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Rigorous Resonant 1-d Nonlinear Geometric Optics

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§1. One phase or simple nonlinear waves.

Consider an k×k system of strictly hyperbolic equations

(1) $$\partial_t u + \sum A_j(t,x,u)\partial u/\partial x_j = f(t,x,u)$$

where $x \in \mathbb{R}^d$, $u=(u_1(t,x),..,u_k(t,x))$ is $\mathbb{C}^k$ valued and the coefficients A_j, f are smooth functions of their arguments. The system is **semilinear** if the A_j do not depend on u and linear if in addition f does not depend on u.

We consider nonlinear generalizations of the familiar asymptotic solutions of linear geometric optics. These have the form

(2) $$u^\varepsilon \sim e^{i\varphi(t,x)/\varepsilon}[a_0(t,x) + \varepsilon a_1(t,x) + \ldots].$$

Here the phase φ is real valued smooth solutions of the eikonal equation with $d\varphi$ nowhere zero.

In the linear theory, the construction of one phase solutions is in three steps. Equations for the amplitudes a_j are derived by plugging the expansion into the equation and setting the coefficients of powers of ε equal to zero. Solving the resulting equations and applying Borel's theorem produces a family v^ε asymptotic to the given series. It follows that $Lv^\varepsilon \sim \varepsilon^{-\infty}$. Standard hyperbolic existence and regularity results then show that there is a family of exact solutions to $Lu^\varepsilon=0$ with $u^\varepsilon \sim v^\varepsilon$.

When nonlinear problems are considered, expressions like $f(u^\varepsilon)$

with u^ε as in (1) must be considered and this immediately introduces terms of the form $e^{in\varphi/\varepsilon}$ with integer n. The example $f(u)=\bar{u}$ shows that negative n occur. This creation of overtones shows that nonlinear problems do not usually have monochromatic solutions as in (1) and one is lead to the more general form

$$(3) \qquad u^\varepsilon \sim \sum_{n\in\mathbb{N}} e^{in\varphi(t,x)/\varepsilon}[a_{0,n}(t,x) + \varepsilon a_{1,n}(t,x) + \dots].$$

$$= \sum_{m\in\mathbb{N}} U_m(t,x,\varphi/\varepsilon)\,\varepsilon^m$$

where we have introduced the profiles

$$(4) \qquad U_m(t,x,\theta) \equiv \sum a_{m,n}(t,x)e^{in\theta}$$

which are periodic in θ. Single phase expansions of this form describe the phenomena of creation and interaction of overtones. Their analysis rests on the theory of stratified solutions which are conormal with respect to the foliation by the level surfaces of the phase φ (see [JR3, JR5, G]). For quasilinear problems the first profile is independent of θ and so is a background state $u_0(t,x)=U_0(t,x,\theta)$. The asymptotic solutions are perturbations of amplitude ε and wavelength ε about this state.

In both semilinear and quasilinear cases there is a background linear operator

$$L(t,x,D) = \partial_t + \sum A_j(t,x,u_0(t,x))\partial/\partial x_j$$

and associated eikonal equation ·

$$(4) \qquad \det(L(t,x,d\varphi) = 0.$$

The utility of such results lies in the fact that the behavior of a family of solutions of solutions of the original equation with singular dependence on the parameter ε is described by the leading profile which is a solution of a system of integrodifferential equations which is not a great deal more complicated than the original equation. One can search for explicit solutions, perform analysis of the equation for the profile, or seek approximate solutions numerically, all without encountering the difficulty of structures of size ε.

§2. Resonance.

For linear problems, when two distinct phases are present one merely adds the two waves. For nonlinear problems there can be nontrivial interaction between the waves and in particular new phases may appear. Such interactions go by the name resonance.

Suppose that waves coexist with phases $\varphi_j(t,x)$, $j=1,..N$ (and background state u_0 in the quasilinear case). Considering nonlinear functions introduces oscillations of the form $e^{i\alpha.\varphi/\varepsilon}$ with $\alpha\in\mathbb{N}^N$ and $\alpha.\varphi\equiv\sum\alpha_j\varphi_j$. We are led to consider expressions of the form $L^{-1}(e^{i\alpha.\varphi/\varepsilon})$. In the unlikely event that $\alpha.\varphi$ satisfies the eikonal equation, such an inversion is described by linear geometric optics and is of the form (1) with φ replaced by $\alpha.\varphi$. In this way the new phase $\alpha.\varphi$ appears in the description of the solution. This is one aspect of resonance. The other is that the the waves with distinct phases interact.

§3. Resonance and transversality.

