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# Complete Mellin symbols and the conormal asymptotics in boundary value problems

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# Complete Mellin symbols and the conormal asymptotics in boundary value problems

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## 0. Introduction

It is well known that the solutions of large classes of boundary value problems (bvp's) or of equations on manifolds with singularities have an asymptotic expansion of the form

$$u(y,t) \sim \sum_{j=0}^{\infty} \sum_{k=0}^{m_j(y)} \zeta_{jk}(y)\, t^{p_j(y)} \log^k t \qquad (0.1)$$

as $t \to 0$, with exponents $p_j \in \mathbb{C}$. The variable $y$ runs tangent to the boundary or on a manifold $M$.

If $u \in C^\infty(M \times \mathbb{R}_+)$ admits an expansion like (0.1) we speak about a conormal asymptotics of the type

$$\mathcal{P} = (p_j, m_j)_{j \in \mathbb{Z}_+}, \tag{0.2}$$

$(p_j, m_j) \in \mathbb{C} \times \mathbb{Z}_+$. For $C^\infty$ regularity up to the boundary, e.g. in classical elliptic bvp's, (0.1) is the Taylor expansion, where $m_j = 0$ and $p_j = j$. On the other hand a general conormal asymptotics of the form (0.1) was observed for

(i) mixed bvp's, i.e. when the boundary conditions may have jumps,

(ii) bvp's for operators of the form

$$A(y, t, D_y, t^\gamma D_t), \ \gamma \leq 1, \tag{0.3}$$

(iii) operators on manifolds with singularities such as conical points, wedges, corners, and bvp's in domains with boundaries that have singularities,

(iv) classes of ordinary differential equations with degenerate symbols (such as of Fuchs type) and coefficients with conormal asymtptotics.

The list of problems can easily be enlarged, cf., for instance, the paper of Lee, Melrose [9]. In many cases one also studies asymptotic expansions of the form (0.1) as $t \to \infty$. The problem of characterizing the type of the asymptotics of the solutions is rather classical and of practical interest in many concrete situations, e.g. in numerical methods. A large bibliography was given in [8].

It became a typical approach to use the Mellin transform or Mellin operators in t direction involved in parametrix constructions. Let us mention e.g. results of Eskin [4] on the half axis for operators (0.3) with $\gamma = 0$, a higher-dimensional theory of the authors [13], [14], and the theory

of Melrose [10] for $\gamma = 1$.

Parametrix constructions for $\psi$DOs on a complete symbolic level in $C^\infty$ spaces with conormal asymptotics were obtained by the authors. Here we sketch some of the results. The details are given in [15], which is the first of a series of papers that employ systematically numerous classes of Mellin operators in the applications mentioned at the beginning.

For classical elliptic bvp's there is a complete symbolic calculus which yields parametrices in $C^\infty$ spaces up to the boundary, cf. [2], [12]. The basic difficulty of constructing a good analogue in the cases (i), (ii), (iii) is that we have to expect individual singularity types (0.2) with a complicated branching behaviour of the $p_j(y)$ and changing of multiplicities $m_j(y)$ under varying $y$. It is the aim of our theory to arrange the symbolic calculus and the negligible operators sensitive enough to reflect that interesting phenomenon and to obtain parametrices in classes of functions with conormal asymptotics.

## 1. Boundary symbolic calculus and Mellin operators

In this section we consider boundary symbols that appear in the analysis of pseudo-differential operators on manifolds with boundary, or locally on the half space $\mathbb{R}^n_+$. The algebra generated by the pseudo-differential action is the model of other variants that belong to the applications mentioned at the beginning. We may deal with symbols of order zero modulo order reducing operators. With $a(\xi) \in C^\infty(\mathbb{R}^n \setminus 0)$, positively homogeneous of order zero, we associate a family of operators on $\mathbb{R}_+$

$$op_\psi(a)(\xi') : L^2(\mathbb{R}_+) \longrightarrow L^2(\mathbb{R}_+), \qquad (1)$$

$\xi' \in \mathbb{R}^{n-1}$, $\xi = (\xi', \tau)$, defined by $op_\psi(a) = r^+ op(a) e^+$, $op(a)(\xi')u = (2\pi)^{-1} \int e^{i(t-s)\tau} a(\xi', \tau)u(s)ds d\tau$, $r^+u = u|_{\mathbb{R}_+}$, $e^+$ as in Section 0. The splitting of the covariables refers to the coordinates $x = (x', t) \in \mathbb{R}^n$, $x' = (x_1, \ldots, x_{n-1})$.

If a has the transmission property, (1) induces a continuous operator $\mathcal{S}(\overline{\mathbb{R}}_+) \longrightarrow \mathcal{S}(\overline{\mathbb{R}}_+)$, $\mathcal{S}(\overline{\mathbb{R}}_+) = \mathcal{S}(\mathbb{R})|_{\overline{\mathbb{R}}_+}$, if not we get an operator between spaces with more general asymptotics. The algebra generated by operators $op_\psi(a)$ also contains so-called Mellin and Green operators.

As usual, the Mellin transform is defined by the formula

$$\tilde{u}(z) = (Mu)(z) = \int_0^\infty t^{z-1} u(t) dt,$$

first for $u \in C_0^\infty(\mathbb{R}_+)$, where $z \in \mathbb{C}$, and then by extension to other function and distribution spaces on $\mathbb{R}_+$, where we have a subset $W \subset \mathbb{C}$ on which $u(z)$ is defined in a natural way. In particular M leads to an isomorphism

$$M : L^2(\mathbb{R}_+) \longrightarrow L^2(\Gamma_{1/2}),$$

$\Gamma_\varrho = \{ z \in \mathbb{C} : \operatorname{Re} z = \varrho \}$. We are dealing with spaces of functions $u \in L^2(\mathbb{R}_+) \cap C^\infty(\mathbb{R}_+)$ with conormal asymptotics near $t = 0$ and $t = \infty$ of the types $\varphi = (p_j, m_j)_{j \in \mathbb{Z}_+}$ and $\eta = (q_k, l_k)_{k \in \mathbb{Z}_+}$, respectively,

$$\operatorname{Re} p_j > -\tfrac{1}{2}, \ \operatorname{Re} p_j \longrightarrow \infty \ \text{as} \ j \longrightarrow \infty,$$

$$\operatorname{Re} q_k < -\tfrac{1}{2}, \ \operatorname{Re} q_k \longrightarrow -\infty \ \text{as} \ k \longrightarrow \infty.$$

The space of these functions may be equipped with a natural semi-norm system under which is becomes a nuclear Fréchet

space. Denote it by $C^{\infty}_{\mathcal{P},\mathcal{Q}}$. Let $\mathcal{S}_{\mathcal{P}}$ be the subspace of all $u \in C^{\infty}_{\mathcal{P},\mathcal{Q}}$ for which $e^{+}(1-\omega)u \in \mathcal{S}(\mathbb{R})$, $\mathcal{S}(\mathbb{R})$ the Schwartz space on $\mathbb{R}$, $\omega$ a cut-off function, i.e. $\omega \in C^{\infty}(\bar{\mathbb{R}}_{+})$, $\omega = 1$ near $t = 0$, $\omega = 0$ near $t = \infty$. $\mathcal{M}_{\mathcal{P},\mathcal{Q}} := M(C^{\infty}_{\mathcal{P},\mathcal{Q}})$ can easily be characterized as a space of meromorphic functions in $\mathbb{C}$ with poles at $z_j = -p_j$ and $z_k = -q_k$ of multiplicity $m_j + 1$ and $l_k + 1$, respectively, which in addition strongly decrease for $|\mathrm{Im}\ z| \longrightarrow \infty$.

