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Preparation theorems for systems

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PREPARATION THEOREMS FOR SYSTEMS

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1. Introduction

The Malgrange preparation theorem is a useful tool in analysis. It is a generalization of the Weierstrass' preparation theorem to C^∞ functions as follows: if $f(t,x) \in C^\infty(\mathbf{R}\times\mathbf{R}^d)$ satisfies

$$0 = f(0,0) = \partial_t f(0,0) = \cdots = \partial_t^{n-1} f(0,0) \quad \text{and} \quad \partial_t^n f(0,0) \neq 0, \tag{1.1}$$

then we can factor

$$f(t,x) = c(t,x)\big(t^n + a_{n-1}(x)t^{n-1} + \cdots + a_1(x)t + a_0(x)\big) \tag{1.2}$$

near (0,0), where $c(0,0) \neq 0$ and $a_j(0) = 0$, $0 \le j < n$. The condition (1.1) means that

$$f(t,0) = c(t)t^n,$$

where $c(0) \neq 0$. A possible generalization of this result to matrix valued functions, is to replace (1.1) by

$$0 = F(0,0) = \partial_t F(0,0) = \cdots = \partial_t^{n-1} F(0,0) \quad \text{and} \quad |\partial_t^n F(0,0)| \neq 0, \tag{1.3}$$

where $F(t,x) \in C^\infty$ is $N \times N$ matrix valued, and $|F|$ is the determinant. Then we should obtain (1.2) with matrix valued $c(t,x)$ and $a_j(x)$, satisfying $|c(t,x)| \neq 0$ and $a_j(0) = 0$, $\forall j$. In the case when $n = 1$ in (1.3), this was proved in [**1**]. But the condition (1.3) is too restrictive, since it does not cover the cases when $F(t,x) = (f_j(t,x)\delta_{jk})$ is diagonal, with diagonal elements f_j satisfying (1.1) with different n (in which case we can use the Malgrange preparation theorem). More generally, we assume that

$$F(t,0) = C(t)\sum_{j=0}^{n} t^j \pi_j, \tag{1.4}$$

where $|C(0)| \neq 0$, and π_j is orthogonal projection on $\mathbf{C}^N$, such that $\pi_j\pi_k = \delta_{jk}\pi_k$ and $\sum_{j=0}^n \pi_j = \mathrm{Id}_N$. This includes condition (1.3), and is equivalent to

$$\mathbf{C}^N = \bigoplus_{j=0}^{n} \operatorname{Im} \partial_t^j F(0,0)\Big|_{E_{j-1}}, \tag{1.5}$$

where $E_k = \bigcap_{0\le j\le k} \operatorname{Ker} \partial_t^j F(0,0)$. This condition is invariant under *left* multiplication of F by elliptic systems. Assuming (1.5), we show in Theorem 2.5 that

$$F(t,x) = C(t,x)\Big(\sum_{j=0}^{n} t^j\pi_j + \sum_{j=0}^{n-1} t^j A_j(x)\Big), \tag{1.6}$$

near (0,0), where $C(t,x)$ and $A_j(x)$ are C^∞ functions satisfying $|C(0,0)| \neq 0$ and $A_j(0) = 0$, $0 \le j < n$. Since we are allowed to make row operations, we obtain that $A_j(x)\pi_k \equiv 0$ when $j \ge k$. The orthogonal projections π_k are uniquely determined in (1.6).

Now condition (1.5) is also too restrictive, for example, it is not satisfied for the polynomials

$$\sum_{j=0}^{n} t^j \pi_j + \sum_{j=0}^{n-1} t^j A_j(x),$$

when $A_j(x)\pi_k \equiv 0$ for $j \ge k$, and $A_j(0) \neq 0$ for some j. But such systems always satify the condition

$$\partial_t^m (\det F)(0,0) \neq 0, \tag{1.7}$$

for some m, thus the determinant does not vanish of infinite order. In Theorem 3.3, we show that condition (1.7) is sufficient for a preparation of F on the form (1.6), with orthogonal projections $\{\pi_j\}$ on $\mathbf{C}^N$, such that $\pi_j\pi_k = \delta_{jk}\pi_k$ and $\sum_{j=0}^{n} \pi_j = \mathrm{Id}_N$. We obtain that $A_j(x)$ satisfies $A_j(x)\pi_k \equiv 0$ when $j \ge k$, and

$$A_j(0) = \sum_{i<j<k} \pi_i A_j(0) \pi_k. \tag{1.8}$$

The projections π_k and matrices $A_j(0)$ are uniquely determined by (1.8). The rank of the projections π_k are determined by the elementary divisors of the Taylor expansion of $F(t,0)$ at $t=0$, but the projections themselves are harder to compute, except for π_0 and π_1 (see Remark 3.4).

