau-s}(VB_M u(s))||_{p'}\, ds \leq \varepsilon(M) C_{V,p}\, \tau^{n-1} \int_{\tau-k}^{\tau} \frac{1}{(\tau-s)^d}||u(s)||_{p'}\, ds$$

$$\leq \varepsilon(M) C_{V,p,k}\, F(t).$$

($F(t)$ is given by (2.1)). If M is sufficiently large, this term is bounded by $\frac{1}{4}F(t)$. Thus the estimate for $B_M u$ is a consequence of the following result, for which the proof is straightforward..

Lemma 3.1. *Let* $V \in C_0^2(\mathbb{R}^n), V \geq 0$. *If* T_l *is the operator corresponding to the Fourier multiplier* $\dfrac{\sin l|\xi|}{|\xi|}, 0 \leq l \leq k,$ *and* B_M *is given by (3.1), then*

$$||T_l(VB_M u(s))||_2 \leq \frac{C_l}{M}||u\,(s)||_2 \, .$$

The remaining low frequency term $A_M u$ satisfies

$$(3.3) \qquad A_M u = (\sin t(-\Delta + V)^{-1/2})(-\Delta + V)^{-1/2} A_M f \, .$$

We write the spectral decomposition of the this operator using the Lippman-Schwinger eigenfunction expansion. (See e.g. Reed-Simon [4].) Since our assumptions on V guarantee that the spectrum of $(-\Delta + V)$ is absolutely continuous, it follows that, for f with support in $\{x: |x| \leq 1\}$, we may write

$$\chi A_M u(x) = \int K(t,x,y)f(y)\,dy\,,$$

(3.4)

$$K(t,x,y) = \chi(x)\chi(y)\int_{S^{n-1}}\int_0^\infty \frac{\sin t\rho}{\rho}\varphi(x,\rho,\omega)\overline{\varphi}(y,\rho,\omega)a(\rho/M)\rho^{n-1}d\rho\,d\omega\,.$$

The Lippman-Schwinger functions $\varphi(x,\rho,\omega)$ are the the analogues of the usual exponentials $\varphi_0(x,\rho,\omega) = c_n e^{i\rho x\cdot\omega}$. They satisfy $(-\Delta + V)\varphi(x,\rho,\omega) = \rho^2\varphi(x,\rho,\omega)$, and are the solutions of the integral equation

(3.5) $$\varphi(x,\rho,\omega) = \varphi_0(x,\rho,\omega) - \int G_n(|x-y|,\rho)V(y)\varphi(y,\rho,\omega)dy\,.$$

The Green's functions $G_n(|x-y|,\rho)$, which invert $(-\Delta - \rho^2)$, are given in terms of Hankel functions (see e.g. Berthier [1]):

(3.6) $$G_n(r,\rho) = c_n\left(\frac{\rho}{r}\right)^{n/2-1}H^{(1)}_{n/2-1}(r\rho)\,.$$

We will estimate $K(t,x,y)$ by integrating by parts with respect to ρ in (3.4). Since x,y, and ρ are all bounded on the support of the integrand, it will suffice to have estimates on $\varphi(x,\rho,\omega)$ of the form

(3.8)
$$|\varphi| \le C_M\,,\quad \rho^{n-2-j}|\partial_\rho^k\varphi\,\partial_\rho^l\varphi| \le C_M \text{ if } j+k+l\le n-1, k\le n-2\,,$$
$$\text{for } |x| \le 1, |y| \le 1, 0\le\rho\le M\,.$$

Lemma 3.2. *Let* $K(t,x,y)$ *be given by* (3.4), *and assume that* (3.8) *holds. Then*
$$|K(t,x,y)| \le \frac{C_{M,n}}{t^{n-1}} \text{ uniformly for } |x| \le 1, |y| \le 1.$$

