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ON THE LAX & PHILLIPS SCATTERING THEORY FOR TRANSPORT EQUATION

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Let Ω be a bounded star shaped domain in $\mathbb{R}^n$ and ρ the radius of a ball B_ρ around the origin which contains all the points of Ω . E_+ the free forward and E_- the free backward point sets are defined as follow:

$$E_+ = \{ (x,v) \in \mathbb{R}^n \times V \mid x.v \geqq \rho \}$$

$$E_- = \{ (x,v) \in \mathbb{R}^n \times V \mid x.v \leqq -\rho \}$$

Corresponding to the subsets $E_\pm$ one defines the incoming subspace D_- and outgoing subspace D_+ by:

$$D_\pm = \{ f \in X \mid \operatorname{supp} f \subset E_\pm \}$$

In [1] Lax & Phillips have taken the unit sphere in $\mathbb{R}^n$ as the velocity space V and $X = L^2(\mathbb{R}^n \times V)$. They have shown that for $U_o(t)$ the one-parameter unitary group defined by $U_o(t)f(x,v) = f(x-vt,v)$ one has

<u>THEOREM 1.</u> The subspaces D_+ and D_- satisfy the following properties:

i)$_+$ $\quad U_o(t)D_+ \subset D_+ \quad$ for $\quad t \geqq o$

i)$_-$ $\quad U_o(t)D_- \subset D_- \quad$ for $\quad t \leqq o$

ii) $\quad \bigcap_{t \in \mathbb{R}^n} U_o(t)D_\pm = \{0\}$

iii) $\quad \bigcup_{t \in \mathbb{R}^n} U_o(t)D_\pm$ is dense in X.

This theorem can be easily generalized to the case when V is an annulus contained in the unit ball of $\mathbb{R}^n$

$$V = \{ v \in \mathbb{R}^n \mid 0 < v_m \leq |v| \leq 1 \}$$

and $X = L^p(\mathbb{R}^n \times V)$ for $1 \leq p < \infty$.

$U_o(t)$ is a strongly continuous positive group generated by free collision transport operator

$$T_o f = -v.\nabla_x f$$

in any $L^p(\mathbb{R}^n \times V)$. For any λ in $\mathbb{C}$ the only function which verifies $T_o\phi = \lambda\phi$ is

$$\phi(x,v) = g(x_\perp, v)\exp\{-\lambda x.v/|v|^2\}$$

where $x_\perp = x - |v|^{-2}(x,v)v$. Hence for any g in $L^\infty(\mathbb{R}^n \times V)$, belongs also to $L^\infty(\mathbb{R}^n \times V)$ if and only if $\lambda = i\beta$ for any real β .

This shows that the nature of the spectrum of T_o depends on the exponent p in L^p . In fact if we denote by $\Sigma(T_o)$ the spectrum of T_o, using $\Sigma_p(T_o)$, $\Sigma_c(T_o)$ and $\Sigma_r(T_o)$ to denote respectively the point spectrum,continuous spectrum and residual spectrum of T_o, we can prove the following peculiar result:

THEOREM 2. a) $\Sigma(T_o) = \Sigma_r(T_o) = i\,\mathbb{R}$ in $L^1(\mathbb{R}^n \times V)$.

b) $\Sigma(T_o) = \Sigma_c(T_o) = i\,\mathbb{R}$ in $L^2(\mathbb{R}^n \times V)$.

c) $\Sigma(T_o) = \Sigma_p(T_o) = i\,\mathbb{R}$ in $L^\infty(\mathbb{R}^n \times V)$.

One of our major aim in this paper is to show when the Lax & Phillips representation theorem (Theorem 1.) is valid in $L^1(\mathbb{R}^n \times V)$ for collision dynamics U(t) the one parameter group generated by linearized Boltzmann operator

$$T f = -v.\nabla_x f - \sigma_a(x,v)f + \int_V k(x,v',v)f(x,v')dv'$$

where σ_a and k are two non-negative measurable functions on $R^n \times V$ and $R^n \times V \times V$ respectively. We define the production cross section σ_p by:

$$\sigma_p(x,v) = \int_V k(x,v,v')dv'$$

and we suppose that the transport system

$$\frac{\partial u}{\partial t} = T\,u \quad , \quad u(x,v,0) = u_o(x,v) \in L^1(\mathbb{R}^n \times V)$$

is admissible . i.e:

i) σ_a and σ_p belong to $L^\infty_+(\mathbb{R}^n \times V)$

ii) There is a compact set K in Ω so that σ_a and σ_p vanish if $x \notin K$.

