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GLOBAL ANALYTIC AND GEVREY SURJECTIVITY OF THE MIZOHATA  
OPERATOR  $D_2 + ix_2^{2k}D_1$  (\*)

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Global analytic surjectivity of all linear constant coefficient partial differential operators on  $\mathbb{R}^2$  has been proved by E. De Giorgi and L. Cattabriga [5], where counterexamples for the case of operators on  $\mathbb{R}^3$  are also indicated (see also E. De Giorgi [4] and for other results for operators on open sets of  $\mathbb{R}^n$ : T.Kawai [10] and L.Hörmander [8]). Global Gevrey surjectivity of all linear constant coefficient partial differential operators on  $\mathbb{R}^2$  was subsequently proved by L. Cattabriga [3] in the case of Gevrey spaces with rational index (see also L. Cattabriga [2], G. Zampieri [15], R.W. Braun-R. Meise-D. Vogt [1] for the case of operators on  $\mathbb{R}^n$ ). As for the case of linear operators with variable coefficients results on global Gevrey surjectivity are given by L.Ehrenpreis [6] and in the analytic case by T.Kawai [11].

Here we consider, as a simple example of an operator on  $\mathbb{R}^2$  with variable coefficients, the Mizohata operator  $D_2 + ix_2^{2k}D_1$ , where  $k$  is a positive integer and  $D_j = -i\partial_j$ ,  $j = 1, 2$ , and prove the following

**Theorem.** Let  $\mathcal{E}^{\{s\}}(\mathbb{R}^n)$ ,  $s \geq 1$ , be the space of all  $C^\infty$  functions  $f$  on  $\mathbb{R}^n$  such that for every compact subset  $K$  of  $\mathbb{R}^n$  there exists a constant  $A > 0$  such that

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$$\sup_{x \in K} \sup_{\alpha \in \mathbb{Z}_+^n} A^{-|\alpha|} \alpha!^{-s} |\partial_x^\alpha f(x)| < +\infty.$$

Then for every  $s \geq 1$

$$i) (D_2 + ix_2^{2k} D_1) \mathcal{E}^{\{s\}}(\mathbb{R}^2) = \mathcal{E}^{\{s\}}(\mathbb{R}^2),$$

$$ii) (D_2 + ix_2^{2k} D_1) \mathcal{E}^{\{s\}}(\mathbb{R}^3) \subsetneq \mathcal{E}^{\{s\}}(\mathbb{R}^3).$$

**Proof.** Let  $f \in \mathcal{E}^{\{s\}}(\mathbb{R}^2)$ ,  $s \geq 1$ , and let  $\sigma > \max\{s, 1\}$ . Since  $f \in \mathcal{E}^{\{\sigma\}}(\mathbb{R}^2)$ , from a result by H. Komatsu [12] it follows that there exists  $v_1 \in \mathcal{E}^{\{\sigma\}}(\mathbb{R}^2)$  such that

$$(1) \quad \begin{cases} v_1(x_1, 0) = 0 \\ \partial_x^\gamma [(D_2 + ix_2^{2k} D_1)v_1 - f](x_1, 0) = 0 \quad \forall \gamma \in \mathbb{Z}_+^2, x_1 \in \mathbb{R}. \end{cases}$$

Let

$$(2) \quad h(x_1, x_2) = f(x_1, x_2) - (D_2 + ix_2^{2k} D_1)v_1(x_1, x_2),$$

and with the new variables in  $\mathbb{R}^2$

$$\begin{cases} y_1 = x_1 \\ y_2 = x_2^{2k+1}/(2k+1), \end{cases}$$

set

$$g(y_1, y_2) = \begin{cases} h(y_1, [(2k+1)y_2]^{1/(2k+1)}) [(2k+1)y_2]^{-2k/(2k+1)} & \text{for } y_2 \neq 0 \\ 0 & \text{for } y_2 = 0. \end{cases}$$

In view of (1),  $g \in C^\infty(\mathbb{R}^2)$ . Hence, by the global  $C^\infty(\mathbb{R}^2)$ -surjectivity of all linear partial differential operators with constant coefficients, there exists  $w \in C^\infty(\mathbb{R}^2)$  such that

$$(D_2 + i D_1)w(y_1, y_2) = g(y_1, y_2) \quad \text{on } \mathbb{R}^2.$$

Letting

$$v_2(x_1, x_2) = w(x_1, x_2^{2k+1}/(2k+1)), \quad (x_1, x_2) \in \mathbb{R}^2,$$

it follows that

$$v_2 \in C^\infty(\mathbb{R}^2)$$

and

$$(3) \quad (D_2 + ix_2^{2k} D_1)v_2(x_1, x_2) = x_2^{2k} g(x_1, x_2^{2k+1}/(2k+1)) = h(x_1, x_2), (x_1, x_2) \in \mathbb{R}^2.$$

Thus from (2) and (3)

$$(D_2 + ix_2^{2k} D_1)(v_1 + v_2) = f \quad \text{on } \mathbb{R}^2,$$

where

$$v = v_1 + v_2 \in C^\infty(\mathbb{R}^2).$$

Recalling now that the Mizohata operator  $(D_2 + ix_2^{2k} D_1)$  is (analytic and)  $s$ -Gevrey hypoelliptic on  $\mathbb{R}^2$  for every  $s \geq 1$  (see for example S.Mizohata [13], L.Rodino [14], and also L.Hörmander [9]), we conclude that  $v$  is in fact in the same space  $\mathcal{E}^{\{s\}}(\mathbb{R}^2)$  as  $f$  is. This proves part i) of the theorem.

To prove part ii), let  $f \in \mathcal{E}^{\{s\}}(\mathbb{R}^3)$ ,  $s \geq 1$ , and define

$$g(x_1, x_2, x_3) = f(x_1, x_2^{2k+1}/(2k+1), x_3) x_2^{2k}, \quad (x_1, x_2, x_3) \in \mathbb{R}^3.$$

Since  $g \in \mathcal{E}^{\{s\}}(\mathbb{R}^3)$ , if  $(D_2 + ix_2^{2k} D_1) \mathcal{E}^{\{s\}}(\mathbb{R}^3) = \mathcal{E}^{\{s\}}(\mathbb{R}^3)$ , there exists

$w \in \mathcal{E}^{\{s\}}(\mathbb{R}^3)$  such that  $(D_2 + ix_2^{2k} D_1)w(x_1, x_2, x_3) = g(x_1, x_2, x_3)$  on  $\mathbb{R}^3$ .

Hence  $v(x_1, x_2, x_3) = w(x_1, [(2k+1)x_2]^{1/(2k+1)}, x_3)$  would be a  $\mathcal{D}'(\mathbb{R}^3)$  solution of  $(D_2 + iD_1)v = f$ . Since the operator  $D_2 + iD_1$  is partially hypoelliptic in  $\mathbb{R}^3$  with respect to the  $(x_1, x_2)$  variables, from the corollary of Theorem 4.1.2 of [7], it follows that  $v \in \mathcal{E}^{\{s\}}(\mathbb{R}^3)$ . So the operator  $D_2 + iD_1$  on  $\mathbb{R}^3$  would be  $s$ -surjective, what is false as it has been proven in [3] and [4].

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