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EIGENVALUE ASYMPTOTICS FOR SCHRÖDINGER AND DIRAC OPERATORS WITH THE CONSTANT MAGNETIC FIELD AND WITH ELECTRIC POTENTIAL DECREASING AT INFINITY

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In this lecture we consider the Schrödinger and Dirac operators in $\mathbf{R}^d$ $(d=2,3)$ with a constant non-zero magnetic field and with electric potential decreasing at infinity and we derive the asymptotics of the eigenvalues of these operators tending to the boundary of the essential spectrum. Moreover, certain generalizations are discussed.

1. Namely, let us consider the Schrödinger operator

$$A_{\mathrm{S}} = P_j g^{jk} P_k + V$$

and Dirac operator

$$A_{\mathrm{D}} = \frac{1}{2}\sigma_l(\omega^{jl}P_j + P_j\omega^{jl}) + \sigma_0 m + V\cdot I$$

where $P_j = D_j - V_j$, $g^{jk} = g^{kj}$, V_j, V are smooth real-valued functions, $\sigma_0, \dots, \sigma_d$ are Dirac matrices (see [7] e.g.), $g^{jk} = \omega^{jl}\omega^{kr}\delta_{lr}$ in the Dirac operator case and we use the Einstein summation rule. We assume that

$$c^{-1}|\xi|^2 \le g^{jk}\xi_j\xi_k \le c|\xi|^2 \qquad \forall x,\xi. \tag{1}$$

Let us introduce tensor intensity $F_{jk} = \partial_k V_j - \partial_j V_k$ of the magnetic field. We assume that $(F_{jk}) \ne 0$.

First of all in the case when g^{jk}, ω^{jl} and F_{jk} are constant, $V=0$ the spectra of these operator are known: for $d=2$ these operators have pure point infinite multiplicity spectra and

$$\mathrm{Spec}\,(A_{\mathrm{S}}) = \{(2j+1)F, j \in \mathbb{Z}^+\},$$

$$\mathrm{Spec}\,(A_{\mathrm{D}}) = \{\zeta\sqrt{2jF+m^2}, j\in \mathbb{Z}^+, \zeta=\pm 1, (j,\zeta)\neq (0,\varsigma)\}$$

where $F = (\frac{1}{2}g^{jk}g^{lr}F_{jl}F_{kr})^{1/2}$ is a scalar intensity of the magnetic field, $\mathbb{Z}^+ = \{0,1,2,...\}$ and we assume without loss of generality that $\sigma_0\sigma_1\sigma_2 = i\varsigma I$, $\det(\omega^{jk}) > 0$ and $F_{12} > 0$. On the other hand for $d = 3$ these spectra are absolutely continuous and $\mathrm{Spec}\,(A_{\mathrm{S}}) = [F,\infty)$, $\mathrm{Spec}\,(A_{\mathrm{D}}) = (-\infty, m]\cup[m,\infty)$. Moreover in the multi-dimensional case these spectra are known (see [8] e.g.); in particular, these spectra are pure point of infinite multiplicity if rank $(F_{jk}) = d$ and they are absolutely continuous otherwise.

Let us perturb these operators by potential V tending to 0 at infinity. Moreover, we can consider more general perturbations: we can assume that g^{jk}, ω^{jl} and F_{jk} tend at infinity to constants. Then for $d = 2$ every former eigenvalue λ (of infinite multiplicity) will be separated into the sequence of eigenvalues λ_n tending to λ and the question is to obtain the asymptotical distribution of λ_n. The asymptotics of the eigenvalue counting function with highly accurate remainder estimates were obtained in [4,6] and it is not difficult to generalize the results of this paper to the case when rank $(F_{jk}) = d$ and λ appears only once in the list of eigenvalues of the non-perturbed operator given by formula (6.1.8) in [8].

