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AN EXAMPLE OF THE HEISENBERG GROUP

by E. M. STEIN

Let us consider the Kohn-Laplacian  $\square_b^{(q)}$  which acts on appropriate  $q$ -forms on the Heisenberg group  $H_n$ . Then as Kohn showed (more generally),  $\square_b^{(q)}$  is  $C^\infty$  hypoelliptic and locally solvable, when  $0 < q < n$ ; however this is not the case when  $q = 0$ , or  $q = n$ , and the latter fact goes back to the fundamental example of Lewy.

If we introduce the coordinates  $(z, t)$   $z \in \mathbb{C}^n$ ,  $t \in \mathbb{R}^n$ , and the complex vectorfields

$$z_j = \frac{\partial}{\partial z_j} + i \bar{z}_j \frac{\partial}{\partial t}, \quad j = 1, \dots, n$$

then one can write

$$\square_b^{(q)} = \mathcal{L}_\alpha \otimes I, \quad \text{with} \quad \alpha = n - 2q$$

$$\mathcal{L}_\alpha = -\frac{1}{2} \sum_j (z_j \bar{z}_j + \bar{z}_j z_j) + i\alpha \frac{\partial}{\partial t},$$

and there is an explicit fundamental solution, given by

$$\psi_\alpha c_\alpha^{-1}, \quad c_\alpha = \frac{c}{\Gamma(\frac{n-\alpha}{2}) \Gamma(\frac{n+\alpha}{2})}$$

$$\text{with} \quad \psi_\alpha = (|z|^2 - it)^{-(n+\alpha)/2} (|z|^2 + it)^{-(n-\alpha)/2}.$$

All of this holds for  $\alpha \neq \pm n, n+2, \dots$  and incidentally shows the analytic hypoellipticity of  $\square_b^{(q)}$ ,  $0 < q < n$ , on the Heisenberg group (see [1]).

Our first question, is what happens when  $q = 0$  (i.e.  $\alpha = n$ )? These is then ([2]) a relative fundamental solution  $\tilde{\psi}$  so that

$$\square_b \tilde{\psi} = I - \mathcal{L}$$

where  $\xi$  is the projection operator on the null space of  $\square_b^0$  (i.e. the Cauchy-Szegö projector) and  $\tilde{\chi}$  has a description similar to the above fundamental solutions.

Finally what happens to  $\square_b^0 + \mu I$ ? (From now on, for simplicity of notation  $\square_b = \square_b^0$ , and  $n = 1$ ).

Theorem : Suppose  $\mu \neq 0$ , the operator  $\square_b + \mu I$  is locally solvable,  $C^\infty$  and analytic hypoelliptic.

The  $C^\infty$  hypoellipticity was already observed by Melin

The idea will be to construct a parametrix (analytic away from the diagonal).

The formal solution to our problem is

$$(\square_b + \mu I)^{-1} = \xi/\mu + \sum_{n=1}^{\infty} (-\mu)^{n-1} \tilde{\chi}^n.$$

The main difficulty is to give a meaning to this infinite series, and prove the appropriate properties of the kernel it represents. This requires some definitions.

$$\text{Let } E(u) = \sum_{n \geq 3}^{\infty} \frac{u^n}{n!(n-3)!}.$$

This is a Bessel function. What is important for us is that  $|E(u)| \leq e^{C|u|^{1/2}}$ ,  $u$  complex. Next let  $w = |z|^2 - it$ . Write

$$P_\mu = C/\mu + K^1 - \mu K^2 + \frac{\bar{w}}{2\mu \pi^2} \int_0^\infty E\left\{2\mu(w+\bar{w}s) \frac{\log s}{s-1}\right\} \log(w+\bar{w}s) \times \frac{s-1}{(w+\bar{w}s)^3} ds$$

$$\text{with } C \text{ the Cauchy-Szegö kernel} = \frac{1}{\pi^2 w^2}$$

$$K = \frac{1}{2\pi} (\log w - \log \bar{w}) w^{-1},$$

$$\text{and } K^2 = \frac{1}{\pi^2} \bar{w} \int_0^\infty (w + \bar{w}s)^{-1} \frac{\log s}{s-1} ds.$$

Proposition : (1)  $P_\mu$  is real-analytic away from the origin  
 (2)  $(\square_b + \mu I)P_\mu = S + R_\mu$ , where  $R_\mu$  is everywhere real-analytic,  
 with  $\delta$  = Dirac function of the origin.

The proof of the proposition is somewhat complicated; its details will appear elsewhere [3].

Bibliography

- [1] Folland and Stein : Comm. Pure and Applied Math. 27 (1974), 429-522.
- [2] Greiner, Kohn and Stein : Proc. Nat. Acad. Sci. 72 (1975), 3287-3289.
- [3] To appear.

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