By allowing right multiplication by elliptic systems, i.e. column operations, we may also obtain that $\pi_k A_j(x) \equiv 0$ when $j \ge k$, and $A_j(0) = 0$ when (1.8) holds (see Proposition 3.5). By duality, we obtain the corresponding results for *right* preparation of F, i.e. left preparation of F^*, in Theorem 4.2. We also prove the generalization of Malgrange's division theorem in Theorems 3.6 and 4.3. The method of proof follows in part Mather [**6**], with the improvements of Hörmander [**2**, Section 7.5]. Observe that, since the proofs of Malgrange [**3**] use *commutative* algebra, they are not directly applicable here.

2. Left Preparation

In what follows, let π_j be (complex) orthogonal projections in $\mathbf{C}^N$, $0 \le j \le n$, such that $\sum_{j=0}^{n} \pi_j = \mathrm{Id}_N$ and $\pi_j\pi_k = \delta_{jk}\pi_k$. This means that $\pi_j^* = \pi_j$, $\forall j$. Put

$$P(t,\mathsf{A}) = \sum_{0\le j\le n} t^j \pi_j + \sum_{0 \le j < n} t^j A_j, \tag{2.1}$$

with $\mathsf{A} = (A_0, \dots, A_{n-1})$, where $A_j \in \mathcal{L}_N = \mathcal{L}(\mathbf{C}^N, \mathbf{C}^N)$ is a complex $N \times N$ matrix satisfying

$$A_j \pi_k = 0 \quad \text{when} \quad j \ge k. \tag{2.2}$$

Let $|A_j| = \det A_j$ be the determinant of A_j, and let $\|\mathsf{A}\| = \sum_j \|A_j\|$, where $\|A_j\|$ is the matrix norm. We are going to divide matrix valued analytic functions with such matrix valued polynomials. Let ω be an open set in $\mathbf{C}$, let $G(t)$ be analytic in $\overline{\omega}$ with values in $\mathcal{L}_N$, and assume $|P(t,\mathsf{A})| \neq 0$ on $\partial\omega \in C^1$. Then

$$G(t) = Q(t)P(t,\mathsf{A}) + R(t) \qquad t \in \omega, \tag{2.3}$$

where

$$(2.4) \qquad Q(t) = (2\pi i)^{-1} \int_{\partial\omega} G(s)P(s,\mathsf{A})^{-1}(s-t)^{-1}\,ds \qquad t \in \omega$$

is analytic in ω, and

$$(2.5) \qquad R(t) = (2\pi i)^{-1} \int_{\partial\omega} G(s)P(s,\mathsf{A})^{-1}(P(s,\mathsf{A}) - P(t,\mathsf{A}))(s-t)^{-1}\,ds$$

is a polynomial of degree $n-1$ in t. We find that

$$(2.6) \quad R(t)\pi_k = (2\pi i)^{-1} \int_{\partial\omega} G(s)P(s,\mathsf{A})^{-1}\big((s^k - t^k)\pi_k + \sum_{j<k}(s^j - t^j)A_j\pi_k\big)(s-t)^{-1}\,ds$$

is a polynomial of degree $< k$ in t. The remainder $R(t)$ is uniquely determined by this condition, if $R(t)P(t,\mathsf{A})^{-1}$ is analytic when $t \notin \omega$.

Let $\mathbf{V} \subseteq \bigotimes_{j=0}^{n-1} \mathcal{L}_N$ be the set of $\mathsf{A} = (A_0, \dots, A_{n-1})$ satisfying (2.2), let $m_k = \operatorname{Rank} \pi_j$ and $m = \sum_{1\le j\le n} j \cdot m_j$. Since $A_k = \sum_{k<j} A_k\pi_j$, A_k lies in a subspace of (complex) dimension $\sum_{k<j} m_j N$ of $\mathcal{L}_N$. This implies that $\mathbf{V} \cong \mathbf{C}^{mN} \cong \mathbf{R}^{2mN}$, since we have $\sum_{0\le k<j\le n} m_j = \sum_{j=1}^{n} j \cdot m_j = m$. We obtain the following division theorem on $\mathbf{R}$.

