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ANALYTIC APPROXIMATION FOR HOMOGENEOUS SOLUTIONS OF LINEAR PDE's.

by M.S. BAOUENDI

Let P(x,D) be a differential operator with analytic coefficients in an open set of $\mathbb{R}^n$. Assume that the principal symbol of P is nowhere identically zero. It is natural to ask the following question :

Is it true that any distribution solution of P(x,D)u = 0 is locally a limit of real analytic solutions of the same equation?

The answer to this question is not known. However an affirmative answer is given in Baouendi-Trêves [3] (see also [2] for first order overdetermined systems) when P has simple (complex) characteristics. An affirmative answer is also given in Baouendi-Rothschild [1] when P is a left invariant operator defined on a general Lie group. Detailed proofs could be found in [1] and [3].

First we state the result obtained in [3]. Denote by t the variable in $\mathbb{R}$, by x the on in $\mathbb{R}^n$. Let Ω be an open set in $\mathbb{R} \times \mathbb{R}^n$ containing the origin. We consider a first order linear differential operator ot the form

$$L = I_N \frac{\partial}{\partial t} - \sum_{j=1}^{n} A_j(t,x) D_{x_j} - A_o(t,x),$$

where A_j are real-analytic in Ω valued in the space of complex N x N matrices, and I_N is the identity matrix. Set

$$a(t,x,\xi) = \sum_{j=1}^{n} A_j(t,x)\xi_j .$$

We assume that for every $(t,x,\xi) \in \Omega \times (\mathbb{R}^n \setminus \{0\})$ the matrix $a(t,x,\xi)$ has N distinct eigenvalues $\lambda_j(t,x,\xi)$, j = 1,...,N.

Theorem 1 :

Let $h \in \mathcal{D}'(\Omega')$, $0 \in \Omega' \subset \Omega$, satisfying Lh = 0. There exist an *open neighborhood of 0, $\Omega'' \subset \Omega'$, and a sequence of analytic functions h_ν in Ω'' satisfying :*

(i) $\quad L\, h_\nu = 0 \quad$ *in* Ω''

(ii) $\quad \lim h_\nu = h$ *in* $\mathcal{D}'(\Omega'')$.

Furthermore if h is of class C^k, then the convergence in (ii) is in $C^k(\Omega'')$.

Now we state the result in [1].

Theorem 2 :

Let P be a left invariant differential operator defined on a Lie group G. For every open set $U \subset G$, neighborhood of the identity $e \in G$, there exists another open neighborhood of e, $W \subset G$, such that if u is a distribution on G satisfying $Lu = 0$ in U, then there exists a sequence u_ν of real analytic functions defined in W and satisfying

(i) $\quad L\, u_\nu = 0 \quad$ *in* W,

(ii) $\quad \lim u_\nu = u$ *in* $\mathcal{D}'(W)$.

Furthermore if u is of class C^k, then the convergence in (ii) is in $C^k(W)$.

We sketch now the proof if theorem 1 in the case of a single complex vector field, i.e. $N = 1$. Set

$$L = \frac{\partial}{\partial t} - \sum_{j=1}^{n} a_j(t,x) \frac{\partial}{\partial x_j} ,$$

where a_j are analytic functions in $\Omega = I \times U$, $0 \in I \subset \mathbb{R}$, $0 \in U \subset \mathbb{R}^n$. Let $h \in C^1(\Omega)$, $L\, h = 0$, and $g \in C_0^\infty(U)$, $g \equiv 1$ near the origin in $\mathbb{R}^n$. Set

$$u(t,x) = g(x)\ h(t,x) \quad , \ L\, u = f.$$

Note that f vanishes in a neighborhood of $x = 0$ for all $t \in I$.

For $j = 1,\dots,n$, denote by $Z_j(t,x)$ the solution of the Cauchy problem

$$L\,Z_j = 0 \qquad Z_j\Big|_{t=0} = x_j\ ,$$

and set $Z = (Z_1,\dots,Z_n)$.

For $\nu \in \mathbb{Z}_+$ define the operator K_ν by

(1) $$(K_\nu u)(t,x) = \left(\frac{\nu}{\sqrt{\Pi}}\right)^n \int_{\mathbb{R}^n} e^{-\nu^2[Z(t,x)-Z(t,y)]^2} \det(Z'_y(t,y))u(t,y)dy.$$

The operator K_ν has the following properties :

(a) $K_\nu(L\,u) = L(K_\nu u)$

(b) $\lim_{\nu\to\infty} K_\nu u = u$ uniformly in a fixed neighborhood of the origin in $\mathbb{R}^{n+1}$

(c) $K_\nu f$ extends holomorphically in x to a fixed neighborhood of the origin in $\mathbb{C}^n$, and there converges to 0.

Assuming (a) (b) and (c), set

$$h_\nu = K_\nu u - v_\nu$$

where v_ν is the solution of

$$L\,v_\nu = K_\nu f \qquad v_\nu\Big|_{t=0} = 0.$$

It follows from (c) thas $\lim_{\nu\to\infty} v_\nu = 0$. Therefore (a) and (b) imply that we have (i) and (ii) of the conclusion of theorem 1.

Q.E.D.

Note that the operator K_ν defined by (1) can be written

(2) $$(K_\nu u)(t,x) = \frac{1}{(2\Pi)^n}\iint e^{i[Z(t,x)\xi-Z(t,y)\xi]-\varepsilon|\xi|^2} \det(Z'_y(t,y))u(t,y)dyd\xi.$$

We limit ourselves to mention that the proof of theorem 1 in the general case (i.e. $N > 1$) is done by reducing the system L to a diagonal one, at least microlocally. Operators similar to (2) are introduced, where $Z(t,x)\xi$ is replaced by $\varphi(t,x,\xi)$ satisfying

$$\partial_t \varphi - \lambda(t,x,\partial_x \varphi) = 0 \ ,$$

$$\varphi|_{t=0} = x.\xi \ ,$$

λ stands for one of the eigenvalues of the matrix a. The exponential function in (2) is multiplied by analytic amplitudes determined by geometrical optics.

The proof of theorem 2 is based on the use of convolution with a suitable Gaussian defined near $e \in G$, and the use of the Campbell-Hansdorff formula in order to prove a result similar to (c) above.

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