In the Lax & Phillips representation theorem the crucial point is the density property iii) . This property is closely related to the local decay property of the dynamics (see [2]). i.e For any compact subset K of R^n and any function f in $L^1(\mathbb{R}^n \times V)$

(LD) $$\int_{K\times V} |U(t)f(x,v)|\,dxdv \to 0$$

as $|t| \to o$. It comes out that this last property is also intimately related with the spectral configuration of the infinitesimal generator T of U(t). In fact one can never get (LD) if $\sigma_p(T) \neq \emptyset$. The following theorem shows that this may happen to our case.

<u>THEOREM 3</u>. In $L^1(R^n \times V)$, $\Sigma(T) = i\,R\ \ \Sigma_p(T)$ where $\Sigma_p(T)$ is either empty or at most a finite set of isolated pointd lying in the strip $\Lambda = \{ z \in C \mid -c \leqq \text{Re}z \leqq c_2 \}$.

Sketch of the proof. Let us denote the operators A_1 and A_2 on $L^1(\mathbb{R}^n \times V)$ by:

$$[A_1 f](x,v) = -\sigma_a(x,v) f(x,v)$$

$$[A_2 f](x,v) = \int_V k(x,v',v) f(x,v') dv'$$

Put $A = A_1 + A_2$ and $T_1 = T_o + A_1$. Since A_1 and A_2 are bounded by $c_1 = \|\sigma_a\|_\infty$ and $c_2 = \|\sigma_p\|_\infty$ respectively. From the theory of semigroups one can deduce that $T = T_o + A$ generates an one-parameter group $U(t)$ and $\Sigma(T)$ lies in Λ with $c = c_1 + c_2$. Let $L_\lambda = (\lambda - T_1)^{-1} A_2$. By virtue of Dunford-Pettis theorem one can show that $\lambda \to L_\lambda^2$ is an analytic compact operator-valued function in C, and we have for $\operatorname{Re} \lambda \neq 0$, $\|L_\lambda^2\| \leq \|A_2\|^2 / |\operatorname{Re} \lambda|^2$. Hence L_λ^2 tends to zero as $|\operatorname{Re} \lambda| \to \infty$. Therefore 1 and -1 are not the eigenvalues for all operators L_λ^2. Thus by applying the analytic Fredholm Theorem $(I - L_\lambda^2)^{-1}$ exists, except at most a countable set of isolated points λ_k, where the function $\lambda \to (I - L_\lambda^2)^{-1}$ has a pole. From the two following algebric identities:

$$(I - L_\lambda)^{-1} = (I + L_\lambda)(I - L_\lambda^2)^{-1}$$
$$(\lambda - T)^{-1} = (I - L_\lambda)^{-1}(\lambda - T_1)^{-1}$$

it follows that for $\operatorname{Re} \lambda \neq 0$ any pole of $(I - L_\lambda^2)^{-1}$ is an eigenvalue of T. The finiteness of the number of these eigenvalues will be proved later.

Going back to the theorem 1. The proof of the assertions i) and ii) is a simple consequence of the following lemma.

LEMMA 4. For any $t \geq o$ $[t \leq o]$ and any $f \in D_+$ $[f \in D_-]$ one has

$$U(t)f = U_o(t)f \quad .$$

Theorem 3 shows that the assertion iii) of Theoreml for U(t) fails to be true in general case , but however we have:

THEOREM 5. The following assertions are equivalent

a) $\bigcup_{t\in R} U(t)D_\pm$ is dense in $L^1(R^n \times V)$

b) The local decay property (LD) holds .

c) The operator T admits neither eigenvalues on the complex plane nor resonances on the imaginary axis.

For implication b) ⟹ a) see [3] . In order to prove c) ⟹ b) we have to introduce the Lax & Phillips semigroup Z(t) for transport equation.

