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The Fermi surface for the discretized Maxwell equations

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1. Introduction

Let $\Gamma = a_1\mathbf{Z} \oplus a_2\mathbf{Z} \oplus a_3\mathbf{Z}$ be a lattice of $\mathbf{R}^3$. The shifted cell problem for Maxwell's system has the following form : For each $k \in \mathbf{R}^3$ one considers

$$\nabla \wedge H = -i\omega\varepsilon E, \nabla \cdot (\varepsilon E) = 0$$

$$-\nabla \wedge E = -i\omega\mu H, \nabla \cdot (\mu H) = 0$$

with boundary conditions

$$E(x+\gamma) = e^{i<k,\gamma>}E(x), H(x+\gamma) = e^{i<k,\gamma>}H(x)$$

for all $\gamma \in \Gamma$, where E (resp.H) are in $H^1_{loc}(\mathbf{R}^3)^3$ and $\varepsilon(x)$,$\mu(x)$ are smooth positive diagonal 3×3 matrices of Γ-periodic functions. Eliminating H and supposing $\mu = 1$ one gets an eigenvalue problem for E :

$$A(\varepsilon)E \overset{def}{=} \varepsilon^{-1}\nabla \wedge (\nabla \wedge E) = \lambda E \tag{1}$$

$$D(\varepsilon)E \overset{def}{=} \nabla \cdot (\varepsilon E) = 0 \tag{2}$$

$$\text{with} \qquad E(x+\gamma) = e^{i<k,\gamma>}E(x) \quad \forall\gamma \in \Gamma. \tag{3}$$

(1) and (3) form a self adjoint boundary value problem yielding a discrete spectrum

$$... \leq E_{-2}(k) \leq E_{-1}(k) \leq E_0(k) = 0 \leq E_1(k) \leq ...$$

where $E_j(k)$ depends continously on k. It is periodic in the dual lattice

$$\Gamma^\sharp = \{b \in \mathbf{R}^3 \mid < b, \Gamma > \subset 2\pi\mathbf{Z}\}.$$

In particular $\lambda = 0$ is an eigenvalue of infinite geometric multiplicity, with eigenspace

$$N(k) = \{E \in L^2_{loc}(\mathbf{R}^3)^3 \mid \nabla \wedge E = 0 \quad \text{and} \quad (3)\}.$$

These eigenvectors do not satisfy $\nabla \cdot (\varepsilon E) = 0$ and if λ is an eigenvalue of (1) different from zero then the corresponding eigenvectors fulfill $\nabla \cdot (\varepsilon E) = 0$. In view of the periodicity with respect to $\Gamma^\sharp$, one can replace (3) by

$$E(x+\gamma) = \xi_1^{\gamma_1}\xi_2^{\gamma_2}\xi_3^{\gamma_3}E(x) \tag{4}$$

where $(\gamma_1, \gamma_2, \gamma_3)$ are the coordinates of γ in Γ; and one defines the (physical) Fermi surface $\mathcal{F}_{phys,\lambda}(\varepsilon)$ as

$$\mathcal{F}_{phys,\lambda}(\varepsilon) = \{(\xi_1, \xi_2, \xi_3) \in (S^1)^3 \mid E_n(\xi) = \lambda \quad \text{for} \quad \text{some} \quad n \neq 0\}.$$

We also consider solutions ξ in $(\mathbf{C}^*)^3$,therefore we define the (complex) Fermi surface for $\lambda \neq 0$

$$\mathcal{F}_\lambda(\varepsilon) = \{(\xi_1, \xi_2, \xi_3) \in (\mathbf{C}^*)^3 \mid \exists E \neq 0 \quad \text{solving} \quad (1),(2),(4)\}.$$

Clearly $\mathcal{F}_{phys,\lambda}(\varepsilon) \subset \mathcal{F}_\lambda(\varepsilon)$. Using regularized determinants and decomposing the operator $A(\varepsilon)$ as in [I] it can be shown that $\mathcal{F}_\lambda(\varepsilon)$ is a complex hypersurface in $(\mathbf{C}^*)^3$. One is interested in the following questions :

- Does $\mathcal{F}_{phys,\lambda}(\varepsilon)$ determines $\mathcal{F}_\lambda(\varepsilon)$?
- Does the geometry of $\mathcal{F}_\lambda(\varepsilon)$ contains isospectral information?
- Does $\mathcal{F}_\lambda(\varepsilon)$ determines (generically) ε?

In order to focus on this geometric aspects we consider a discrete approximation. Here the analogue of the Fermi surface is an algebraic variety.