Write for abbreviation $\mathcal{Z} = (\mathcal{P},\mathcal{Q})$, $\mathcal{Z} = (w_j, n_j)_{j \in \mathbb{Z}}$. If $\gamma \in \mathbb{C}$ we set $T^{\gamma}\mathcal{Z} := (w_j + \gamma, n_j)_{j \in \mathbb{Z}}$ and for a function $h(z)$

$$(T^{\gamma}h)(z) = h(z + \gamma).$$

If $\mathcal{Z} = (w_j, n_j)_{j \in \mathbb{Z}}$ is an arbitrary sequence in $\mathbb{C} \times \mathbb{Z}_{+}$ with $\mathrm{Re}\ w_j \longrightarrow \pm\infty$ as $j \longrightarrow \pm\infty$, we define $\mathcal{M}_{\mathcal{Z}} = \{h : T^{\gamma}h \in \mathcal{M}_{T^{\gamma}\mathcal{Z}}\}$, where $\gamma$ is choosen in such a way that $\mathrm{Re}(w_j + \gamma) \neq \frac{1}{2}$ for all $j$.

Let $h \in \mathcal{M}_{\mathcal{Z}}$ and $w_j \notin \Gamma_{1/2}$, $j \in \mathbb{Z}$. Then

$$op^{\circ}_{M}(h)u := op_{M}(h)u := M^{-1}(hMu)$$

induces a continuous operator

$$op^{\circ}_{M}(h) : L^2(\mathbb{R}_{+}) \longrightarrow L^2(\mathbb{R}_{+}).$$

If $\omega$ is a cut- off function the operator

$$\omega\, op^{\circ}_{M}(h)\omega : \mathcal{S}_{\underline{\mathcal{P}}} \longrightarrow \mathcal{S}_{\underline{\mathcal{P}}}$$

is continuous, too. Here $\mathcal{S}_{\underline{\mathcal{P}}} = \varinjlim \mathcal{S}_{\mathcal{P}}$ means the inductive limit over all singularity types $\mathcal{P}$. It is obvious how to calculate the precise singularity type of $\omega\, op^{\circ}_{M}(h)\omega u$ for given $u \in \mathcal{S}_{\mathcal{P}}$. If $\mu \geq 0$ we choose a $\gamma$, $0 \leq \gamma \leq \min(1, \mu)$ and set

$$\mathrm{op}_M^{-\mu}(h)u = t^{\mu-\gamma}\,\mathrm{op}_M(T^{\gamma}h)t^{\gamma}u, \tag{2}$$

where we admit $w_j \in \Gamma_{1/2}$ for a finite number of points $w_j$ and $\gamma$ is assumed to satisfy $w_j + \gamma \notin \Gamma_{1/2}$ for all $j \in \mathbb{Z}$, such that $T^{\gamma}h$ is holomorphic near $\Gamma_{1/2}$. It can be proved that another choice of $\gamma$ only affects the action $\omega \mathrm{op}_M^{-\mu}(h)\omega$ modulo a so-called Green operator, cf. the definition below.

Now consider sums of the form

$$\mathcal{M} = \sum_{j=0}^{\infty} \omega_j \,\mathrm{op}_M^{-j}(h_j)\,\omega_j, \tag{3}$$

$\omega_j(t) = \omega(c_j t)$, with a sequence of constants $c_j > 0$ and $h_j \in \mathcal{M}_{\mathcal{Z}^j}$, where we assume convergence in $\mathcal{L}(\mathcal{S}_{\varkappa}, \mathcal{S}_{\varkappa'})$ for given $\varkappa$ and the associated $\varkappa'$ that is determined by the $\mathcal{Z}^j$.

1. Definition. An operator of the form (3) is called a Mellin operator on $\mathbb{R}_+$ and

$$\sigma_N^{-j}(\mathcal{M}) = h_j,\ j \in \mathbb{Z}_+,$$

its conormal symbol of order $-j$.

Let $g(t,s) \in L^2(\mathbb{R}_+ \times \mathbb{R}_+)$ $(= L^2(\mathbb{R}_+) \otimes_H L^2(\mathbb{R}_+)$ with $\otimes_H$ as the Hilbert space tensor product) and

$$\mathrm{op}_G(g)u(t) = \int_0^{\infty} g(t,s)u(s)ds \tag{4}$$

the associated Hilbert-Schmidt operator.

2. Definition. $\mathcal{G} = \mathrm{op}_G(g)$ for $g \in \mathcal{S}_{\varkappa} \otimes_{\pi} \mathcal{S}_{\varrho}$ is called a Green operator, $\varrho$, $\varkappa$ certain singularity types.

$\mathcal{S}_{\varkappa} \otimes_{\pi} \mathcal{S}_{\varrho}$ is the projective tensor product and a nuclear Fréchet space, again. The notation Green operator comes from the tradition in boundary value problems, cf.[2], where in the context of operators with the transmission property

the kernels belong to $\mathcal{S}(\bar{\mathbb{R}}_+) \hat{\otimes}_\pi \mathcal{S}(\bar{\mathbb{R}}_+)$. Observe that (4) with $g \in \mathcal{S}_{\mathcal{U}} \hat{\otimes} \mathcal{S}_{\mathcal{O}}$ induces continuous operators

$$op_G(g) : L^2(\mathbb{R}_+) \longrightarrow \mathcal{S}_{\mathcal{U}} \,, \quad \mathcal{S}_{\mathcal{P}} \longrightarrow \mathcal{S}_{\mathcal{U}}$$

for arbotrary $\mathcal{P}$ . We set $\sigma_N^{-j}(g) = 0$, $j \in \mathbb{Z}_+$.