PROPOSITION 2.1. *Let $F(t) \in \mathcal{S}(\mathbf{R})$ have values in $\mathcal{L}_N$. Then we can find $Q(t,\mathsf{A},F) \in C^\infty(\mathbf{R}\times\mathbf{V})$ and $R_j(\mathsf{A},F) \in C^\infty(\mathbf{V})$, $0 \le j < n$, with values in $\mathcal{L}_N$ and depending linearly on $F(t)$, such that $R_j(\mathsf{A})\pi_k \equiv 0$ when $j \ge k$, and*

$$(2.7) \qquad F(t) = Q(t,\mathsf{A},F)P(t,\mathsf{A}) + \sum_{j=0}^{n-1} t^j R_j(\mathsf{A},F) \quad \textit{when} \quad \|\mathsf{A}\| < 1, \quad t \in \mathbf{R}.$$

We also get global estimates on all the derivatives of Q and R_j. The proof of Proposition 2.1 follows the proof of [**2**, Lemma 7.5.4]. Thus, first we show that we may divide bounded analytic functions by $P(t,\mathsf{A})$ in a strip containing $\mathbf{R}$, with uniform bounds. Then, we get the result by a Fourier decomposition of F.

REMARK 2.2. If $F(t,x) \in \mathcal{S}(\mathbf{R}\times\mathbf{R}^n)$ depends on parameters x, then $Q(t,\mathsf{A},F(\cdot,x)) \in C^\infty(\mathbf{R}\times\mathbf{V}\times\mathbf{R}^n)$ and $R_j(\mathsf{A},F(\cdot,x)) \in C^\infty(\mathbf{V}\times\mathbf{R}^n)$. In fact, by linearity and continuity, we may differentiate directly on F.

Next, we shall compute some invariants. Let $F(t)$ be a C^∞ function on $\mathbf{R}$ with values in $\mathcal{L}_N$. Put $E_{-1} = \mathbf{C}^N$ and

$$(2.8) \qquad E_k = \bigcap_{0\le j\le k} \operatorname{Ker} \partial_t^j F(0), \quad k \ge 0.$$

PROPOSITION 2.3. *If*

$$(2.9) \qquad \mathbf{C}^N = \bigoplus_{j=0}^{n} \operatorname{Im} \partial_t^j F(0)\Big|_{E_{j-1}},$$

then it follows that $E_n = \{0\}$. We find that the spaces E_k, $0 \le k \le n$, and condition (2.9) are invariant under left multiplication of F by elliptic systems.

PROOF: Assume that $C(t)$ is elliptic, then $\operatorname{Ker} CF(0) = \operatorname{Ker} F(0)$. Now, we have by Leibniz' rule

$$\partial_t^k(CF)(0) = \sum_{j=0}^{k} \binom{k}{j} \partial_t^{k-j} C(0)\partial_t^j F(0),$$

so by induction we obtain

$$\bigcap_{0\le j\le k} \operatorname{Ker} \partial_t^j CF(0) = \Big(\bigcap_{0\le j<k} \operatorname{Ker} \partial_t^j F(0) \Big) \cap \operatorname{Ker} \partial_t^k (CF)(0) = \bigcap_{0\le j\le k} \operatorname{Ker} \partial_t^j F(0),$$

which gives the invariance of E_k, $\forall\, k$. We also obtain that

$$\operatorname{Im} \partial_t^k (CF)(0)\Big|_{E_{k-1}} = C(0) \operatorname{Im} \partial_t^k F(0)\Big|_{E_{k-1}}. \tag{2.10}$$

Since $|C(0)| \neq 0$, this gives the invariance of condition (2.9).

