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ON THE WELLPOSED SINGULAR BOUNDARY VALUE PROBLEMS FOR HEAT OPERATOR

by S. MIZOHATA

§ 1 - INTRODUCTION -

In [2] , S. Itô treated the following initial-boundary value problem for heat operator :

$$
\text{(I.B.P.)} \quad \begin{cases}
(1) \quad \dfrac{\partial u}{\partial t} = \Delta u \ , \ \text{in } \Omega \subset \mathbb{R}^n \ , \\[2ex]
(2) \quad Bu = a(x) \dfrac{\partial u}{\partial n} + b(x)u = 0 \ , \ x \in \partial\Omega, \\
\qquad \text{where } \dfrac{\partial u}{\partial n} \text{ is the derivative in the direction of outer-} \\
\qquad \text{normal, and,} \\[2ex]
\qquad a(x) \geqslant 0, \ b(x) \geqslant 0, \ a(x) + b(x) = 1, \\[2ex]
(3) \quad u\big|_{t=o} = u_o(x)
\end{cases}
$$

He proved that this problem is well-posed by constructing explicitly fundamental solution. His method is fairly complicated. Later several authors (cf. [1] , [3] , [4] , [6]), treated the elliptic boundary value problem with the boundary condition (2), using the methods of functional analysis. Fairly recently, M. Terakado pointed out that in the paper of S. Itô, a crucial lemma is not clear. From that time, the author discussed often with him about this problem. Our purpose is to consider what conditions should be imposed on a(x) and b(x) in (2), in order that the problem (I.B.P.) to be wellposed.

To be more precise, we consider the problem under the following situation.

$$
\begin{cases}
\Omega = \{(x,y) \in \mathbb{R}^2 \,|\, y < 0 \ , \ a(x) \dfrac{\partial u}{\partial y} + b(x)u\big|_{y=o} = 0 \ , \\[2ex]
a(x) \text{ and } b(x) \text{ are real-valued ; } a(x)^2 + b(x)^2 = 1 \ ; \\[2ex]
a(x), \ b(x) \in C^{\infty}, \text{ and } \underline{\text{bounded}} \text{ with all their derivatives.}
\end{cases}
$$

We are concerned with $H^\infty(\Omega)$-wellposedness for (I.B.P.), imposing the compatibility conditions. Namely, denoting

(4) $$Bu \equiv a(x) \frac{\partial u}{\partial y} + b(x)u\Big|_{y=o} ,$$

(5) $$B(\Delta^j u_o(x,y)) = 0, \text{ for } j = 0,1,2,\ldots$$

The result is the following.

Theorem. For the above problem to be H^∞-wellposed, the following conditions are necessary and sufficient :

(i) a(x) does not change the sign. We assume therefore $a(x) \geq 0$,

(ii) On the set $\{x;\ a(x) = 0\}$, $b(x) > 0$.

We are concerned here with the necessity. The sufficiency is proved fairly easily. First we observe that, if we put

$$u(.,t) = T_t\, u_o ,$$

T_t is a semi-group. T. Komura obtained the necessary and sufficient condition to the infinitesimal generator in Fréchet spaces [5]. However it seems difficult to apply her method to the actual problem. Instead of that, suggested by this article, we use the truncated Laplace transform :

$$\hat{u}(.,\lambda) = \int_o^1 e^{-\lambda t} u(.,t)dt.$$

§ 2 - PRELIMINARIES -

We assume the problem (I.B.P.) is H^∞-wellposed, and from this assumption some basic facts.

1)- Continuity :

By the assumption of the wellposedness, by Banach, there exist an integer q and the constant C(T) such that

(2.1) $\sup_{t\in[o,T]} ||u(.,t)||_{H^2} \leqslant C(T)||u(.,o)||_{H^{2q}}$.

In this inequality, the initial data should satisfy the compatibility conditions :

(2.2) $B(\Delta^j u(.,o)) = 0 \ , \ 0 \leqslant j \leqslant q-1.$

2)- Truncated Laplace transform :

Let :

(2.3) $\hat{u}(.,\lambda) = \int_o^1 e^{-\lambda t} u(.,t)dt$,

we obtain

(2.4) $(\lambda-\Delta)\ \hat{u}(.,\lambda) = u(.,o) - e^{-\lambda}u(.,1).$

We take u(.,o) in the following form: Let $f_o(x) \in C_o^\infty$, and put

$$u_1(x,y) = (\sum_{j=o}^{q} (\lambda - \frac{d^2}{dx^2})^j \ f_o(x).\ \frac{y^{2j}}{(2j)!})\alpha(g),$$

where $\alpha(y) \in C_o^\infty$, which takes the value 1 in a neighborhood ot the origin.