3. Theorem (cf. [15]). The class of operators of the form $w + g$, where $w$ is a Mellin and $g$ a Green operator, form an algebra, with the Green operators as a two sided ideal. For the conormal symbols we have

$$\sigma_N^{-l}(w_1 w_2) = \sum_{j+k=l} (T^{-k}\sigma_N^{-j}(w_1))\, \sigma_N^{-k}(w_2), \tag{5}$$

$l \in \mathbb{Z}_+$.

To (1) we also associate a sequence of conormal symbols $\sigma_N^{-l}(a) := \sigma_N^{-l}(op_\psi(a))$, $l \in \mathbb{Z}_+$. Define $g^-(z) = (1-e^{2\pi i z})^{-1}$, $g^+(z) = 1-g^-(z)$.

$$\sigma_N^{0}(a)(z) = a_0^+ g^+(z) + a_0^- g^-(z), \tag{6}$$

$$\sigma_N^{-l}(a)(\xi', z) = \left\{a_1^+(\xi')g^+(z) + a_1^-(\xi')g^-(z)\right\} \prod_{k=1}^{l} (k-z)^{-1}, \tag{7}$$

$l \geq 1$.

Here

$$a_k^{\pm}(\xi') = \frac{i^k}{k!} \frac{\partial^k}{\partial \rho^k} \, a(\rho\xi', \pm 1)\Big|_{\rho=0}$$

are homogeneous polynomials in $\xi'$ of order k.

Set $\mathcal{P}^0 = (k,0)_{k \in \mathbb{Z}_+}$. Further denote by $\mathcal{P} + \mathcal{O}$ that singularity type which appears under the multiplication of $\tilde{u}, \tilde{v}$ for $u \in \mathcal{S}_{\mathcal{P}}$ , $v \in \mathcal{S}_{\mathcal{O}}$. Then it can be proved that

$$op_\psi(a) : \mathcal{S}_{\mathcal{P}} \longrightarrow \mathcal{S}_{\mathcal{P}+\mathcal{P}^0} \tag{8}$$

is continuous for every singularity type $\mathcal{P}$ with $T^j\mathcal{P} \subseteq \mathcal{P}$,

$j \in \mathbb{Z}_+$.

Let

$$\mathcal{A} = op_\psi(a) + \mathcal{M} + \mathcal{G} \tag{9}$$

($\xi' \neq 0$ fixed), $\mathcal{M}$ a Mellin, $\mathcal{G}$ a Green operator, and

$$\sigma_N^{-j}(\mathcal{A}) = \sigma_N^{-j}(a) + \sigma_N^{-j}(\mathcal{M}), \quad j \in \mathbb{Z}_+ . \tag{10}$$

The conormal symbols (10) are meromorphic but not necessarily strongly decreasing for $|\operatorname{Im} z| \to \infty$. Set $\sigma_\psi(\mathcal{A}) = a$.

4. Theorem (cf. [15]). The class of operators (9) form an algebra, where those of the form $\mathcal{M} + \mathcal{G}$ and $\mathcal{G}$ form two-sided ideals. For the symbols we have

$$\sigma_N^{-l}(\mathcal{A}_1\mathcal{A}_2) = \sum_{j+k=l} (T^{-k}\sigma_N^{-j}(\mathcal{A}_1))\,\sigma_N^{-k}(\mathcal{A}_2),$$

$$\sigma_\psi(\mathcal{A}_1\mathcal{A}_2) = \sigma_\psi(\mathcal{A}_1)\ \sigma_\psi(\mathcal{A}_2).$$

This assertion is not a priori obvious, since at a first glance it may seem that the $\psi$DOs of the form $r^+F^{-1}aFe^+$ (F the Fourier transform) and the Mellin operators containing $M^{-1}$ h M do not fit together. Fortunately we have a Mellin expansion of the pseudo-differential action, cf. Theorem 5, which actually leads to the desired algebra.

5. Theorem (cf. [15]). For every $N \in \mathbb{Z}_+$ there is a $k = k(N)$, $k(N) \to \infty$ as $N \to \infty$, such that for any cut-off function

$$\omega\, op_\psi(a)(\xi')\omega = \omega \sum_{j=0}^{N} t^j op_M(\sigma_N^{-j}(a))(\xi')\omega + op_G(g)(\xi') \tag{11}$$

with a kernel $g(t,s,\xi') \in C^\infty(\mathbb{R}^{n-1}_{\xi'} \setminus 0,\ C^k(\mathbb{R}_+ \times \mathbb{R}_+))$.

Note that both the Green kernels and the Mellin symbols are functions of $\xi' \in \mathbb{R}^{n-1} \setminus 0$, where the Mellin symbols in

$op_M^{-j}(h)$ are polynomials in $\xi'$ of order j. On the half axis the parameter dependence may be neglected, but it plays a role in the higher dimensional calculus, where the operators on $\mathbb{R}_+$ are considered as boundary symbols.

The boundary symbols are of the form

$$\begin{pmatrix} \alpha & k \\ b & q \end{pmatrix}(x',\xi') : \begin{matrix} \mathcal{S}_+ \otimes \mathbb{C}^{N_1} \\ \oplus \\ \mathbb{C}^{N_1'} \end{matrix} \longrightarrow \begin{matrix} \mathcal{S}_+ \otimes \mathbb{C}^{N_2} \\ \oplus \\ \mathbb{C}^{N_2'} \end{matrix}, \qquad (12)$$

$(x',\xi') \in \Omega \times \mathbb{R}^{n-1}$, $\Omega \subseteq \mathbb{R}^{n-1}$ open, and on principal symbolic level the left upper corners are as in Theorem 4 , whereas the other operators being finite-dimensional express the boundary and coboundary conditions. In this sense we have an extension of the concept of Boutet de Monvel, cf.[2], here for operators without the transmission property. (The case of t depending interior symbols can easily be included). In particular we have the notion of ellipticity of objects of the form (12), namely that for the principal symbols the $L^2(\mathbb{R}_+)$ closures have to be bijective for every $(x',\xi')$, which necessarily implies that $\sigma_\psi(\alpha)(x',\xi') : \mathbb{C}^{N_1} \to \mathbb{C}^{N_1}$ and $\overset{\circ}{\sigma}_N(\alpha)(x',\xi',z)|_{\mathrm{Re}\, z=\frac{1}{2}} : \mathbb{C}^{N_1} \to \mathbb{C}^{N_1}$ are bijective. In that case both on principal and complete symbolic level we have the inverses in our class, the latter modulo negligible boundary symbols.