Let us define the projections $P_\pm$ on $L^1(\mathbb{R}^n \times V)$ by $P_\pm f = \chi_\pm^{\prime} f$ where $\chi_\pm$ are the characteristic functions of $E_\pm$ and $\chi_\pm^{\prime} = 1 - \chi_\pm$. We define the Lax & Phillips semigroup by:

$$Z(t) = P_+ U(t) P_-$$

Let us consider K a subspace of $L^1(\mathbb{R}^n \times V)$ consisting of functions f which are identically zero on $E_+ \cup E_-$.

THEOREM 6. The operators $\{ Z(t) \mid t \geq o \}$ map K into itself and form strongly continuous semigroup on K. Furthermore it is a differentiable and compact semigroup for sufficiently large t .

The eigenvalues of B the infinitesimal generator of Z(t) are called *resonances* and the compactness of Z(t) implies that the spectrum of B is constituted of pure resonances . Furthermore the differentiability of Z(t) implies that these resonances are lying in a logarithmic region of the form

$$\Lambda = \{ \lambda \in \mathbb{C} \mid \operatorname{Re}\lambda < a - b\log|\lambda| \}$$

where a is real and $b > o$. Thus the following theorem proves the finiteness of $\Sigma_p(T)$. This theorem is based on the fact that any eigenfunction of T vanishes out of Ω .

<u>THEOREM 7.</u> $\Sigma_p(T) \subset \Sigma(B)$.

Here the fundamental problem of the existence of such resonances arises In order to prove that $\Sigma(B) \neq \emptyset$ we will look to the interior transport problem which was posed by Jörgens [4] . He proved that in some circumstances the interior transport operator T^J admits eigenfunctions verifying the interior boundary condition:

$$\phi(x,v) = 0 \quad \text{for} \quad x \in \partial\Omega \quad \text{and} \quad n(x).v < 0$$

where $n(x)$ is the exterior normal to Ω at x . By an extension of these eigenfunctions to whole space we prove

<u>THEOREM 8.</u> $\Sigma(T^J) = \Sigma(B)$.

This extension shows that the asymptotic form of these eigenfunctions look like $\exp\{-\mu x.v/|v|^2\}$ when $n(x).v \geq 0$. According to Lax & Phillips terminology we will call them *generalized eigenfunctions*.

By an analysis based on a complex residues computation we prove an eigenfunction expansion for $Z(t)$ which is asymptotically valid for large t. i.e: By arranging the eigenvalues μ_j of B in decreasing order of their real parts and denote by P_j the projection into the j^{th} eigenspace and D_j^k the corresponding nilpotent operator of order k, one has

$$Z(t) \simeq \sum_{\mu_j \in \Sigma(B)} e^{\mu_j t}(P_j + \sum_k \frac{t^k}{k!} D_j^k)$$

The following version of the above formula was suggested by Melrose[5], for wave equation which is more coherent to our setting.

THEOREM 9. For any f in $L^1(\mathbb{R}^n \times V)$ there exist a sequence μ_j in $\mathbb{C}$ and generalized eigenfunctions $w_{j,k}$, $k = o,\ldots,m_{j-1}$ such that for any $n \in \mathbb{N}$ and ε, $o < \varepsilon < \mathrm{Re}\mu_n - \mathrm{Re}\mu_{n+1}$

$$\sup_{(x,v)\in \Omega\times V} \left| [U(t)f](x,v) - \sum_{j=1}^{n} e^{\mu_j t} \sum_{k=o}^{m_j-1} t^k w_{j,k}(x,v) \right| \leq c \left| e^{(\mu_n-\varepsilon)t} \right|$$

for sufficiently large t . The constant c depends only on n and ε .

This theorem yields the implication c) $\Rightarrow$ b) in theorem 5. We deduce also from compactness of Z(t) and the fact that $\{0\} \notin \Sigma_p(T)$ (see [6]) that a) $\Rightarrow$ c).

Finally we give a physically relvant situation in which the property b) of Theorem 5 occurs. This situation is presented by Hejtmanek [7] . He showed when the Dyson-Phillips expansion of U(t) is finite , which physically means that the system is of finite collisions then the spectrum of T does not exceed the imaginary axis . We can conclude under the above condition Lax & Phillips representation theorem is fully valid.

R E F E R E N C E S

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