Then :

$$g(x,g) = (\lambda-\Delta)u_1 = (\lambda - \frac{d^2}{dx^2})^{q+1} f_o(x).\ \frac{y^{2q}}{(2q)!},$$

in a neighborhood of the origin. g satisfies

$B(\Delta^j g) = 0 \ (0 \leqslant j \leqslant q-1)$, and

(2.5) $B\ u_1 = -\ b(x)\ f_o(x).$

Observe that g is determined by $f_o(x)$. We take

$f_o(x) \in H^{4q+2}$.

We put $u(.,o) = g(x,y)$. Then by hypothesis, there exists a unique solution $u(.,t)$ with initial data g. Thus $\hat{u}(.,\lambda)$ is defined by (2.3). Since $(\lambda-\Delta)u_1 = u(.,o)$, from (2.4) we get

$$(\lambda-\Delta)\ (\hat{u}(.,\lambda) - u_1) = - e^{-\lambda}\ u(.,1).$$

Moreover, since $B\hat{u}(.,\lambda) = 0$, we obtain

$$B(\hat{u}-u_1) = - Bu_1 = - b(x)\ f_o(x).$$

We proceed further. Let $u_D(x,y)$ be the solution of

$$\begin{cases} (\lambda-\Delta)u_D = u(.,1) \\ u_D\big|_{y=o} = 0. \end{cases}$$

Then denoting

$$v(x,y) = \hat{u}(.,\lambda) - u_1(x,y) - e^{-\lambda}u_D \tag{2.6}$$

$$\begin{cases} (\lambda-\Delta)\ v(x,y) = 0 \\ B\ v = b(x)\ f_o(x) + e^{-\lambda}\ \tilde{f}(x)\ , \end{cases} \tag{2.7}$$

where $\tilde{f}(x) = a(x)\ \frac{\partial}{\partial y}\ u_D\big|_{y=o}$.

<u>3) - Functional equation on the boundary</u> :

If we use the Poisson representation of $v(x,y)$,

$$v(x,y) = \frac{1}{2\Pi} \int_{-\infty}^{\infty} e^{ix\xi}\ e^{\sqrt{\lambda+\xi^2}\ y}\hat{v}(\xi,o)d\xi\ ,$$

putting $v(x,o) = v(x)$, the above equation can be written

$$A(x,D)\ v(x) = (a(x)\ \sqrt{\lambda+D^2} + b(x))\ v(x) = f(x) + e^{-\lambda}\ \tilde{f}(x) \tag{2.8}$$

where $f(x) = b(x)\ f_o(x)$.

We assume hereafter $b(x) \neq 0$, in a neighborhood of the origin, and $f_o(x)$ has its support in a small neighborhood of the origin. We obtain :

Proposition 1 :

$$\tilde{A}(x,D) = a(x) + b(x)\,\sqrt{\lambda+D^2}^{\,-1} \equiv a(x) + b(x)\,\Lambda^{-1}. \tag{2.9}$$

Then for any $f(x) \in H^{4q+2}$, there exists a solution $w(x) \in H^{\frac{1}{2}}$

$$\tilde{A}(x,D)\,w(x) = f(x) + e^{-\lambda}\,\tilde{f}(x)\ , \tag{2.10}$$

satisfying the following type inequalities.

$$||w(x)||_{\frac{1}{2}}\ ,\ ||\tilde{f}(x)||_{\frac{3}{2}} \leqslant \text{const. } ||f(x)||_{4q+2}.$$

Remark :

Hereafter we denote $||.||_s = ||.||_{H^s}$, and

$$||f(x)||_s^2 = \int (\lambda+\xi^2)^s\ |\hat{f}(\xi)|^2\ d\xi.$$

§ 3 - ALMOST NULL SOLUTIONS -

For simplicity, we consider the problem under the following assumptions :

$$\begin{cases} \text{(i)} & a(x) = x^2 \text{ in a neighborhood } V \text{ of the origin,} \\ \text{(ii)} & b(x) = -1,\ x \in \mathbb{R}. \end{cases} \tag{3.1}$$

We consider a solution $\psi_o(x)$ of the following equation

$$(x^2 - \Lambda^{-1})\ \psi_o(x) = 0. \tag{3.2}$$

By taking the Fourier transformation,

$$\left(\frac{d^2}{d\xi^2} + (\lambda + \xi^2)^{-\frac{1}{2}}\right)\ \hat{\psi}_o(\xi) = 0. \tag{3.3}$$