## 2. Mixed boundary problems and branching of exponents

The asymptotic properties of solutions of boundary problems are local in nature and we may deal with the half space case $\mathbb{R}^{n+1}_+ = \{x_{n+1} > 0\}$. Let us illustrate how mixed boundary

problems lead to a calculus as sketched in Section 1 (cf. (i) of the Introduction). For simplicity consider $\psi$DOs of order zero. Let

$$\mathcal{A} = \begin{pmatrix} A \\ T_+ \\ T_- \end{pmatrix} : L^2(\mathbb{R}^{n+1}, E) \longrightarrow \begin{matrix} L^2(\mathbb{R}^{n+1}_+, E) \\ \oplus \\ L^2(\mathbb{R}^n_+, G_+) \\ \oplus \\ L^2(\mathbb{R}^n_-, G_-) \end{matrix} \qquad (1)$$

be a mixed boundary problem for an elliptic $\psi$DO A with the transmission property with respect to $\partial\mathbb{R}^{n+1}_+ = \mathbb{R}^n \ni x=(x',x_n)$, $x' = (x_1,\dots,x_{n-1})$, and $T_\pm = r^\pm T^\pm$, $T^\pm$ elliptic boundary conditions in open neighbourhoods of $\mathbb{R}^n_\pm$, $\mathbb{R}^n_\pm = \{x_n \gtrless 0\}$, $r^\pm u = u|_{\mathbb{R}^n_\pm}$, $E \cong \mathbb{C}^M$, $G_\pm \cong \mathbb{C}^N$. Further let

$$\mathcal{A}_0 = \begin{pmatrix} A \\ T_c \end{pmatrix} : L^2(\mathbb{R}^{n+1}_+, E) \longrightarrow \begin{matrix} L^2(\mathbb{R}^{n+1}_+, E) \\ \oplus \\ L^2(\mathbb{R}^n, G) \end{matrix}$$

be elliptic and of the class as in [2]. Reducing $\mathcal{A}$ to $\partial\mathbb{R}^{n+1}_+$ by means of $\mathcal{A}_0$ leads to a $\psi$DO on $\mathbb{R}^n$ with a jump on $\mathbb{R}^{n-1} = \{x \in \mathbb{R}^n : x_n = 0\}$

$$D := \begin{pmatrix} r^+T^+C_0e^+ & r^+T^+C_0e^- \\ r^-T^-C_0e^+ & r^-T^-C_0e^- \end{pmatrix} : \begin{matrix} L^2(\mathbb{R}^n_+, G_+) \\ \oplus \\ L^2(\mathbb{R}^n_-, G_-) \end{matrix} \longrightarrow \begin{matrix} L^2(\mathbb{R}^n_+, G_+) \\ \oplus \\ L^2(\mathbb{R}^n_-, G_-) \end{matrix}.$$

Here $C_0$ is the coboundary part of a parametrix $\mathcal{A}_0^{(-1)} = (P_0, C_0)$ of $\mathcal{A}_0$ and $e^\pm : L^2(R^n_\pm, G_\pm) \longrightarrow L^2(R^n, G)$ extends functions by zero to the opposite half space, $G|_{\mathbb{R}^n_\pm} = G_\pm$. The reflection $\varepsilon : R^n_+ \longrightarrow R^n_-$, $\varepsilon(x',x_n) = (x',-x_n)$ induces an operator

$$\begin{pmatrix} 1 & 0 \\ 0 & \varepsilon^* \end{pmatrix} D \begin{pmatrix} 1 & 0 \\ 0 & \varepsilon^* \end{pmatrix}^{-1} : L^2(\mathbb{R}^n_+, \mathbb{C}^{2N}) \longrightarrow L^2(\mathbb{R}^n_+, \mathbb{C}^{2N}), \qquad (2)$$

which, in general, is of the class $\mathfrak{L}^0$ studied in [13]. A general theory of mixed boundary problems in Sobolev spaces and the principal symbolic calculus were developed in [14]. Any assertion on the solvability of (2) gives rise to a result on (1). It is easily seen that the concrete choice of $T_0$ is not essential. So we may assume that $T_0$ coincides with $T^-$ in a neighbourhood of $\mathbb{R}^n_-$. Then

$$r^-T^-C_0e^+ = 0, \quad r^-T^-C_0e^- = 1$$

and it suffices to consider

$$r^+T^+C_0e^+ : L^2(\mathbb{R}^n_+,\mathbb{C}^N) \longrightarrow L^2(\mathbb{R}^n_+,\mathbb{C}^N) \tag{3}$$

which is a $\psi$DO on $\mathbb{R}^n_+$ of the form $r^+Pe^+$ with the action of the zero order operator $P = T^+C_0$ on $L^2(\mathbb{R}^n,\mathbb{C}^N)$ in the usual sense. In general P has not the transmission property with respect to $\mathbb{R}^{n-1}$.

As a (higher order) example consider the Laplacian $\Delta$ in $\mathbb{R}^{n+1}_+$ under jumping oblique derivatives

$$T^{\pm} = \sum_{i=1}^{n+1} b_i^{\pm}(x)\, D_{x_i} .$$

For an action on $H^s(\mathbb{R}^{n+1}_+)$ we set $s = s_0+k$, $k \in \mathbb{Z}_+$ fixed, $s_0 = \frac{3}{2} + \varepsilon$, $0 < \varepsilon < \frac{1}{2}$.

Then a reduction of orders and reduction to the boundary leads to the symbol

$$p(x,\xi) = \left( \frac{1 - i\xi_n}{1 + i\xi_n} \right)^{k+\varepsilon} \frac{c^+(x,\xi)}{c^-(x,\xi)} , \tag{4}$$

$x = (x', x_n)$, $\xi = (\xi', \xi_n)$.

$$c^{\pm}(x,\xi) = \sum_{i=1}^{n} b_i^{\pm}(x)\,\xi_i + ib_{n+1}^{\pm}|\xi| \ .$$

Let us disregard here the non-homogeneity of p which causes some extra arguments in the precise calculus for the strip $|\xi'| \leq 1$. For simplicity assume that for large $x$ all coefficients are independent of $x$.