Let us consider the case $d = 3$. In this case there arises a sequence of eigenvalues tending to the bottom F of $\mathrm{Spec}_{\mathrm{ess}}(A_{\mathrm{S}})$ or to boundaries $\pm m$ of $\mathrm{Spec}_{\mathrm{ess}}(A_{\mathrm{D}})$. So the question is to obtain the asymptotical distribution of these eigenvalues. In the case when V decreases slowly enough the asymptotics of the eigenvalue counting function with accurate remainder estimates were obtained in [2,3,5]. In particular, under appropriate conditions in these notes was treated the case when V[1] decreases as $|x|^{2p}$ with $p\in(-1,0)$ and the asymptotics of $N^-(\bar{F}-\varepsilon)$ were derived for the Schrödinger operator and asymptotics of $N(m-\epsilon_0, m-\varepsilon)$ and of $N(-m+\varepsilon, -m+\epsilon_0)$ were derived for the Dirac operator where $N(\lambda_1,\lambda_2)$ is the number of eigenvalues lying in the interval $[\lambda_1,\lambda_2)$ with $\lambda_2 > \lambda_1$, $N^-(\lambda) = N(-\infty,\lambda)$, $\varepsilon\to +0$ and $\epsilon_0 > 0$ is a small enough constant. The principal parts of these asymptotics are $\asymp \varepsilon^{(3+p)/2p}$ and the remainder estimates are $O(\varepsilon^{1/p})$. So in the limit case $r = -1$ we obtain *formally* estimates $N^-(\bar{F}-\varepsilon) = O(\varepsilon^{-1})$ and $N(m-\epsilon_0, m-\varepsilon) = O(\varepsilon^{-1})$. Compare with the classical case $F = 0$ when the principal parts of asymptotics are $\asymp \epsilon^{d(p+1)/2p}$ and the remainder estimates are $O(\varepsilon^{(d-1)(p+1)/2p})$ and in the limit case $p = -1$ one has *formally* $O(1)$. This difference is linked with the following principal difference: *In the classical case $F = 0$ the ex-*

[1] For the Schrödinger operator one should replace V here and in what follows at this subsection by $V^* = V + F - \bar{F}$, $\bar{F} = \lim_{|x|\to\infty} F$.

ponent $p = -1$ is critical in the sense that for $p < -1$ there is only finite number of eigenvalues. In our case (under appropriate condition to the sign of V or V^*) there is infinite number of eigenvalues for every $p < 0$ and the exponent $r = -1$ is critical only in the sense that asymptotics derived in [2,3,5] *fail for $p \le -1$.* So the problem is to derive asymptotics in the case $r \le -1$. And in the remainder part of this lecture *asymptotics in the case $d = 3$, $p < -1$ will be presented.* The asymptotics of this type were obtained few years ago by Alexander Sobolev [14]; see also the related papers [10-13,15-17]. However he obtained no remainder estimate and so the problem was to derive asymptotics with accurate remainder estimates. During few years I considered this problem as a challenge because I was unable even to reprove the results of A.Sobolev by my methods. Concluding this section I would like to notify that results of the type "$d = 3$, $p > -1$ can be generalized to the case $d > 3$ provided $|F_{jk}| \le c$ and $\operatorname{Spec}(F_{jk}) \cup (-\epsilon_0, \epsilon_0) = \emptyset$ for all x with some constant $\epsilon_1 > 0$. The open problem is to drop this condition.

2. So let us assume that $d = 3$. Moreover, for the Dirac operator let us assume that $m > 0$ (otherwise $\operatorname{Spec}_{\text{ess}}(A) = \mathbb{R}$). Without loss of generality one can assume that $m = 1$ and that σ_j are 4×4-matrices (see [7,9]). For the Dirac operator we treat eigenvalues tending to $1 - 0$.

Let us introduce the vector intensity F^j of the magnetic field: $F^j = \frac{1}{2}\epsilon^{jkl}F_{kl}$ where ϵ^{jkl} is absolutely skew-symmetric pseudo-tensor with $\epsilon^{123} = 1/\sqrt{g}$, $g = \det(g^{jk})^{-1}$. Let us assume that

$$F^2 = F^3 = 0, \quad g^{13}g^{23} = 0 \qquad \text{for } |x| \ge c \tag{2}$$

(under reasonable conditions we can reach it by the appropriate change of variables). Without loss of generality one can assume that $V_3 = 0$ (one can always reach it by appropriate gradient transform; see [8,9] e.g.)

Moreover let us assume that

$$|D^\alpha(g^{jk} - \delta_{jk})| \le c\langle x\rangle^{-\delta-|\alpha|}, \tag{3}$$

$$|D^\alpha(F^3 - \bar{F})| \le c\langle x\rangle^{-\delta-|\alpha|}, \tag{4}$$

$$|D^\alpha V^*| \le c\langle x\rangle^{2p-|\alpha|} \qquad \forall\alpha : |\alpha| \le K \tag{5}$$

where K is large enough, $\langle x\rangle = (|x|^2 + 1)^{1/2}$, $\delta > 0$, $p < -1$, $\bar{F} = \text{const} \ne 0$ (without loss of generality one can assume that $\bar{F} = 1$) and

V^* is an effective potential: $V^* = V + F - F$ for the Schrödinger operator, $V^* = (V^2 - m^2)/2m$ for the Dirac operator.

Let us consider the auxiliary operator $\mathcal{A} = g^{33}D_3^2 + V'$ on $\mathbf{R} \ni x_3$ depending on $x' = (x_1, x_2)$.