When $\alpha.\varphi$ does not satisfy the eikonal equation, $L^{-1}(e^{i\alpha.\varphi/\varepsilon})$ leads to oscillatory integrals with phases which are stationary only at points of the set

(5) $$\mathcal{S} \equiv \Big\{(t,x)\ :\ \det(t,x,L(t,x,d(\alpha.\varphi)=0\Big\}.$$

To see the importance of the size of the set S, we consider a family of 3x3 semilinear equations,

$$X_1u_1 = 0$$

(6) $$X_2u_2 = 0 \qquad\qquad X_j \equiv \partial_t + \lambda_j(t,x)\partial_j$$

$$X_3u_3 = u_1u_2 .$$

For j=1,2 take

(7) $$u_j = a_j(t,x)e^{i\varphi_j(t,x)/\varepsilon} \quad \text{with} \quad X_j\varphi_j = X_ja_j = 0.$$

Then if $u_3=0$ for $t<0$ we have

(8) $$u_3 = X_3^{-1}(a_1a_2e^{i(\varphi_1+\varphi_2)/\varepsilon}).$$

The stationary points of this oscillatory integral are the set of points

(9) $$\Gamma \equiv \left\{(t,x)\in\mathbb{R}^2 \ : \ X_3(\varphi_1+\varphi_2)=0 \right\}.$$

It is not difficult to verify that if Γ is a set of $\mathbb{R}^2$-Lebesgue measure equal to zero (≡weak transversality) then u_3 is o(1) in $L^p_{loc}(\mathbb{R}^2)$ for all $p<\infty$. Note however that if Γ contains an integral curve of X_3 then $u_3=X_3^{-1}(a_1a_2)e^{i(\varphi_1+\varphi_2)/\varepsilon}$ along that curve so in particular is not o(1) in L^∞.

It is also not hard to show that if for every integral curve σ of X_3, the intersection $\sigma\cap\Gamma$ has one dimensional Lebesgue measure zero (≡strong transversality), then $u_3=o(1)$ in $L^\infty_{loc}(\mathbb{R}^2)$.

In the extreme opposite case where $\Gamma=\mathbb{R}^2$, then $u_3=X_3^{-1}(a_1a_2)e^{i(\varphi_1+\varphi_2)/\varepsilon}$ is an oscillatory solution with the new phase $\varphi_1+\varphi_2$. The three phases $\varphi_1,\varphi_2,\varphi_1+\varphi_2$ are resonant.

examples. For the first two

$$X_1=\partial_t+\partial_x \qquad X_2=\partial_t-\partial_x \qquad X_3=\partial_t \ .$$

Then the phases

$$\varphi_1=x-t \qquad \varphi_2=x+t \qquad \varphi_1+\varphi_2=x$$

are resonant. In fact these are, up scalar multiples, the only resonant phases for these fields. For example the phases

$$\varphi_1=(x-t)^2 \qquad \varphi_2=(x+t)^2$$

satisfy the strong transversality condition.

The fields

$$X_1=\partial_t+x\partial_x \qquad X_2=\partial_t-x\partial_x \qquad X_3=\partial_t+a\partial_x \quad a\in\mathbb{R}\backslash 0$$

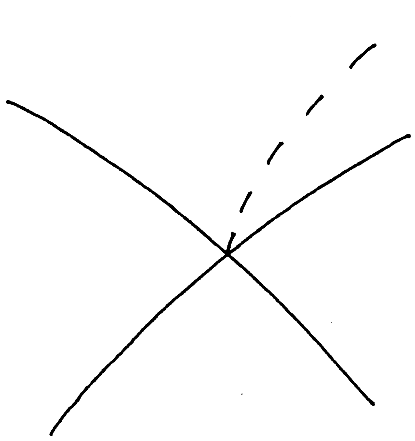
have no resonant phases. That is the overdetermined system

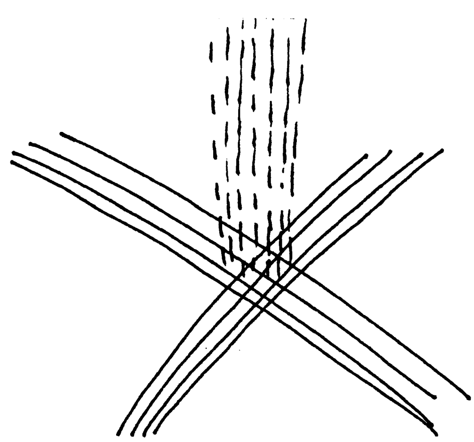
$$X_1\varphi_1=0 \qquad X_2\varphi_2=0 \qquad X_3(\varphi_1+\varphi_2)=0$$

has no nontrivial solutions.

Note that the study of the existence of resonances leads to the study of an overdetermined system of linear partial differential equations.