Let, for instance, $k = 0$. Then the operator

$$r^+Pe^+ : L^2(\mathbb{R}^n_+) \longrightarrow L^2(\mathbb{R}^n_+) \tag{5}$$

$P = Op(p)$, may happen to be Fredholm without boundary conditions on $\mathbb{R}^{n-1}$. This depends on the coefficients $b_n^{\pm}(x)$, $b_{n+1}^{\pm}(x)$ near $x_n = 0$. Since $b_{n+1}^{\pm}$ is supposed to be $\neq 0$ (the ellipticity of $T^{\pm}$), we may set $b_{n+1}^{\pm} = 1$. The curve $\alpha = -\frac{1}{\beta}$ defines three connection components $\omega_0, \omega_+, \omega_-$ in the real $(\alpha, \beta)$ plane, $\omega_0$ containing the origin and $\omega_{\pm}$ the point $(\mp 2, \pm 2)$. For

$$(b_n^+(x),\ b_n^-(x)) \in \omega_0 \quad \text{near } x_n = 0 \tag{6}$$

(and $\varepsilon > 0$ sufficiently small) (5) is Fredholm. For $(b_n^+(x),\ b_n^-(x)) \in \omega_{\pm}$ we have to pose one boundary or one coboundary condition on $\mathbb{R}^{n-1}$, according to the sign. The details are elaborated in [14]. Now assume, for instance, that (6) holds. Then we may hope to get a regularity of (5) in the sense that

$$r^+Pe^+ : \mathcal{S}(\mathbb{R}^{n-1}, \mathcal{S}_{\underline{p}}) \longrightarrow \mathcal{S}(\mathbb{R}^{n-1}, \mathcal{S}_{\underline{p}}) \tag{7}$$

is Fredholm, too ($\mathcal{S}(\mathbb{R}^{n-1},E)$ being the Schwartz space of E-valued functions) and that the kernel of (5) belongs to $\mathcal{S}(\mathbb{R}^{n-1}, \mathcal{S}_{p})$ and the kernel of the adjoint to $\mathcal{S}(\mathbb{R}^{n-1}, \mathcal{S}_{q})$ $p, q$ certain singularity types. Moreover a parametrix of

(7) should be obtained by a higher dimensional version of the calculus in Section 1. This is indeed true, provided the singularity types are independent of x'. The precise theorems of this sort for rather general classes of mixed problems (arbitrary order, systems) will be published in Chapter 3 of [15].

Here we only want to discuss the qualitative phenomenon. The main difficulty in dealing with variable coefficients is that the singularity types are functions of x' and that the exponents $p_j(x')$ may have a complicated branching behaviour which is individual for any concrete problem (a situation similar to eigenvalues of parameter depending operators). This makes the spaces in (7) highly suspect, and it turns out that basically we have to replace them by more general spaces with conormal asymptotics. These spaces are briefly discussed in Section 3.

Now let us show that the branching points actually occur.

First it is known that the singularity type of a solution of (5) for a smooth right hand side contains $(\varkappa+j,j)_{j\in\mathbb{Z}_+}$, $\varkappa$ the factorization index of (4), cf.[4]. In [14] is proved a similar behaviour for right hand sides in $\mathcal{S}_{\underline{\varphi}}$. It is clear that $\varkappa$ is a function of x'.

In the case of systems we have not always a good factorization, but we can state the

1. Theorem. Let P be elliptic and u a solution of $r^+Pe^+u = f$, $f \in \mathcal{S}(\mathbb{R}^{n-1}, \mathcal{S}_{\sigma} \otimes \mathbb{C}^N)$, $\sigma$ some singularity type. Then $u \in \mathcal{S}(\mathbb{R}^{n-1}, \mathcal{S}_{\varphi} \otimes \mathbb{C}^N)$ for a singularity type $\varphi$ that contains

both $\varphi$ and the points $(r_j, l_j)$, where $r_j \in \mathbb{C}$ runs through the zeros of $\det \sigma_N^0(r^+Pe^+)$ and $l_j+1$ denotes the multiplicity of $r_j$.

The $\chi$ can be described more precisely, but we drop the details here, since they would require much more technical background.

Consider the following mixed problem

$$\mathcal{A} = \begin{pmatrix} \Delta & 0 \\ 0 & \Delta \\ T_+^1 & r^+R \\ 0 & T_+^2 \\ T_-^1 & r^-R \\ 0 & T_-^2 \end{pmatrix} : H^s(\mathbb{R}^{n+1}, \mathbb{C}^2) \longrightarrow \begin{matrix} H^{s-2}(\mathbb{R}^{n+1}_+, \mathbb{C}^2) \\ \oplus \\ H^{s-\frac{3}{2}}(\mathbb{R}^n_+, \mathbb{C}^2) \\ \oplus \\ H^{s-\frac{3}{2}}(\mathbb{R}^n_-, \mathbb{C}^2) \end{matrix}$$

Here $T_\pm^i$ are as in the previous example for $i = 1,2$ and $R$ is a smooth non-vanishing vector field on $\partial\mathbb{R}^{n+1}_+$.

Reduction of orders and reduction to the boundary leads to an elliptic operator

$$r^+Pe^+ : L^2(\mathbb{R}^{n-1}, \mathbb{C}^2) \longrightarrow L^2(\mathbb{R}^{n-1}, \mathbb{C}^2),$$

cf. (3), with the property that $\det \sigma_N^0(r^+Pe^+)(x',z) = 0$

$$\Leftrightarrow \quad \sigma_N^0(r^+P_ke^+)(x',z) = 0, \ k = 1,2 .$$

Here $P_k = \mathrm{Op}(p_k)$, with

$$p_k(x,\xi) = \left( \frac{1 - i\xi_n}{1 + i\xi_n} \right)^{\varepsilon} \frac{c_k^+(x,\xi)}{c_k^-(x,\xi)}, \ k = 1,2,$$

associated to $T_\pm^k$, $k = 1,2$, in the sense of (4). It is obvious that $\det \sigma_N^0(r^+Pe^+)$ has a two-fold zero for those $x'$, where the vector fields involved in $T_\pm^1$, $T_\pm^2$ coincide. On the other hand, the zeros are simple if the factorization index of $p_1$ is different from that of $p_2$ which is usually the

case. Thus, applying Theorem 1, we observe a branching behaviour of the singularities. The example is typical and easy to generalize, for instance, to mixed boundary problems for the Lamé system in any dimension. Besides, even for the first example, the Laplacian with jumping oblique derivatives, we can obtain branching singularities. This requires a more careful discussion of the structure of the parametrix.

In Section 3 we give an idea how to arrange the spaces that are adapted to the general situation. Here let us formulate a result on (1) under the condition that the zeros in $\mathbb{C}$ of $\sigma_N^0(r^+Pe^+)$ in Theorem 1 are independent of $x'$.

A theorem on general mixed problems (cf. [15], Chapter 4) says that (under a certain weak topological condition) $\mathcal{A}$ can be extended to a Fredholm operator

$$\mathcal{B} = \begin{pmatrix} A & 0 \\ T_+ & R_+ \\ T_- & R_- \\ S & Q \end{pmatrix} : \begin{matrix} L^2(\mathbb{R}^{n+1}_+,E) \\ \oplus \\ L^2(\mathbb{R}^{n-1},H) \end{matrix} \longrightarrow \begin{matrix} L^2(\mathbb{R}^{n+1}_+,E) \\ \oplus \\ L^2(\mathbb{R}^n_+,G_+) \\ \oplus \\ L^2(\mathbb{R}^n_-,G_-) \\ \oplus \\ L^2(\mathbb{R}^{n-1},K) \end{matrix} \tag{8}$$

with $H = \mathbb{C}^h$, $K = \mathbb{C}^k$ for certain $h,k \in \mathbb{Z}_+$. $S$ and $R_\pm$ are extra boundary and coboundary conditions, respectively, with respect to $\mathbb{R}^{n-1}$, and $Q$ is a $\psi$DO on $\mathbb{R}^{n-1}$.