It remains to prove that $\dim E_n = 0$. Let $m_k = \dim E_k$, so that $m_{-1} = N$. Then, we find

$$\dim\Big(\operatorname{Im} \partial_t^k F(0)\Big|_{E_{k-1}} \Big) = \dim E_{k-1} - \dim\Big(\operatorname{Ker} \partial_t^k F(0)\Big|_{E_{k-1}} \Big) = m_{k-1} - m_k.$$

Thus we find from (2.9) that

$$N \le \sum_{j=0}^{n} (m_{k-1} - m_k) = N - m_n.$$

This means that $m_n \le 0$, which proves the result. ▮

PROPOSITION 2.4. *Let $\mathbf{C}^N = E_{-1} \supseteq E_0 \supseteq \cdots \supseteq E_n = \{0\}$, and let π_k be the orthogonal projection on $E_k^\perp \bigcap E_{k-1}$, for $0 \le k \le n$. Then it follows that $\pi_j \pi_k = \delta_{jk} \pi_k$, and*

$$\bigoplus_{j=0}^{k} \operatorname{Im} \pi_j = E_k^\perp, \quad 0 \le k \le n. \tag{2.11}$$

In particular, we obtain $\bigoplus_{0\le j\le n} \operatorname{Im} \pi_j = \mathbf{C}^N$, which implies $\sum_{j=0}^n \pi_j = \mathrm{Id}_N$.

PROOF: Clearly, $\operatorname{Ker} \pi_k = (\operatorname{Im} \pi_k)^\perp = E_k \bigoplus E_{k-1}^\perp$, so

$$\operatorname{Im} \pi_j \subseteq E_{j-1} \subseteq E_k \subseteq \operatorname{Ker} \pi_k \quad \text{if} \quad j > k.$$

Thus, $\pi_k \pi_j \equiv 0$ when $j > k$, and by taking adjoints we obtain this when $j < k$, which implies $\pi_j \pi_k = \delta_{jk} \pi_k$.

By taking orthogonal complements, we find that (2.11) is equivalently to

$$\bigcap_{j=0}^{k} \operatorname{Ker} \pi_j = E_k \quad 0 \le k \le n. \tag{2.12}$$

We find $\operatorname{Ker} \pi_0 = E_0 \oplus E_{-1}^\perp = E_0$. Assume by induction that (2.12) holds for some $k \ge 0$. Then we find that

$$\bigcap_{0\le j\le k+1} \operatorname{Ker} \pi_j = E_k \bigcap (E_{k+1} \oplus E_k^\perp) = E_{k+1},$$

since $E_{k+1} \subseteq E_k$, thus by induction we obtain (2.12) for all k. Since $E_n = \{0\}$ we find that $\sum_{j=0}^n \pi_j$ is bijective, and since $(\sum_{j=0}^n \pi_j)^2 = \sum_{j=0}^n \pi_j$, it is equal to the identity. ▮

Now, we can state the following generalization of the Malgrange preparation theorem.

THEOREM 2.5. *Let $F(t,x)$ be a C^∞ function of (t,x) in a neighborhood of the origin of $\mathbf{R}\times\mathbf{R}^d$, with values in $\mathcal{L}_N$, and assume that*

$$\mathbf{C}^N = \bigoplus_{j=0}^{n} \operatorname{Im} \partial_t^j F(0,0)\Big|_{E_{j-1}}, \tag{2.13}$$

where $E_{-1} = \mathbf{C}^N$ and $E_k = \bigcap_{0\le j\le k} \operatorname{Ker} \partial_t^j F(0,0)$. Let π_k be the orthogonal projection on $E_k^\perp \bigcap E_{k-1}$ for $0 \le k \le n$. Then, we may factor

$$F(t,x) = C(t,x)\Big(\sum_{j=0}^{n} t^j \pi_j + \sum_{j=0}^{n-1} t^j A_j(x)\Big) = C(t,x)P(t,\mathsf{A}(x)) \tag{2.14}$$

near (0,0), where $C(t,x)$ and $A_j(x)$ are C^∞ functions with values in $\mathcal{L}_N$, satisfying $A_j(x)\pi_k \equiv 0$, $j \ge k$. We also find $|C(0,0)| \ne 0$ and $A_j(0) = 0$, $0 \le j < n$. If F is real (matrix) valued, we may choose C and A_j real (matrix) valued (and the projections π_k are real).

This is proved by using Proposition 2.1 and Remark 2.2, to divide $F(t,x)$ by the polynomials $P(t,\mathsf{A})$. Then, we use the implicit function theorem, to choose $\mathsf{A}(x) \in C^\infty$ making the remainder $\sum_j t^j R_j(\mathsf{A}(x), F) \equiv 0$.

Proposition 2.3 and (2.13) imply that $E_n = \{0\}$. Thus, we obtain from Proposition 2.4 that $\pi_j\pi_k = \delta_{jk}\pi_k$ and $\sum_j \pi_j = \mathrm{Id}_N$. Since (2.14) implies

$$F(t,0) = C(t,0)\sum_{j=0}^{n} t^j \pi_j,$$

we obtain from Proposition 2.3 that the condition (2.13) is necessary for the preparation (2.14).