Note also that nonlinear interaction of singularities, with the creation of singularities as in the left hand figure is generic.





On the other hand, the production of oscillations as in the right hand figure is nongeneric. For generic systems it will not occur. Even for systems for which it is possible, generic phases will not lead to creation.

§4. Form of profiles for interaction.

Next consider system (6) with incoming states $u_j \sim U_j(t,x,\varphi_j/\varepsilon)$ $j=1,2$, with U_j (not to be confused with (4)) periodic in the last variable. Interaction yields $n\varphi_1+m\varphi_2$ $n,m\in\mathbb{Z}$ as candidate phases with corresponding sets of stationarity $S_{n,m}$. There are an infinity of distinct phases to be examined. The sets of stationarity almost never satisfy uniformity conditions. This renders somewhat surprising the the fact that asymptotic expansions can be justified. It renders reasonable the fact that the errors are rarely better than o(1). Note that nondegenerate stationary points yield $O(\sqrt{\varepsilon})$ in (8) while phases without singular points yield $O(\varepsilon^{-\infty})$. However, there are examples without stationary points for which the errors in the expansion are $>c\varepsilon^{\rho}$ with $\rho>0$ as small as one wants [JMR1].

Next consider the system (6) with a third incoming wave $u_3=a_3(t,x)e^{i\varphi_3/\varepsilon}$ in $t<0$. Several possibilities exist.

i. There is no resonance, and the behavior is essentially linear.

ii. There is a resonance $n\varphi_1+m\varphi_2=\alpha\varphi_3$ with $\alpha\notin\mathbb{Q}$. Then u_3 will be quasi periodic in φ_3.

iii. There is a resonance $n\varphi_1+m\varphi_2=\psi_3$ with $X_3\psi_3=0$ and ψ_3 not proportional to φ_3. Then u_3 will oscillate with two phases, $u_3\sim U(t,x,\varphi_3/\varepsilon,\psi/\varepsilon)$.

iv. There is a resonance $n\varphi_1+m\varphi_2=\varphi_3+c$. Then u_3 will oscillate with the phases φ_3, φ_3+c, $u_3\sim U(t,x,\varphi_3/\varepsilon,1/\varepsilon)$.

These considerations lead us admit almost periodic profiles, allow more than one phase for each mode, and, add the "constant phase" 1 as supplementary phase variable.

§5. Asymptotics for the semilinear Cauchy problem.

Consider the semilinear Cauchy problem in one space dimension with initial data of the form

$$(10) \qquad u^{\varepsilon}(0,x) = h^{\varepsilon}(x) = H(x,\psi(x)/\varepsilon) + o(1) \text{ in } L^{\infty}(\mathbb{R}),$$

where $H(x,.)$ is a continuous function of x with values in the almost periodic functions on $\mathbb{R}$.

Suppose that ψ is real valued with $d\psi$ nowhere zero and let $\varphi_1,\ldots,\varphi_k$ be the solutions of the eikonal equation defined on a bounded open set Ω which is deterministic in the sense that $\Omega\cap\{t<t_1\}$ is contained in the domain of determinacy of $\Omega\cap\{t<t_2\}$ for all $t_2<t_1$.

Definition. We say that the weak transversality hypothesis is satisfied if $\alpha\in\mathbb{R}^k$ and $\det(L(t,x,d(\alpha.\varphi)))$ is not identically zero in Ω, then the set S in (5) has $\mathbb{R}^2$ measure zero. The strong transversality hypothesis is satisfied if $\alpha\in\mathbb{R}^k$ and $X_j(\alpha.\varphi)$ is not identically zero then on every j-characteristic curve σ, the set of points on σ such that $X_j(\alpha.\varphi)=0$ is a set of one dimensional measure zero. Here $X_j=\partial_t+\lambda(t,x)\partial_x$ is the j^{th} characteristic direction.

Unique continuation for C^{ω} functions shows that if the equation and initial phase are real analytic, then the weak transversality hypothesis is automatically satisfied.

It is not hard to show that for this Cauchy problem the possibility described in iv. above does not occur and that it is not necessary to introduce the extra phase 1.

Theorem. If the weak transversality hypothesis holds then, there is a $T>0$ and an $\varepsilon_1>0$ such that for all $0<\varepsilon<\varepsilon_1$ the semilinear initial value problem (1),(10) has a bounded solution on $\Omega\cap\{0\le t<T\}$ the bound being uniform in ε. In addition, there is a profile $U(t,x,\theta_1,\theta_2,..,\theta_k)$ which is a continuous function of t,x in Ω with values in the almost periodic functions on $\mathbb{R}^k$ such that

$$(11)\qquad u^{\varepsilon}(t,x) = U(t,x,\varphi_1/\varepsilon,..,\varphi_k/\varepsilon) + o(1) \quad L^p(\Omega)\ \forall p<\infty.$$

If the strong transversality hypothesis is satisfied, then one can take $p=\infty$

§6. Averaging operators and the determination of the profile.