Introduce polar coordinates $(t,\varphi)$ in the $(x_n,x_{n+1})$ plane, such that $t=0$ corresponds to the origin and $0 \leq \varphi \leq \pi$ to $x_{n+1} \geq 0$. For any singularity type $\mathcal{P}$ we can define the spaces $C^\infty([0,\pi],\mathcal{S}_\mathcal{P})$ and $\mathcal{S}(\mathbb{R}^{n-1},C^\infty([0,\pi],\mathcal{S}_\mathcal{P}))$ and also the spaces based on $\mathcal{S}_{\underline{\mathcal{P}}}$.

Set

$$\mathcal{S}_1 = \begin{matrix} \mathcal{S}(\mathbb{R}^{n-1}, C^\infty([0,\pi], \mathcal{S}_{\underline{p}} \otimes E)) \\ \oplus \\ \mathcal{S}(\mathbb{R}^{n-1}, H) \end{matrix}, \quad \mathcal{S}_2 = \begin{matrix} \mathcal{S}(\mathbb{R}^{n-1}, C^\infty([0,\pi], \mathcal{S}_{\underline{p}} \otimes E)) \\ \oplus \\ \mathcal{S}(\mathbb{R}^{n-1}, \mathcal{S}_{\underline{p}}^+ \otimes G_+) \\ \oplus \\ \mathcal{S}(\mathbb{R}^{n-1}, \mathcal{S}_{\underline{p}}^- \otimes G_-) \\ \oplus \\ \mathcal{S}(\mathbb{R}^{n-1}, K) \end{matrix},$$

$\mathcal{S}_{\underline{p}}^{\pm} := \mathcal{S}_{\underline{p}}(R_\pm)$.

2. Theorem. [15, Chapter 3] Under the mentioned conditions $\mathcal{B}$ induces a Fredholm operator $\tilde{\mathcal{B}} : \mathcal{S}_1 \to \mathcal{S}_2$ with the same index as $\mathcal{B}$ and $\ker \mathcal{B} \subset \mathcal{S}_1$, $\ker \mathcal{B}^* \subset \mathcal{S}_2$. A parametrix of $\mathcal{B}$ can be obtained by a higher-dimensional analogue of the calculus in Section 1.

3. General singularity types

As noted in Section 2 for dealing with mixed boundary problems or boundary problems for operators without the transmission property in $C^\infty$ spaces we have to introduce a new type of asymptotics, first on the half axis near $t = 0$, and then in $\mathbb{R}^{n+1}_+$ near $\mathbb{R}^{n-1}$ (where the conditions have a jump) or in $\mathbb{R}^n_+$ near $\mathbb{R}^{n-1}$.

The spaces in higher dimensions are defined in terms of spaces on $\mathbb{R}_+$. In a similar way the actions of operators are obtained by actions on $\mathbb{R}_+$ composed with certain more standard operations in the other variables. So the main point is the calculus on the half axis in the general classes with asymptotics.

Let us sketch the idea, first of the definition of the spaces on $\mathbb{R}_+$ with general asymptotics. Let $\Lambda$ be a closed

set in $\mathbb{C}$ being the disjoint union of compact sets $\Lambda_j$, $j \in \mathbb{Z}_+$, $\Lambda \subset \{\operatorname{Re} z < \frac{1}{2}\}$, $\sup\{\operatorname{Re} z : z \in \Lambda_j\} \longrightarrow -\infty$ as $j \longrightarrow \infty$. Denote by $\mathcal{A}'(\Lambda_j)$ the space of analytic functionals carried by $\Lambda_j$. Then we define a space $\mathcal{S}_\Lambda$ of functions in $C^\infty(\mathbb{R}_+) \cap L^2(\mathbb{R}_+)$ for which there is a sequence $\zeta_j \in \mathcal{A}'(\Lambda_j)$ with

$$u(t) \sim \sum_{j=0}^{\infty} \langle \zeta_j(w), t^{-w} \rangle \text{ as } t \longrightarrow 0 \qquad (1)$$

and $e^+(1-\omega(t))u(t)$ a Schwartz function on $\mathbb{R}$ for any cut-off function $\omega$. The functional $\zeta_j$ acts with respect to $w$ and is uniquely determined by $u$. This definition can be given in terms of a countable system of semi-norms under which $\mathcal{S}_\Lambda$ becomes a nuclear Fréchet space.

In a similar way we can define a space $C^\infty_{\Lambda,\Xi}$, where $\Xi \subset \mathbb{C}$ is the disjoint union of compact sets $\Xi_j$ in $\{\operatorname{Re} z > \frac{1}{2}\}$, $\inf\{\operatorname{Re} z : z \in \Xi_j\} \longrightarrow \infty$ as $j \longrightarrow \infty$, with analytic functionals $\rho_j \in \mathcal{A}'(\Xi_j)$, where in addition to (1)

$$u(t) \sim \sum \langle \rho_j(w), t^{-w} \rangle \text{ as } t \longrightarrow \infty .$$

The definition of $\mathcal{S}_\Lambda$, $C^\infty_{\Lambda,\Xi}$ can be generalized to arbitrary closed sets $\Lambda \subset \{\operatorname{Re} z < \frac{1}{2}\}$, $\Xi \subset \{\operatorname{Re} z > \frac{1}{2}\}$ that intersect each strip of the form $\{\alpha_1 < \operatorname{Re} z < \alpha_2\}$, $\alpha_1, \alpha_2 \in \mathbb{R}$, in a compact set. Clearly $\mathcal{S}_\Lambda \subset C^\infty_{\Lambda,\Xi}$. Set $Z = \Lambda \cup \Xi$ and denote by $\mathfrak{M}_Z$ the space of holomorphic functions in $\mathbb{C} \setminus Z$ being the Mellin image of some $f \in C^\infty_{\Lambda,\Xi}$. If $Z^1 \subset \mathbb{C}$ is another set in $\mathbb{C}$ with the property $T^{-\gamma} Z^1 = Z$, $\gamma \in \mathbb{R}$ fixed, we have not necessarily $\Gamma_{1/2} \cap Z^1 = \emptyset$. Then we define $\mathfrak{M}_{Z^1} = \{T^{-\gamma} h : h \in \mathfrak{M}_Z\}$.