EXAMPLE 2.6. Let $F(t,x)$ be a C^∞ function of (t,x) with values in $\mathcal{L}_N$, and assume that

$$|\partial_t^n F(0,0)| \ne 0 \quad \text{and} \quad \partial_t^j F(0,0) \equiv 0, \quad 0 \le j < n.$$

Then we obtain from Theorem 2.5

$$F(t,x) = C(t,x)\big(t^n \,\mathrm{Id}_N + \sum_{0\le j<n} t^j A_j(x)\big),$$

where $C(t,x)$ and $A_j(x)$ are C^∞ functions with values in $\mathcal{L}_N$, $|C(0,0)| \ne 0$ and $A_j(0) = 0$, $0 \le j < n$. (The case when $n = 1$ was proved in [**1**, Theorem A.3].)

3. THE PREPARATION THEOREM

The condition (2.13) in Theorem 2.5 is still too restrictive. In fact, the systems $P(t,\mathsf{A}(x))$ in (2.14) do not satisfy condition (2.13) when $\mathsf{A}(0) \ne 0$, but will be acceptable normal forms when $\mathsf{A}(x) \in \mathbf{V}$, i.e. $A_j(x)\pi_k \equiv 0$ for $j \ge k$. As before, we assume that π_j is orthogonal projection in $\mathbf{C}^N$, $0 \le j \le n$, such that $\sum_{j=0}^{n} \pi_j = \mathrm{Id}_N$ and $\pi_i\pi_j = \delta_{ij}\pi_j$. First, we consider the necessary condition for such a preparation.

PROPOSITION 3.1. *Let $F(t) \in C^\infty(\mathbf{R})$ with values in $\mathcal{L}_N$, and assume that*

$$F(t) = C(t)\Big(\sum_{j=0}^{n} t^j \pi_j + \sum_{j=0}^{n-1} t^j A_j\Big) = C(t)P(t, \mathsf{A}) \tag{3.1}$$

where $|C(0)| \neq 0$ and $A_j \pi_k = 0$ when $j \geq k$. Then it follows that

$$\partial_t^m (\det F)(0) \neq 0, \tag{3.2}$$

for some m. We also find

$$E_n = \bigcap_{0 \leq k \leq n} \operatorname{Ker} \partial_t^k F(0) = \{\, 0 \,\}. \tag{3.3}$$

PROOF: Since the spaces E_k are invariant under multiplication from left by elliptic systems by Proposition 2.3, we may replace $F(t)$ by $P(t, \mathsf{A})$ in (3.3). Now $\partial_t^k P(0, \mathsf{A}) = k!(\pi_k + A_k)$, where $A_k = \sum_{k<j} A_k \pi_j$. Thus, we find that $\operatorname{Ker} \partial_t^n F(0) = \operatorname{Ker} \pi_n$. By induction we have

$$\bigcap_{j=k}^{n} \operatorname{Ker}(\pi_j + A_j) = \Big(\bigcap_{j=k+1}^{n} \operatorname{Ker} \pi_j\Big) \cap \operatorname{Ker}(\pi_k + A_k) = \bigcap_{j=k}^{n} \operatorname{Ker} \pi_j,$$

for $0 \leq k \leq n$, which proves (3.3). It is also clear that condition (3.2) is invariant under multiplication by elliptic systems. By a (constant) orthogonal base change, we may assume that

$$\operatorname{Im} \pi_k = \{\, (z_1, \dots, z_N) : z_j \neq 0 \Rightarrow \sum_{i=0}^{k-1} \operatorname{Rank} \pi_i < j \leq \sum_{i=0}^{k} \operatorname{Rank} \pi_i \,\}, \quad 0 \leq k \leq n.$$