2. The discrete model

Inside $\mathbf{Z}^3$ we take the lattice $\Gamma = \bigoplus_{j=1,2,3} \mathbf{Z} a_j e_j$, where e_j is the j-th standard basis vector and all the a_j are distinct, greater two and relatively prime. Let $\varepsilon = (\varepsilon_i \delta_{ij})$ with $\varepsilon_i : \mathbf{Z}^3 \to \mathbf{R}_+$ be periodic with respect to Γ. The operators $\varepsilon A(\varepsilon)$ and $D(\varepsilon)$ are discretized by replacing the partial derivates ∂_i by the operators $S^{e_i} - S^{-e_i}$, where S^α is the shift operator acting on functions $\mathbf{Z}^3 \to \mathbf{C}$ by

$$(S^\alpha f)(m) = f(m + \alpha).$$

We don't change the notation for the discretized operators.

For $\lambda \neq 0$ the Fermi surface is

$$\mathcal{F}_\lambda(\varepsilon) = \{(\xi_1, \xi_2, \xi_3) \in (\mathbf{C}^*)^3 \mid \exists E \neq 0 \quad \text{with} \quad A(\varepsilon)E = \lambda E,$$

$$D(\varepsilon)E = 0, S^{a_i e_i} E = \xi_i E, i = 1,2,3\}.$$

Due to the boundary conditions, the vector E is determined by its $a_1 a_2 a_3$ values on the fundamental domain of Γ . So $\mathcal{F}_\lambda(\varepsilon)$ translates into an eigenvalue problem for a $3a_1a_2a_3 \times 3a_1a_2a_3$ matrix, and $\mathcal{F}_\lambda(\varepsilon)$ is then given by the zero set of a polynomial in the variables $\xi_1, \xi_1^{-1}, \xi_2, \xi_2^{-1}, \xi_3, \xi_3^{-1}$.

3. Results

We have

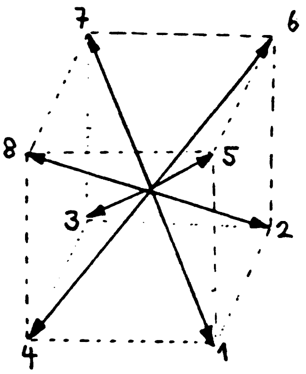
Theorem 1. *Assume* $\varepsilon_1(m) < \varepsilon_2(m) < \varepsilon_3(m) \quad \forall m \in \mathbf{Z}^3$ *then* $\mathcal{F}_\lambda(\varepsilon)$ *is irreducible.*

It follows, that if $\mathcal{F}_{phys,\lambda}(\varepsilon)$ contains a piece of a two-dimensional real surface, then $\mathcal{F}_{phys,\lambda}(\varepsilon)$ determines $\mathcal{F}_\lambda(\varepsilon)$.

The idea of the proof is to construct a compactification $\overline{\mathcal{F}_\lambda(\varepsilon)}$ of $\mathcal{F}_\lambda(\varepsilon)$, such that the generic points added at "infinity" are smooth points of $\overline{\mathcal{F}_\lambda(\varepsilon)}$.

Naively one could try to compactify $\mathcal{F}_\lambda(\varepsilon)$ by embedding $(\mathbf{C}^*)^3$ in $(\mathbf{P})^3$ and closing the Fermi surface in there. This doesn't work, since the new points added to $\mathcal{F}_\lambda(\varepsilon)$ are highly singular. Instead we construct, motivated by an idea of Mumford (see [M]), as in [B1] an intrinsic compactification of $\mathcal{F}_\lambda(\varepsilon)$ by embedding the algebraic torus $T = (\mathbf{C}^*)^3$ in the toroidal compactification X_Σ of T corresponding to the fan Σ in $\mathbf{R}^3$ of the cones over the faces of the 6 prisms of the following picture :

$$1 \overset{def}{=} (+a_1,+a_2,+a_3), 2 \overset{def}{=} (-a_1,+a_2,+a_3)$$
$$3 \overset{def}{=} (-a_1,-a_2,+a_3), 4 \overset{def}{=} (+a_1,-a_2,+a_3)$$
$$5 \overset{def}{=} (+a_1,+a_2,-a_3), 6 \overset{def}{=} (-a_1,+a_2,-a_3)$$
$$7 \overset{def}{=} (-a_1,-a_2,-a_3), 8 \overset{def}{=} (+a_1,-a_2,-a_3)$$