The model of the new quality of Mellin operators are objects of the form

$$op_M^{-j}(h_j) := \sum_{k=1}^{N_j} t^{j-\gamma_{jk}} op_M(T^{\gamma_{jk}} h_{jk}) t^{jk}$$

with functions $h_{jk} \in \mathcal{M}_{Zjk}$ for which $T^{\gamma_{jk}} h_{jk}$ are holomorphic near $\Gamma_{1/2}$ and $0 \leq \gamma_{jk} \leq j$.

$$h_j := \sum_{k=1}^{N_j} h_{jk} \qquad (2)$$

is holomorphic in $\mathbb{C} \setminus (\bigcup_{k=1}^{N_j} Z^{jk})$. If $h'_{jk} \in \mathcal{M}_{Zjk}$ is another system of functions with $h_j = \sum h'_{jk}$, the associated operator, multiplied from the left and the right by a cut-off function, only changes modulo a Green operator, here defined as an integral operator with a kernel in $\mathcal{S}_{\Lambda^1} \hat{\otimes}_\pi J\mathcal{S}_{\Lambda^2}$ for certain sets $\Lambda^1, \Lambda^2 \subset \mathbb{C}$, $J$ the complex conjugation.

The expression 1 (3) can also be defined for $h_j$ as in (3) and we can use all notations of Section 1. The operators of the form 1 (9) form an algebra, again, and we have an analogue of 1. Theorem 4. Any operator of the form 1 (9) with the more general Mellin and Green operators induces a continuous operator

$$op_\varphi(a) + \mathcal{M} + \mathcal{G} : \mathcal{S}_{\Lambda^1} \longrightarrow \mathcal{S}_{\Lambda^2}$$

for any singularity type $\Lambda^1$ and a resulting singularity type $\Lambda^2$ that also depends on $\mathcal{M}$ and $\mathcal{G}$.

The boundary symbols mentioned in Section 1 can also be defined in the $\mathcal{S}_\Lambda$ setting. The corresponding higher dimensional calculus leads to an analogue of 2. Theorem 2, here for the spaces $\mathcal{S}_{\underline{\Lambda}} = \varinjlim \mathcal{S}_\Lambda$ (the inductive limit

over all singularity types). The non-classical ingredients are the Mellin and Green operators. Let us mention, for instance, the complete Mellin boundary symbols on $\mathbb{R}^{n+1} \times \mathbb{R}_+$, involved in the calculus of operators $r^+Pe^+$, P a $\psi$DO without the transmission property with respect to $\mathbb{R}^{n-1}$. They have the form of asymptotic sums

$$\mathcal{M}(x',\xi') \sim \sum_{j=0}^{\infty} \mathcal{M}_{\mu-j}(x',\xi'),$$

$\mu \in \mathbb{R}$ the tangential order,

$$\mathcal{M}_{\mu-j}(x',\xi') = \sum_{l=0}^{\infty} \omega(c_l t\zeta(\frac{\xi'}{c_l}))\mathrm{op}_M^{-l}(h_{l,\mu-j})(x',\xi')\,\omega(c_l t\zeta(\frac{\xi'}{c_l})),$$

$$h_{l,\mu-j} \in \sum_{k=1}^{N_j} \mathcal{M}_{Z_l^{jk}} \otimes_\pi S_{cl}^{l+\mu-j}(\mathbb{R}^{n-1} \times \mathbb{R}^{n-1}).$$

Here $\zeta \in C^\infty(\mathbb{R}_+)$, $\zeta > 0$, $\zeta(t) = t$ for $t \geq 1$, and the $c_l > 0$ are constants, increasing sufficiently fast as $l \to \infty$.

Now the idea of treating the branching of singularities is to represent them as smooth families of analytic functionals. The parametrices that we obtain can be applied to functions with discrete asymptotics in the sense of the Sections 1,2, and the resulting solutions just inherit the branchings, induced by the zeros of the principal conormal symbol. Although their behaviour may be rather capricious, it can actually be described in terms of smooth functions with values in $\mathcal{A}'(\mathbb{C})$. The details are given in [15], Chapter 3.

## 4. Notes on the spectral asymptotics

The relation between the asymptotic properties of distributions on $\mathbb{R}_+$ and the behaviour of the Mellin image can be

interpreted for the counting function $N(t) = \{ \# \lambda_k : \lambda_k \leq t \}$ of the eigenvalues $\lambda_k$ of some self-adjoint positive operator.

Denote by $\mathbb{L}^2_{(b)}$ the space of those holomorphic functions in $\operatorname{Re} z > \frac{1}{2}$ which are the Mellin image of some $u \in L^2_{(b)}$,

$$L^2_{(b)} := \{ v \in L^2(\mathbb{R}_+) : \operatorname{supp} v \quad \text{bounded} \} .$$

Then for every $\delta \geq 0$, $\lambda \geq \delta$, $r \in \mathbb{Z}$

$$T^{-\delta}(\lambda+z)^{-r} Mu \in \mathbb{L}^2_{(b)}$$

$$\Leftrightarrow \quad u \in t^{\delta}(-\lambda + t\tfrac{d}{dt})^r L^2_{(b)} .$$

Let $u$ be an extendible distribution on $\mathbb{R}_+$ with bounded support and $Mu(z)$ its Mellin transform (a definition of $M$ on distributions was given in [6]). Then for any extendible distribution $u$ with bounded support on $\mathbb{R}_+$ the following conditions are equivalent:

(i) There exist constants $\xi_{jk} \in \mathbb{C}$, $0 \leq k \leq m_j$, $j \in \mathbb{Z}_+$, such that for every $\delta \geq 0$ there is an $N = N(\delta) \in \mathbb{Z}_+$ and an $r = r(\delta) \in \mathbb{Z}$ such that

$$u - \sum_{j=0}^{N} \sum_{k=0}^{m_j} \xi_{jk}\, t^{p_j} \log t^k\, \omega(t) \in t^{\delta}(-\lambda + t\tfrac{d}{dt})^r L^2_{(b)} \qquad (1)$$

$\omega$ any cut-off function, $\lambda \geq \delta$ ,

(ii) $Mu(z)$ is a meromorphic function in $\mathbb{C}$ with poles at $z_j = -p_j$ of multiplicity $m_j + 1$, and for every $\delta \geq 0$ there are $N = N(\delta)$, $r = r(\delta)$ with

$$T^{-\delta}(\lambda+z)^{-r}\Big(Mu(z) - \sum_{j=0}^{N} \sum_{k=0}^{m_j} \xi_{jk}\, f^{p_j,k}(z)\Big) \in \mathbb{L}^2_{(b)},$$

$f^{p_j,k}(z) = M(t^{p_j} \log t^k \omega(t))(z)$, $\lambda \geq \delta$ .