Since $P(t, \mathsf{A})\pi_k = (t^k + \sum_{j<k} t^j A_j)\pi_k$ we find that

$$\partial_t^m (\det P)(0, \mathsf{A}) = \Big|\sum_{k=0}^{n} k! \pi_k\Big| \neq 0,$$

if $m = \sum_{j=1}^{n} j \cdot \operatorname{Rank} \pi_j$, which proves (3.2). ∎

The factorization (3.1) is not unique, according to the following

EXAMPLE 3.2. Let

$$P_1(t) = \begin{pmatrix} t & 1 \\ 0 & t \end{pmatrix} = t \operatorname{Id}_2 + A_0$$

$$P_2(t) = \begin{pmatrix} t^2 & 0 \\ t & 1 \end{pmatrix} = \pi_0 + t^2 \pi_2 + t B_1$$

and $Q(t) = \begin{pmatrix} t & -1 \\ 1 & 0 \end{pmatrix}$. Then we have $Q(t)P_1(t) = P_2(t)$, and $|Q(t)| \equiv 1$. Since $B_1 \pi_0 = 0$, it is clear that $P_1(t)$ and $P_2(t)$ are on the form (2.1)–(2.2).

Now we are ready to state the main preparation theorem.

THEOREM 3.3. *Let $F(t,x)$ be a C^∞ function of (t,x) in a neighborhood of the origin of $\mathbf{R}\times\mathbf{R}^d$ with values in $\mathcal{L}_N$, and assume that*

$$\partial_t^m(\det F)(0,0)\neq 0 \quad \text{and} \quad \partial_t^k(\det F)(0,0)=0, \quad 0\le k<m. \tag{3.4}$$

Then we may factor

$$F(t,x)=C(t,x)\Big(\sum_{j=0}^{n} t^j\pi_j+\sum_{j=0}^{n-1} t^jA_j(x)\Big)=C(t,x)P(t,\mathsf{A}(x)) \tag{3.5}$$

near (0,0), where π_j is orthogonal projection in $\mathbf{C}^N$, $0\le j\le n$, such that $\pi_j\pi_k=\delta_{jk}\pi_k$ and $\sum_{j=0}^n\pi_j=\mathrm{Id}_N$. Here $C(t,x)$ and $A_j(x)$ are C^∞ functions with values in $\mathcal{L}_N$, satisfying $|C(0,0)|\neq 0$, $A_j(x)\pi_k\equiv 0$ when $j\ge k$ and

$$A_j(0)=\sum_{i<j<k}\pi_iA_j(0)\pi_k, \tag{3.6}$$

which implies $A_0(0)=0$. The projections π_k and matrices $A_j(0)$ are uniquely determined by the condition (3.6), and it follows that $m=\sum_j j\cdot\mathrm{Rank}\,\pi_j$ in (3.4). If F is real (matrix) valued, we may choose $C(t,x)$, π_k and $A_j(x)$ real (matrix) valued.

Theorem 3.3 is proved by reducing to the case of Theorem 2.5. Since condition (2.13) is necessary for that preparation, and is invariant under *left* multiplications, we must also multiply F from the right. Then, we have to be careful not to destroy the normal form $P(t,\mathsf{A}(x))$.

REMARK 3.4. The rank of the projections π_k are determined by the elementary divisors of the Taylor expansion of $F(t,0)$ at $t=0$. In fact, let d_k be the determinant factors for $1\le k\le N$, i.e. the greatest common divisor of the minors of order k of the Taylor expansion. Then $e_k=d_k/d_{k-1}$ are the elementary divisors, and $\mathrm{Rank}\,\pi_j$ is the number of k such that e_k is divisible by t^j but not by t^{j+1} (see [**8**, § 85]). The projections π_j are harder to compute, except for $j=1$, 2, since in these cases $\mathrm{Ker}\,\pi_0=\mathrm{Ker}\,F(0,0)$, and

$$\mathrm{Ker}\,\pi_0\cap\mathrm{Ker}\,\pi_1=\mathrm{Ker}\left[\partial_tF(0,0)\Big|_{\mathrm{Ker}\,F(0,0)}\mapsto \mathbf{C}^N/\,\mathrm{Im}\,F(0,0)=\mathrm{Coker}\,F(0,0)\right].$$

By multiplication from right with invertible matrices, i.e. column operations, we may also obtain that $\pi_kB_j(x)\equiv 0$ when $j\ge k$, and $\mathsf{B}(0)=0$ in (3.5), according to the following