We see that, there exists the solution satisfying

$$(3.4)\qquad \begin{cases} |\hat{\psi}_o(\xi)| \sim \lambda^{-\frac{1}{8}} (\lambda + \xi^2)^{-\frac{1}{8}}, \ (\lambda \to \infty) \\ |(\frac{d}{d\xi})^p \hat{\psi}_o(\xi)| \leqslant C_p(\lambda + \xi^2)^{-\frac{1}{8} - \frac{1}{8}p} \lambda^{-\frac{1}{8}} \end{cases}$$

Observe that :

$$(3.5)\qquad \begin{cases} \psi_o(x) \in H^{-\delta} \ (\delta > \frac{3}{4}) \\ ||\Lambda^{-2} \psi_o||_1 \sim C_1 \lambda^{-\frac{1}{4}} \ (C_1 \text{ : positive constant}) \end{cases}$$

Take a $\beta(x) \in C_o^\infty(V)$, which takes the value 1 in a neighborhood of the origin. From (3.2), it follows

$$(a(x)-\Lambda^{-1})(\beta\psi_o) - [\beta,\Lambda^{-1}]\psi_o = 0.$$

Let us introduce the symbol "≈ 0". $f \approx 0$ means that, for any k and any p,

$$||\Lambda^k f||_o \leqslant C_{kp}/\lambda^p \text{ when } \lambda \text{ is large.}$$

From the property (ii) of (3.4), it follows

$$[\beta,\Lambda^{-1}]\psi_o \approx 0 \ , \ [a(x)-\Lambda^{-1}](1-\beta(x))\psi_o \approx 0.$$

Therefore, it holds

$$(3.6)\qquad (a(x)-\Lambda^{-1})\psi_o(x) \approx 0.$$

Following the notation (2.9), we denote $\tilde{A} = a(x)-\Lambda^{-1}$. Using the results of proposition 1, we get.

<u>Proposition 2</u> :

<u>For $f(=\Lambda^{-2}\psi_o) \in H^1$, there exists an element $w \in H^1$, satisfying</u>

(i) $||\tilde{A}w - f||_1 \leqslant \frac{1}{2}\,||f||_1$,

(ii) <u>There exist positive constant k_o and C such that</u>

$$||w||_1 \leqslant C\,\lambda^{k_o} \quad \text{(if } \lambda \text{ is large).}$$

Now it is easy to see that these properties are <u>not compatible</u> with (3.6), which shows, in the case (3.1). The problem (I.B.P.) is not well-posed.

Infact, in view of (3.5) :

$$\text{(3.7)} \qquad \mathrm{Re}(\tilde{A}w,\Lambda^{-2}\psi_o)_1 \geqslant \frac{1}{2}\,||\Lambda^{-2}\psi_o||_1^2 \sim \frac{1}{2}\,C_1^2\,\lambda^{-\frac{1}{2}}$$

On the other hand, the left-hand side is equal to

$$\mathrm{Re}\ \langle\tilde{A}w,\overline{\psi}_o\rangle = \mathrm{Re}\ \langle w, \overline{\tilde{A}\psi_o}\rangle.$$

This is estimated from above by $||w||_1\ ||\tilde{A}\psi_o||_{-1}$. By the property of w, and (3.6), we see that, this last quantity is estimated of the form $C'_p\,\lambda^{-p}$ for any p. This is not compatible with (3.7).

<u>§ 4 - PROOF OF PROPOSITION 2</u> -

We define $\tilde{f}, w_o$, w in the following order

$$f \to \tilde{f} \to w_o \to w$$

$$\text{(4.1)} \qquad \begin{cases} \text{(i)} & \tilde{f} = \rho_\delta * f \\ \text{(ii)} & \tilde{A}\,w_o = \tilde{f} + e^{-\lambda} f_1 \\ \text{(iii)} & w = \rho_\varepsilon * w_o \end{cases}$$

where ρ_δ, ρ_ε are mollifiers.

(i) $\qquad ||f - \tilde{f}||_1 \leqslant C\delta^s||f||_{1+s}$ $(o<s<\frac{1}{4})$. We suppose here $\delta \leqslant \lambda^{-\frac{1}{2}}$;

C is independent of δ. Since $||f||_{1+s} \sim C\,\lambda^{\frac{s}{2}}||f||_1$, we take $\delta = \varepsilon_o\,\lambda^{-\frac{1}{2}}$ (ε_o being a small constant). This, are have

(4.2) $$||f-\tilde{f}||_1 \leqslant \frac{1}{6}||f||_1.$$

(ii) Observe $\tilde{f} \in H^{\infty}$, and

$$||\tilde{f}||_{4q+2} \leqslant \text{cont. } \delta^{-(4q+2)}||f||_1 \leqslant \text{cont. } \lambda^{2q+1}||f||_1\ ,$$

(since $\delta = \varepsilon_o\,\lambda^{-\frac{1}{2}}$).