A smooth change of dependent variable casts the equation (1) in the diagonal form

$$(12)\qquad (\partial_t+\lambda_j(t,x)\partial_x)u_j = f_j(t,x,u) \quad j=1,..,k.$$

For all t,x, the profile $U(t,x,.)$ has its spectrum contained in the set of α such that $\alpha.\varphi$ satisfies the eikonal equation. Even more is true. Introduce the averaging operators E_j on almost periodic functions of $\theta\in\mathbb{R}^k$ by

$$(13)\qquad E_j(e^{i\alpha.\theta}) = \begin{cases} e^{i\alpha.\theta} & \text{if } X_j(e^{i\alpha.\varphi}) = 0 \\ 0 & \text{otherwise} \end{cases} .$$

The set of α such that $X_j(e^{i\alpha.\varphi})=0$ is a linear space. Denote by Ψ its annihilator in $\mathbb{R}^k$. Let $d\mu$ denote Lebesgue measure on Ψ. Then, for any bounded open set $\mathcal{O}\subset\Psi$ of measure one one has

$$(14) \qquad (E_jU)(\theta) = \lim_{r\to\infty} r^{-\dim(\Psi)} \int_{r\mathcal{O}} U \, d\mu$$

It follows that E_j maps the almost periodic functions to themselves and is continuous in $L^\infty(\mathbb{R}^k)$ norm and satisfies $E_j^2=E_j$.

One then has the following polarization for the profile,

$$(15) \qquad U = (U_1,..,U_k) \quad \text{and,} \quad E_jU_j = U_j \ , \quad j=1,..,k.$$

(Do not confuse with (4).) The profile for the solution of the Cauchy problem in §5 must satisfy in addition the initial condition

$$(16) \qquad U(0,x,\theta) = H(x,\theta)$$

so that (10) will be satisfied. The prescription of U is then completed by the integrodifferential equation,

$$(17) \qquad (\partial_t+\lambda_j(t,x)\partial_x)U_j = E_j(f_j(t,x,U(t,x,\theta))) \quad j=1,..,k,$$

where E_j on the right acts in the θ variables.

§7. Lifetimes and an application.

The equations for u^ε and for the profile U are nonlinear and the solutions may blowup. If we assume the strong transversality hypothesis, the blowup times are related as follows [JMR2].

If the profile equations (15),(16),(17) have a solution U continuous on $\Omega\cap\{0\le t\le T\}$ with values in the almost periodic functions
then there is an $\varepsilon_2\in]0,\varepsilon_1]$ such that for $0<\varepsilon<\varepsilon_2$ the solution u^ε exists on $\Omega\cap\{t\le T\}$, the family u^ε is bounded in $L^\infty(\Omega\cap\{t\le T\})$ and the relation (11) holds. Conversely if the u^ε exist and are bounded

uniformly $\Omega\cap\{0\le t\le T\}$, then the profile equation is solvable up to time T and (11) holds in $\Omega\cap\{t\le T\}$.

The quasilinear version of this lifetime result is also valid. In that case blowup means blowup of the first derivatives.

There is a nice application of this to gas dynamics whose history is as follows. Majda, Rosales and Schonbek [MRS] studied the behavior of the profile U in the case of gas dynamics by approximately solving the system (15,16,17) on a computer.

The oscillatory initial data for the solution u^ε are of the form $u_0(0,x)+\varepsilon H(x,\psi(x)/\varepsilon)$ so that sup $|\partial_x u^\varepsilon(0,x)|$ is independent of ε. This suggests that shocks should form in time independent of ε.

The numerical experiments indicated long time existence of profiles. When derivatives started to grow in one component of U instead of seeing the waves break the steepened wave would recede and another component would steepen and so on. They conjectured that the profile equation had nontrivial global smooth solutions, and shortly thereafter Pego [P] showed that there were in fact explicit smooth solutions periodic in time.

Our results on lifetimes applied to initial data with profiles given by the Pego profiles show that for any T>0, the solutions of the gas dynamics equations exist and u^ε and $\nabla_{t,x}u^\varepsilon$ are bounded in $L^\infty([0,T]\times\mathbb{R})$ uniformly in ε. In particular, the time of shock formation is indefinitely postponed by the resonant interaction of high frequency wave trains. This conjecture of [MRS] is therefore rigorously established.

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