LEMMA 1. *Let all the conditions* (1) – (5) *be fulfilled. Then*

(i) The number of negative eigenvalues of $\mathcal{A}$ is finite;

(ii) For $|x'| > C$ the number of negative eigenvalues of $\mathcal{A}$ doesn't exceed 1;

(iii) If

$$W(x') = -\int_{-\infty}^{\infty} V'(x', x_3)dx_3 \geq c^{-1}\langle x'\rangle^{1+2p} \tag{6}$$

then for $|x'| \geq C$ operator $\mathcal{A}$ has exactly one negative eigenvalue $\lambda(x')$ and

$$|D^\alpha \lambda| \leq c\langle x'\rangle^{2+4p-|\alpha|} \qquad \forall\alpha : |\alpha| \leq K - 2, \tag{7}$$

$$|\lambda + \frac{1}{4}W(x')^2| \leq \langle x'\rangle^{4p-\delta'} \tag{8}$$

with $\delta' > 0$.

The principal result of this lecture is the following

THEOREM 2. *Let conditions* (1) – (4) *be fulfilled and $\delta > 2$. Moreover, let condition* (5) *be fulfilled for all $x : |x| \geq c$ and let us assume that*

$$|dW| \geq c^{-1}|x'|^{2p} \qquad \forall x' : |x'| \geq c. \tag{9}$$

Then asymptotics

$$\nu(\varepsilon) = (2\pi)^{-1}\bar{F}\int_{\{-\lambda(x')\geq\varepsilon\}} dx' + O(1) \tag{10}$$

holds as $\varepsilon \to +0$ where $\nu(\varepsilon) = N(1-\varepsilon)$ and $\nu(\varepsilon) = N(1-\epsilon_0, 1-\varepsilon)$ for Schrödinger and Dirac operators respectively, $\epsilon_0 > 0$ is a small enough constant and

$$\nu(\varepsilon) \asymp \varepsilon^{1/(1+2p)}. \tag{11}$$

Moreover, these asymptotics remain true if one replaces g^{33} by 1 in the definition of $\mathcal{A}$ and λ. Moreover, these asymptotics remain true if one replaces λ by $-\frac{1}{4}W^2$.

There are few open problems: 1. To treat the case $\delta \leq 2$.

2. To treat the case $d > 3$ provided rank $(F_{jk}) = d-1$, $|F_{jk}| \leq c$ and $\mathrm{Spec}(F_{jk}) \cup (-\epsilon_0, \epsilon_0) = \emptyset$ for all $x : |x| \geq c$ with some constant $\epsilon_1 > 0$. It is very likely that $\nu(\varepsilon) = O(1)$ in the case rank $(F_{jk}) < d-1$.

3. To treat the case $p = -1$. Moreover, to treat the case $p \geq -1$ and to improve the remainder estimates.

References

1. V.Ivrii, Journées "Équat. aux Deriv. Part., Saint Jean de Monts, 1987. Exp. XX, 7 pp.

2. V.Ivrii, Proc. Intern. Cong. Math., Berkeley, 1986, 1084-1093.

3. V.Ivrii, Soviet Math. Doklady, **36** (1988) N 3, 1043-1046.

4. V.Ivrii, Soviet Math. Doklady, **39** (1989) N 3, 431-441.

5. V.Ivrii, Dokl. AN SSSR, **310** (1989) N 4, 788-792.

6. V.Ivrii, Dokl. AN SSSR, **311** (1990) N 5, 1102-1105.

7. V.Ivrii, Semiclassical microlocal analysis and precise spectral asymptotics. Chapter 4. Preprint 2. Ecole Polytechnique, Preprint M 969.0191, January 1991.

8. V.Ivrii, Semiclassical microlocal analysis and precise spectral asymptotics. Chapter 6. Preprint 4. Ecole Polytechnique, Preprint M 974.0591, May 1991.

9. V.Ivrii, Semiclassical microlocal analysis and precise spectral asymptotics. Chapter 7. Preprint 5. Ecole Polytechnique, Preprint M 980.0691, June 1991.

10. G.D.Raikov, Comm. Math. Phys., **124** (1989) 665-685.

11. G.D.Raikov, Comm. Part. Differ. Equat., **15** (1990) N 3, 407-434.

12. G.D.Raikov, Letters Math. Phys., **21** (1991) (to appear).

13. G.D.Raikov, J. reine angew. Math. (to appear).

14. A.Sobolev, J. Soviet Math., **35** (1986) 2201-2211.

15. A.Sobolev, Probl. Math. Phys., **11** (1986) 232-248.

16. A.Sobolev, Operator Theory: Adv. and Appl., (Birkhäuser) **46** (1990) 27-31.

17. S.Solnyshkin, Selecta Math. Soviet., **5** (1986) 297-306.