Proof. We write

$$K(t,x,y) = \chi(x)\chi(y)\int_{S^{n-1}}\int_0^\infty \sin t\rho\,\{\varphi(x,\rho,\omega)\overline{\varphi}(y,\rho,\omega)a(\rho/M)\rho^{n-2}\}\,d\rho\,d\omega\,,$$

and integrate by parts in ρ a total of $n-1$ times. No boundary terms appear at $\rho = 0$ until the final integration. We obtain an expression of the form

$$K = \frac{\chi(x)\chi(y)}{t^{n-1}} \int_{S^{n-1}} \{\beta(x,y,\omega) + \int_0^\infty \cos t\rho \, [\sum_{l=0}^{n-2} \beta_l(x,y,\omega)\partial_\rho^{n-2-l}a(\rho/M)] \, d\rho \} d\omega \,,$$

if n is even, and

$$K = \frac{\chi(x)\chi(y)}{t^{n-1}} \int_{S^{n-1}} \int_0^\infty \sin t\rho \, [\sum_{l=0}^{n-2} \beta_l(x,y,\omega)\partial_\rho^{n-2-l}a(\rho/M)] \, d\rho \} d\omega \,,$$

if n is odd. In either case, the estimate on K follows from the fact that the terms $\beta(x,y,\omega)$, $\beta_l(x,y,\omega)$ are bounded, by (3.8).　　　/////

From 3.4, it follows that we have the estimate

$$\|\chi A_M u(t)\|_{p'} \le \frac{C}{t^{n-1}} \|A_M f\|_p \,.$$

(In fact, the stronger estimate $\|\chi A_M u(t)\|_\infty \le \frac{C}{t^{n-1}} \|A_M f\|_1$ holds.) If we combine this property with the earlier estimate of $B_M u(t)$, we see that the analogue of Proposition 2.2 is established, and hence Proposition 1.1 yields Theorem 1 in the case of $V \ge 0$. It remains to establish (3.8).

The Lippman-Schwinger functions may be analyzed as in Berthier [1]. In particular, if we write $V = V_1 V_2$, with $V_i \in L^2(\mathbb{R}^n)$ and set $\psi = V_2\varphi$, (3.5) may be replaced by an equation for functions in $L^2(\mathbb{R}^n)$:

(3.9)　　$\psi(x,\rho,\omega) = \psi_0(x,\rho,\omega) - \int V_1(x)G_n(|x-y|,\rho)V_2(y)\psi(y,\rho,\omega)dy \,.$

If we set

(3.10)　　　　　$W_z = V_1(-\Delta-z)^{-1}V_2$

for $\{Im\, z > 0\}$, and extend this definition in the natural fashion to $\{Im\, z = 0\}$, then for $z = \rho^2$ we have

(3.11)　　　　　$\psi(x,\rho,\omega) = (I+W_z)^{-1}\psi_0(x,\rho,\omega).$

As long as the V_i have sufficiently rapid decay at ∞, the analytic Fredholm alternative applies to yield the existence of $(I+W_z)^{-1}$ outside an exceptional set

of measure zero in $\{Re\ z > 0\}$ consisting of eigenvalues corresponding to L^2 eigenfunctions for $(-\Delta + V)$; our assumptions on V guarantee that this set is empty, and that $\psi(x, \rho, \omega)$ and hence $\varphi(x, \rho, \omega)$ are uniformly bounded on the set $|x| \leq 1, |y| \leq 1, 0 \leq \rho \leq M$.

If (formally) the derivative ∂_ρ is applied to (3.11), we obtain the expression $\partial_\rho \psi = (I + W_z)^{-1}\partial_\rho \psi_0 + (I + W_z)^{-1}\partial_\rho W_z (I + W_z)^{-1}\psi_0(x, \rho, \omega)$. For the values of x, y, and ρ in question, $\partial_\rho \psi_0$ behaves like ψ_0. For $z = \rho^2$, the operator $\partial_\rho W_z$ has kernel $V_1(x)\partial_\rho G_n(|x - y|, \rho)V_2(y)$. Explicit estimates for the Hankel functions (3.6), as in Watson [6], allow the following conclusions about the ρ - behavior of these kernels and the operators W_z.

Proposition 3.3. *Let W_z be given by* (3.10)*, and assume that* $|V_i(x)| \leq \dfrac{C}{(1 + |x|)^m}$.

a) If n is odd and $k \geq 0$, then $\rho^k \partial_\rho^j W_{\rho^2}$ is a bounded operator on $L^2(\mathbb{R}^n)$ uniformly in $0 \leq \rho \leq M$, as long as $m > max(j - k - 2, (n + 1)/4)$.
b) If n is even and $k \geq max(0, j - (n - 2))$, then the same result holds, except that for the case $\{j = n - 2, k = 0\}$, the estimate is valid for $(1 + |log\ \rho|)^{-1}\partial_\rho^j W_{\rho^2}$.

Repeated differentiations of (3.11) are then justified, and the required local estimates (3.9) on the Lippman-Schwinger functions follow.

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