Note that the functions $f^{p_j,k}$ are strongly decreasing as $|\mathrm{Im}\ z| \longrightarrow \infty$ . In this way, roughly speaking, any extendible distribution u on $\mathbb{R}_+$, supp u bounded, admits an asymptotic expansion of the form (1) as $t \longrightarrow 0$ provided the Mellin image is meromorphic and does not increase faster than some polynomial in $\mathrm{Im}\ z > \frac{1}{2} - \delta$ as $|\mathrm{Im}\ z| \longrightarrow \infty$ . A similar property can be formulated with respect to the half planes $\mathrm{Im}\ z > \frac{1}{2} + \delta$. Meromorphic functions of that type are called of finite order.

Now our remark on the asymptotics of the counting function N(t) as $t \longrightarrow \infty$ is that we can apply an analogue of (1) for $t \longrightarrow \infty$ if we know that the $\zeta$ function of the operator is of finite order.

Set

$$L^2_{(a)} = \left\{ u \in L^2(\mathbb{R}_+) : \mathrm{dist}\ (0, \mathrm{supp}\ u) > 0 \right\} .$$

Then $v \longrightarrow s^{-1} v(s^{-1})$ induces an bijection $\varkappa : L^2_{(b)} \longrightarrow L^2_{(a)}$, and

$$\int_0^\infty t^{z-1} v(t)dt = \int_0^\infty s^{w-1} (\varkappa v)(s)ds, \quad w = 1-z.$$

In the Mellin image we get a correspondence between $L^2_{(a)}$ and $\mathbb{L}^2_{(a)} := \left\{ g(1-w) : g \in \mathbb{L}^2_{(b)} \right\}$. For a distribution u on $\mathbb{R}_+$, dist (0,supp u) > 0, extendible with respect to t = 0 under the substitution $t = s^{-1}$, the condition

$$u - \sum_{j=0}^{N} \sum_{k=0}^{m_j} \zeta_{jk}\, s^{p_j} \log s^k\, \chi(s) \in t^{-\gamma} (\rho + s\tfrac{d}{ds})^r\, L^2_{(a)},$$

$\gamma \geq 0$, $\rho \geq \gamma$, $N = N(\gamma)$, $r = r(\gamma)$, $\chi(s) = \omega(s^{-1})$,

is equivalent to Mu(w) meromorphic with poles at $w_j = -p_j$ of multiplicity $m_j+1$ and

$$T^{\gamma}(-\rho + w)^{-r}(Mu(w) - \sum_{j=0}^{N} \sum_{k=0}^{m_j} \zeta_{jk} f^{p_j,k}(w)) \in \mathbb{L}^2_{(a)} .$$

Let A be an elliptic pseudo-differential operator on a closed compact manifold X, $n = \dim X$, $m = \operatorname{ord} A > 0$, and suppose that A is self-adjoint and positive. Denote by $\zeta_A(z) = \sum_{k=1}^{\infty} \lambda_k^z$ the $\zeta$-function of A, $0 < \lambda_1 \leq \lambda_2 \leq \dots$ being the eigenvalues of A. Then

$$\zeta_A(z) = \int_0^{\infty} t^z \, dN(t) = -z \int_0^{\infty} t^{z-1} N(t) dt, \qquad (2)$$

$\operatorname{Re} z < -\frac{n}{m}$, with the counting function N(t).

It is well known that $\zeta_A(z)$ has a meromorphic extension with simple poles at $z_j = -\frac{n-j}{m}$, $j \in \mathbb{Z}_+$, cf. Seeley [17], and the residues can be expressed in terms of the complete symbol of A. In many cases it is known that $\zeta_A(z)$ is of finite order, in other words, there exists a real function $\mu(x)$, $\mu(x_1) \leq \mu(x_2)$ for $x_1 \leq x_2$, such that for suitable $c(x) > 0$, $a(x) > 0$

$$|\operatorname{Im} z|^{-\mu(x_0)} |\zeta_A(z)| \leq c(x_0) \qquad (3)$$

for $\operatorname{Re} z \leq x_0$, $|\operatorname{Im} z| \geq a(x_0)$, $x_0 \in \mathbb{R}$.

For the Riemann $\zeta$ function this is a classical result, but it is very hard to get precise bounds for $\mu$. For scalar $\psi$DOs in Duistermaat, Guillemin [3] it was proved that $\zeta_A$ is of finite order. In the case of systems this follows from the results of Ivriĭ [5], in a similar manner as in [3]. In all these cases we obtain the following

1. <u>Proposition</u>. Let $\zeta_A(z)$ be of order $\mu(x_0) < \infty$ for

$\mathrm{Re}\ z \leqq x_0$, $x_0 \in \mathbb{R}$. Then the counting function $N(t)$ of $A$ has an asymptotic expansion of the form

$$N(t) - \sum_{j=0}^{q} \zeta_j\, t^{\frac{n-j}{m}}\, \chi(t) \in t^{-\gamma}\Big(\rho + t\frac{d}{dt}\Big)^{\mu(x_0)} L^2_{(a)} \qquad (4)$$

$\gamma = x_0 - \frac{1}{2}$, $\rho \geqq \gamma$, $q = \left\{ \#\, j : -\frac{n-j}{m} \leqq x_0 \right\}$, $\chi \in C^\infty(\mathbb{R}_+)$, $\chi = 0$ near $t = 0$, $\chi = 1$ for $t \geqq 1$, $\zeta_j = \operatorname{Res}_{z = -\frac{n-j}{m}} \zeta_A$.

Indeed, (2) and (3) imply

$$|\mathrm{Im}\ z|^{-\mu(x_0)+1}\, |M(N)(z)| \leqq c(x_0),$$

$|\mathrm{Im}\ z| \geqq a(x_0)$, $\mathrm{Re}\ z \leqq x_0$, and hence $T^{-\gamma}(\rho+z)^{-\mu(x_0)}(M(N)(z) - \sum_{j=0}^{q} \zeta_j\, f^{\frac{n-j,0}{m}}(z)) \in L^2_{(a)}$, $x_0 = \frac{1}{2} + \gamma$, $\rho \geqq \gamma$. The formula for the coefficients follows from $M(t^p) = (z+p)^{-1}$.

The formula (4) can be interpreted as a substitute of the expansion of $N(t)$ which is usually obtained by a Tauber argument. The poles of the $\zeta$ function do not lead to all terms in the classical precise form, but we always have the asymptotics in the mean, provided the $\zeta$ function is of finite order. So it is a task to find precise bounds for $\mu(x)$.

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