PROPOSITION 3.5. *Assume that π_j are orthogonal projections in $\mathbf{C}^N$, $0\le j\le n$, such that $\pi_j\pi_k=\delta_{jk}\pi_k$ and $\sum_{j=0}^n\pi_j=\mathrm{Id}_N$. Let*

$$P(t,\mathsf{A}(x))=\sum_{j=0}^{n}t^j\pi_j+\sum_{j=0}^{n-1}t^jA_j(x) \tag{3.7}$$

where $A_j(x)$ are C^∞ functions with values in $\mathcal{L}_N$, satisfying $A_j(x)\pi_k\equiv 0$ for $j\ge k$. Then we may find $C(t,x)\in C^\infty$ with values in $\mathcal{L}_N$, such that $|C(t,x)|\neq 0$ and

$$P(t,\mathsf{A}(x))C(t,x)=P(t,\mathsf{B}(x)), \tag{3.8}$$

where $\pi_kB_j(x)\equiv B_j(x)\pi_k\equiv 0$ when $j\ge k$. At the points x_0 where $A_j(x_0)$ satisfies (3.6), $\forall j$, i.e. $\pi_kA_j(x_0)=0$ when $k\ge j$, we obtain that $B_k(x_0)=0$, $\forall k$. When $P(t,\mathsf{A})$ is real (matrix) valued, we may take $C(t,x)$ and $B_j(x)$ real (matrix) valued.

We also obtain the following generalization of the division theorem.

THEOREM 3.6. *Let $F(t,x)$ satisfy the hypothesis in Theorem 3.3. If $G(t,x)$ is a C^∞ function in a neighborhood of $(0,0)$ with values in $\mathcal{L}_N$, then we can write*

$$G(t,x) = Q(t,x)F(t,x) + \sum_{j=0}^{n-1} t^j R_j(x) \tag{3.9}$$

near $(0,0)$. Here $Q(t,x)$ and $R_j(x)$ are C^∞ functions with values in $\mathcal{L}_N$, satisfying $R_j(x)\pi_k \equiv 0$ when $j \geq k$, for the projections π_k in Theorem 3.3. If condition (2.13) also is satisfied, then π_k is the orthogonal projection on $E_k^\perp \bigcap E_{k-1}$ for $0 \leq k \leq n$, where $E_{-1} = \mathbf{C}^N$ and $E_k = \bigcap_{0\leq j\leq k} \operatorname{Ker} \partial_t^j F(0,0)$, $k \geq 0$.

PROOF: By Theorem 3.3, we may assume that

$$F(t,x) = \sum_{j=0}^{n} t^j \pi_j + \sum_{j=0}^{n-1} t^j A_j(x) = P(t, \mathsf{A}(x)),$$

where $A_j(x)\pi_k \equiv 0$ when $j \geq k$. Since it is no restriction to assume $G(t,x) \in C_0^\infty$, the statement follows from Proposition 2.1 and Remark 2.2, with $Q(t,x) = Q(t, \mathsf{A}(x), G(\cdot,x))$ and $R_j(x) = R_j(\mathsf{A}(x), G(\cdot,x))$ in (2.7). When $F(t,x)$ satisfies (2.13) also, we find that π_k is the orthogonal projection on $E_k^\perp \bigcap E_{k-1}$.

4. RIGHT PREPARATION

In Theorems 2.5 and 3.3, we have only done left preparation of matrix valued functions. By taking transposes we also obtain the corresponding results for right preparation. We first examine what condition we get on F, when (2.13) holds for F^*. Let $F(t)$ be a C^∞ function on $\mathbf{R}$ with values in $\mathcal{L}_N$, put $E_{-1}^* = \mathbf{C}^N$, and

$$E_k^* = \bigcap_{0\leq j\leq k} \operatorname{Ker} \partial_t^j F^*(0), \quad k \geq 0. \tag{4.1}$$

Let F_k be the mapping

$$F_k : \quad \mathbf{C}^N \ni w \longmapsto \partial_t^k F(0) w \qquad (\operatorname{mod} I_{k-1}) \quad k \geq 0, \tag{4.2}$$

where $I_{-1} = \{ 0 \}$, and

$$I_k = \bigoplus_{0\leq j\leq k} \operatorname{Im} \partial_t^j F(0), \quad k \geq 0.$$

PROPOSITION 4.1. *The condition*

$$\mathbf{C}^N = \bigoplus_{k=0}^{n} \operatorname{Im} \partial_t^k F^*(0)\Big|_{E_{k-1}^*} \tag{4.3}$$

is equivalent to

$$\{ 0 \} = \bigcap_{0\leq k\leq n} \operatorname{Ker} F_k, \tag{4.4}$$

and implies

$$\mathbf{C}^N = \bigoplus_{0\leq k\leq n} \operatorname{Im} \partial_t^k F(0). \tag{4.5}$$

We find that condition (4.4) and the spaces $I_k = \bigoplus_{0\le j\le k} \operatorname{Im} \partial_t^j F(0)$, $0 \le k \le n$, are invariant under right multiplication of F by elliptic systems.