The corresponding toroidal "octahedron" is a singular complete algebraic variety with one-dimensional singular locus. The latter is stratified into 18 T-orbits, 12 of dimension 1 and 6 of dimension 0. The one-dimensional orbits correspond to the codimension one cones over the 8 edges of the above cube. These curves have transversal A_k type, with $k = 2a_i - 1$ $(i = 1,2,3)$. The zero dimensional orbits in the closure of the one-dimensional orbits correspond to the zero-codimensional faces. Take now the closure of $\mathcal{F}_\lambda(\varepsilon)$ in the octahedron X_Σ . The resulting variety is always singular in , assuming $\varepsilon_1(m) < \varepsilon_2(m) < \varepsilon_3(m)$ for all $m \in \mathbf{Z}$, $12 \cdot 4$ points , where it meets the one-dimensional singular locus of the toroidal embedding. Blowing-up these singular points in the octahedron gives the compactified Fermi surface $\overline{\mathcal{F}_\lambda(\varepsilon)}$.

One shows that the divisor $\overline{\mathcal{F}_\lambda(\varepsilon)} - \mathcal{F}_\lambda(\varepsilon)$ is a connected union of reduced, irreducible curves, intersecting transversally. Furthermore $\overline{\mathcal{F}_\lambda(\varepsilon)}$ is smooth on the smooth points of $\overline{\mathcal{F}_\lambda(\varepsilon)} - \mathcal{F}_\lambda(\varepsilon)$. This induces Theorem 1.

Observe now that the Fermi surface $\mathcal{F}_\lambda(\varepsilon)$ is the locus of points in $(\mathbf{C}^*)^3$, where the operators

$$A(\varepsilon) - \lambda 1, D(\varepsilon), S^{a_i e_i} - \xi_i 1 \; (i = 1,2,3)$$

have a common kernel in the space $F = \{E : \mathbb{Z}^3 \to \mathbb{C}^3\}$. This means that $\mathcal{F}_\lambda(\varepsilon)$ is the support of the subsheaf $\mathcal{L}_\lambda$ of the trivial bundle $\mathcal{F}_\lambda(\varepsilon) \times F$ given by

$$\mathcal{L}_\lambda = \{((\xi_1, \xi_2, \xi_3), E) \in (\mathbb{C}^*)^3 \times F \mid \text{the above operators have a common kernel}\}.$$

Theorem 2. *$\mathcal{L}_\lambda$ can be extended to a sheaf over his compactification $\overline{\mathcal{F}_\lambda(\varepsilon)}$.*

By this the curves at "infinity" occurs as the support of one-dimensional spectral problems. For this we introduce the well known (see [vM-M]) one-dimensional Bloch variety $\mathcal{B}_a(W)$ defined by

$$\mathcal{B}_a(W) \stackrel{def}{=} \{(\xi, \lambda) \in \mathbb{C}^* \times \mathbb{C} \mid \text{there exists a nontrivial solution } \psi : \mathbb{Z} \to \mathbb{C} \text{ solving}$$

$$-[\psi(m-2) - 2\psi(m) + \psi(m+2)] + W(m)\psi(m) = \lambda\psi(m), \psi(m+a) = \xi\psi(m)\}$$

where $W : \mathbb{Z} \to \mathbb{C}$ has period a, a odd. $\mathcal{B}_a(W)$ is a double covering of a hyperelliptic curve of arithmetic genus $2a - 2$.

One then has, again under the assumption of Theorem 1 :

Theorem 3. *$\overline{\mathcal{F}_\lambda(\varepsilon)} - \mathcal{F}_\lambda(\varepsilon)$ contains the Bloch varieties $\mathcal{B}_{a_i}(W_i)$ with*

$$W_i(m_i) = \frac{1}{a_j a_k} \sum_{m_j, m_k} \varepsilon_i(m_1, m_2, m_3), \quad (i, j, k) \in S_3$$

4. Sketch of the proof of Theorem 3

$\mathcal{B}_{a_1}(W_1)$ is in the chart V of the blown-up octahedron. This chart is generated by the coordinates $(x, z, \mu) \in \mathbb{C}^* \times \mathbb{C} \times \mathbb{C}$. On $V \cap (\mathbb{C}^*)^3$ we have

$$x = \xi_1^{-1}, z = \xi_2^{y_0} \xi_3^{z_0}, \mu z^2 = 1 + \xi_2^{-2a_3} \xi_3^{2a_2}$$

where $(y_0, z_0) \in \mathbb{Z}^2$ with $a_2 y_0 + a_3 z_0 = 1$. Furthermore the fiber F over V is glued with the fiber F on $(\mathbb{C}^*)^3$ by

$$E(m_1, m_2, m_3) = z^{m_2+m_3} E^V(m_1, m_2, m_3).$$

Finally one has $V - (V \cap (\mathbb{C}^*)^3) = \{z = 0\}$.