By proposition 1, there exists $w_o \in H^{\frac{1}{2}}$ such that

$$\tilde{A}\,w_o = \tilde{f} + e^{-\lambda}f_1\ ,$$

where

$$||w_o||_{\frac{1}{2}} \leqslant \text{const. } ||\tilde{f}||_{4q+2} \leqslant \text{const. } \lambda^{2q+1}||f||_1\ ,$$

$$||f_1||_{\frac{s}{2}} \leqslant \text{const. } ||\tilde{f}||_{4q+2} \leqslant \text{const. } \lambda^{2q+1}||f||_1.$$

Observe that $||\tilde{A}\,w_o - \tilde{f}||_1 = e^{-\lambda}||f_1||_1 \leqslant \text{const. } e^{-\lambda}\lambda^{2q+1}||f||_1$, which is negligible.

(iii) $$f - \tilde{A}\,w = (f-\tilde{f}) + (\tilde{f} - \tilde{A}\,w_o) + (\tilde{A}\,w_o - \tilde{A}\,w).$$

We are concerned with the last term. Observe $\tilde{A}\,w_o \in H^{\frac{3}{2}}$, and

(4.3) $$||\tilde{A}\,w_o||_{\frac{3}{2}} \leqslant ||\tilde{f}||_{\frac{3}{2}} + e^{-\lambda}||f_1||_{\frac{3}{2}}\ ,$$

where the last term is negligible when $\lambda \to \infty$. Now,

$$||\tilde{f}||_{\frac{3}{2}} \leqslant \text{const. } \frac{1}{\sqrt{\delta}}\,||f||_1 \leqslant \text{const. } \lambda^{\frac{1}{4}}\,||f||_1.$$

$$\tilde{A}\, w_o - \tilde{A}(\rho_\varepsilon * w_o) = (\tilde{A}\, w_o - \rho_\varepsilon * \tilde{A}\, w_o) + (\rho_\varepsilon * \tilde{A}\, w_o - \tilde{A}(\rho_\varepsilon * w_o)).$$

The first term is estimated by

$$||\tilde{A}\, w_o - \rho_\varepsilon * (\tilde{A}\, w_o)||_1 \leqslant \text{const. } ||\tilde{A}\, w_o||_{\frac{3}{2}} \sqrt{\varepsilon}.$$

By virtue of (4.5), this is estimated, when λ is large, by

$$\text{const. } (\lambda^{\frac{1}{4}} \sqrt{\varepsilon})\ ||f||_1.$$

Now put, $\varepsilon = \varepsilon_1' \lambda^{-(4q+2)}$, thus

$$||[\rho_\varepsilon * , \tilde{A}]\ w_o||_1 \leqslant \frac{1}{6}\ ||f||_1 ,$$

which implies

$$||\tilde{A}\, w_o - \tilde{A}\, w||_1 \leqslant (\frac{1}{6} + \text{const. } \lambda^{-2q})||f||_1,$$

which completes the proof of proposition 2.

§ 5 - FINAL REMARKS -

In the case when a(x) has a finite order of zero at the origin, we can argue essentially in the same way. However the technical complication arises when a(x), and b(x) are general. In the case when a(x) has a infinite order of zero at the origin, we can argue in a fairly different way.

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REFERENCES :

[1] : K. HAYASHIDA : "On the singular boundary value problem for elliptic equations". Trans. Amer. Math. Soc. 184 (1973) p. 205-221.

[2] : S. ITÔ : "Fundamental solutions of parabolic differential equations and boundary value problems". Jap. J. Math. 27 (1957), 55-102.

[3] : A. KAJI : "On the degenerate oblique derivative problems". Proc. Japan Acad. 50(1974), p. 1-5.

[4] : Y. KATO : "Another approach to a non-elliptic boundary problem". Nagoya Math. J. 66(1976), p. 13-22.

[5] : T. KOMURA : "Groups of operators in locally convex spaces". J. Functional Anal. 7 (1968), p. 258-296.

[6] : K. TAIRA : "Sur le problème de dérivée oblique I". J. Math. Pure Appl. 57 (1978), p. 379-395.