PROOF: We have by duality that

$$I_k = \bigoplus_{0\le j\le k} \operatorname{Im} \partial_t^j F(0) = \Big(\bigcap_{0\le j\le k} \operatorname{Ker} \partial_t^j F^*(0) \Big)^{\perp} = (E_k^*)^{\perp}. \tag{4.6}$$

Let π_k be the orthogonal projection on $I_k \bigcap I_{k-1}^{\perp} = (E_k^*)^{\perp} \bigcap E_{k-1}^*$, then we find $\operatorname{Ker} F_k = \operatorname{Ker} \pi_k \partial_t^k F(0)$ and

$$\operatorname{Im} \partial_t^k F^*(0)\Big|_{E_{k-1}^*} = \operatorname{Im} \partial_t^k F^*(0)\pi_k = \big(\operatorname{Ker} \pi_k \partial_t^k F(0)\big)^{\perp}. \tag{4.7}$$

By Proposition 2.3, condition (4.3) is invariant under multiplication of F by invertible systems from right, and it is equivalent to (4.4) by (4.7). We also obtain from Proposition 2.3 that the spaces $E_k^* = I_k^{\perp}$ are invariant under right multiplication of F by invertible systems. Since condition (4.3) implies $E_n^* = \{0\}$ by Proposition 2.3, we obtain (4.5). ∎

Now we obtain from Theorems 2.5 and 3.3 the following result.

THEOREM 4.2. *Let $F(t,x)$ be a C^∞ function of (t,x) in a neighborhood of the origin of $\mathbf{R} \times \mathbf{R}^d$ with values in $\mathcal{L}_N$ satisfying (3.4). Then we may factor*

$$F(t,x) = \Big(\sum_{j=0}^{n} t^j \pi_j + \sum_{j=0}^{n-1} t^j A_j(x) \Big) C(t,x) = P(t, \mathsf{A}(x)) C(t,x) \tag{4.8}$$

near (0,0), where π_j is orthogonal projection in $\mathbf{C}^N$, $0 \le j \le n$, such that $\pi_j \pi_k = \delta_{jk}\pi_k$ and $\sum_{j=0}^{n} \pi_j = \operatorname{Id}_N$. Here $C(t,x)$ and $A_j(x)$ are C^∞ functions with values in $\mathcal{L}_N$, satisfying $|C(0,0)| \ne 0$, $\pi_k A_j(x) \equiv 0$ when $j \ge k$, and

$$A_j(0) = \sum_{i>j>k} \pi_i A_j(0) \pi_k, \tag{4.9}$$

which implies $A_0(0) = 0$. The projections π_k and matrices $A_j(0)$ are uniquely determined by condition (4.9), and $m = \sum_j j \cdot \operatorname{Rank} \pi_j$ in (3.4). If also condition (4.4) is satisfied, we find that $A_j(0) = 0$, $0 \le j < n$, and π_k is the orthogonal projection on $I_k \bigcap I_{k-1}^{\perp}$ for $0 \le k \le n$, where $I_{-1} = \{0\}$, $I_k = \bigoplus_{0\le j\le k} \operatorname{Im} \partial_t^j F(0,0)$. If F is real (matrix) valued, we may choose C, π_k and A_j real (matrix) valued.

It is clear that condition (3.4) is necessary for the preparation (4.8), and condition (4.4) is necessary when $\mathsf{A}(0) \equiv 0$. We also obtain the following version of the division theorem from Theorem 3.6 by duality.

THEOREM 4.3. *Let $F(t,x)$ satisfy the hypothesis in Theorem 4.2. If $G(t,x)$ is a C^∞ function in a neighborhood of $(0,0)$ with values in $\mathcal{L}_N$, then we can write*

$$G(t,x) = F(t,x)Q(t,x) + \sum_{j=0}^{n-1} t^j R_j(x) \tag{4.10}$$

near $(0,0)$. Here $Q(t,x)$ and $R_j(x)$ are C^∞ functions with values in $\mathcal{L}_N$, satisfying $\pi_k R_j(x) \equiv 0$ when $j \ge k$, for the projections π_k in Theorem 4.2.

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