Now $S^{a_1 e_1} E = \xi_1 E$ transforms to

$$S^{-a_1 e_1} E^V = x E^V. \tag{5}$$

Since $S^{(0, a_2 y_0, a_3 z_0)} E = \xi_2^{y_0} \xi_3^{z_0} E = zE$, using the transition function we have

$$S^{(0, a_2 y_0, a_3 z_0)} E_1^V = E_1^V. \tag{6}$$

A straightfoward calculation shows, that putting the transition function in

$$A(\varepsilon)S^{2(0,a_2y_0,a_3z_0)}E=\lambda z^2E$$

gives on $z=0$

$$(-S^{-2e_2}-S^{-2e_3})E_1^V=0,-S^{-2e_3}E_2^V+S^{-(e_2+e_3)}E_3^V=0 \tag{7}$$

and $D(\varepsilon)S^{(0,a_2y_0,a_3z_0)}E=0$ translates on $z=0$ to

$$S^{(0,-1,0)}(\varepsilon_2E_2^V)+S^{(0,0,-1)}(\varepsilon_3E_3^V)=0. \tag{8}$$

From (7) and (8) it follows, using $\varepsilon_1(m)<\varepsilon_2(m)<\varepsilon_3(m)$ for all $m\in\mathbf{Z}^3$, that

$$E_2^V=E_3^V=0 \quad\text{and}\quad S^{(0,-2,2)}E_1^V=-E_1^V. \tag{9}$$

Observe now that $S^{(0,-a_2a_3,a_2a_3)}E=(\mu z^2-1)E$, i.e. we get on $z=0$ $S^{(0,-a_2a_3,a_2a_3)}E_1^V=-E_1^V$. Since a_2 and a_3 are relatively prime and different from 2 , it follows with (9) that

$$S^{(0,-1,1)}E_1^V=\kappa E_1^V \quad\text{with}\quad \kappa^2=-1. \tag{10}$$

This shows that we have the boundary conditions for E_1^V given by :

$$S^{-a_1e_1}E_1^V=xE_1^V,S^{(0,a_2y_0,a_3z_0)}E_1^V=E_1^V,$$

$$S^{(0,-1,1)}E_1^V=\kappa E_1^V.$$

Now we also have $z^{-2}(1+S^{(0,-2a_2a_3,2a_2a_3)})E=\mu E$. But

$$1+S^{(0,-2a_2a_3,2a_2a_3)}=\sum_{i=0}^{a_2a_3-1}(-1)^i(S^{i(0,-2,2)}+S^{(i+1)(0,-2,2)}). \tag{11}$$

Using $A(\varepsilon)E=\lambda E$ and $D(\varepsilon)E=0$ one gets after some calculation

$$(S^{i(0,-2,2)}+S^{(i+1)(0,-2,2)})E_1^V=$$

$$z^2(-S^{(-2,0,2)}+2S^{(0,0,2)}-S^{(2,0,2)})S^{i(0,-2,2)}E_1^V+z^2S^{i(0,-2,2)}S^{(0,0,2)}(\varepsilon_1E_1^V)+z^3(...).$$

Since by (9) $S^{i(0,-2,2)}E_1^V=(-1)^iE_1^V$ we have for (11) on $z=0$

$$z^{-2}(1+S^{(0,-2a_2a_3,2a_2a_3)})E_1^V=$$

$$a_2a_3(-S^{(-2,0,0)}+2-S^{(2,0,0)})E_1^V+(\sum_{i=0}^{a_2a_3-1}\varepsilon_1(m_1,m_2-2i,m_3+2i))E_1^V=\mu E_1^V$$

i.e.

$$(-S^{(-2,0,0)} + 2 - S^{(2,0,0)})E_1^V + \frac{1}{a_2 a_3}\Big(\sum_{m_2,m_3} \varepsilon_1(m_1,m_2,m_3) \Big)\, E_1^V = \mu E_1^V .$$

This shows that one gets the Bloch variety $\mathcal{B}_{a_1}(W_1)$.

5. Related results

The questions posed in the introduction were answered for the operator $-\Delta + V$ in dimension 2 and 3.

Gieseker, Knörrer, Trubowitz have shown that in dimension 2 the Bloch variety is irreducible (in the discrete case [GKT], in the continous case [KT]). Moreover for the discrete model for generic potentials V the Bloch variety determines the potential up to obvious symmetries. This has been generalized by Kappeler in [K] to higher dimensions.

There exists for the discretized model also using toroidal embeddings an intrinsic compactification of the Bloch variety in dimension 2 and for the Fermi surface in dimension 3 (see [B1], [B2]).

For an overview of these and more stronger results consider [P] .

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