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PSEUDO-DIFFERENTIAL OPERATORS WITH SYMBOLS ADMITTING NEGATIVE VALUES

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Let $a(x,\xi) \in S^2(\mathbb{R}^n \times \mathbb{R}^n)$ be a real symbol of order 2 and A be the associated pseudo-differential operator. In this article we shall consider the problem of finding conditions on $a(x,\xi)$ which would imply that A is positive, i.e., satisfies an inequality of the form

$$\mathrm{Re}\langle Au,u\rangle \geqslant 0 \tag{1}$$

for all $u \in C_0^\infty(\mathbb{R}^n)$.

Such estimates are interesting for many reasons, of which we shall mention two :

(a) If $A_t(x,D)$ is a pseudo-differential operator in x depending on a parameter t and $u(x,t)$ is a solution of the evolution equation

$$\frac{du}{dt} + A_t(x,D)u = 0$$

positivity of $A_t(x,D)$ would allow us to control the size of u, since

$$\begin{aligned} d\|u(t)\|^2/dt &= 2\mathrm{Re}\langle du/dt,u\rangle \\ &= -2\mathrm{Re}\langle A_t(x,D)u,u\rangle \leqslant 0. \end{aligned}$$

(b) If $L_1,\dots,L_N$ are pseudo-differential operators of order 1, then the subelliptic estimate

$$\sum_{j=1}^{N} \|L_j u\|^2 \geqslant c\|u\|^2_{(\varepsilon)} \qquad \varepsilon > 0 \tag{2}$$

is equivalent to the positivity of the operator

$$A = \sum_{j=1}^{N} L_j^* L_j - c(I - \Delta)^{\varepsilon}. \tag{3}$$

In fact (b) is just one application of the general and simple observation that any estimate can be reduced to a positivity result, and hence we see immediately the value of good criteria for positivity.

The symbol $a(x,\xi)$ when A is the operator in (3) will usually take negative values ($a(x,\xi)$ can be considered to be real if we neglect in its expansion the purely imaginary symbols of order 1 and the symbols of order 0). In some cases, for example when the symbols of L_j are all real, one can avoid the study of negative symbols by microlocalizing (2) to a cube of size $1 \times M$ in phase space and view (3) as equivalent to saying that the first eigenvalue of the operator $L_o = \sum_{j=1}^{N} L_j^* L_j$ (which has a positive symbol) is always greater than $cM^{2\varepsilon}$. The results of [3] [5] (see also (b) below) will then apply. However, if $N = 1$ and $\mathrm{Symbol}(L_1) = p + iq$ the symbol of L_o is $p^2 + q^2 + \{p,q\}$ and already will not always remain positive. Thus in general we shall have to deal directly with negative symbols.

With these applications in view, it is evident that it suffices in fact to establish inequalities of the form

$$\mathrm{Re}\langle Au,u\rangle \geqslant -C_\delta \| u\|^2_{(\delta)} \tag{4}$$

for some δ, $0 \leqslant \delta \ll 1$. In [7], Hörmander established (4) with $\delta = 0$ for $a \in S^{6/5}$ satisfying the condition $a + \frac{1}{2}\mathrm{Trace}^+ a'' \geqslant 0$. $\mathrm{Trace}^+ a''$ is a positive quantity associated to the Hessian of a which was introduced earlier by Melin [8] and which reduces to $2\sum_{j=1}^{n} \lambda_j \mu_j$ if $a(x,\xi) = \sum_{j=1}^{n} \lambda_j^2 \xi_j^2 + \mu_j^2 x_j^2$. Here we shall look instead for conditions not on the Taylor coefficients of $a(x,\xi)$ but rather on the symplectic geometry of the set $S_K = \{(x,\xi) \in T^*(\mathbb{R}^n)\ ;\ a(x,\xi) < K\}$. The uncertainty principle and the theorem of Egorov indicate that to each canonically twisted cube (i.e., the image $\phi(Q_o)$ of the unit cube $Q_o = \{|x| \leqslant 1, |\xi| \leqslant 1\}$ by a canonical transformation ϕ) contained in S_K, corresponds roughly an eigenstate of A with eigenvalue $< K$. In the case of positive symbols, this heuristic principle is largely justified and we now know that under this condition

(a) $\mathrm{Re}\,\langle Au,u\rangle \geqslant -C\|u\|^2$

(b) $\mathrm{Re}\,\langle Au,u\rangle \geqslant c\lambda \|u\|^2$ if $\lambda = \min_{\phi}(\max_{(x,\xi)\in\phi(Q^o)} a(x,\xi))$

is large ;

(c) The number of eigenvalues $< K$ can be estimated in terms of the number of canonically twisted cubes disjointly imbedded in S_{CK}.

For precise statements we refer to the original articles [2] [3] [4] and to the more detailed exposition in [5].

When $a(x,\xi)$ takes negative values, the above considerations lead naturally to the following conjecture :

<u>Conjecture</u> : Given $\delta > 0$ there exists a constant c_δ such that

$$\mathrm{Re}\,\langle Au,u\rangle \geqslant -C\|u\|^2_{(\delta)}$$

if $a(x,\xi)$ satisfies with $\varepsilon = c_\delta$

<u>Condition</u> $(+)_\varepsilon$: The set $S_o = \{(x,\xi) \in T^*(\mathbb{R}^n) \,;\, a(x,\xi) < 0\}$ does not contain the image of the cube $Q_\varepsilon = \{(x,\xi) \in \mathbb{R}^n \times \mathbb{R}^n \,;\, |x|,|\xi| < \varepsilon\}$ by any canonical transformation.

<u>Remarks</u> : It is of course advantageous to have ε as large as possible. By modifying the size of ε it is not difficult to see that Condition $(+)_\varepsilon$ is basically equivalent to the condition formulated in the last section of [3]. The strength of the conjecture can be assessed from the fact that essentially Assertion (b) above would follow easily by letting $a(x,\xi) = b(x,\xi) -$ (small constant) $(\min_\psi \; \max_{(x,\xi)\in\phi(Q)} b(x,\xi))$ if $b(x,\xi)$ is a positive symbol. Finally it should be noted that in fact not all canonical transformations need be considered in Condition $(+)_\varepsilon$, but only those with certain bounds.

<u>Theorem</u> : The conjecture is true when $n = 2$.

The proof of the theorem is too lengthy to be described here in any detail. We shall just sketch some important steps in a very simple case. First observe that the result can be microlocalized to a cube of size $1 \times M$ and that $a(x,\xi)$ may be assumed to be a polynomial of fixed degree since errors of the form $C_\delta\|u\|^2_{(\delta)}$ are negligible; next apply the cutting and stopping procedures of [3] (this can be done in view of the $S^{M,m}_{\phi,\varphi}$ calculus of pseudo-differential operators of Beals and Fefferman [1]). Since S_o does not admit any canonically twisted cube it follows that the cubes $\{Q_\nu\}$ thus obtained still fall into three categories :

(1) Either $a(x,\xi) \geqslant c(\mathrm{diam}_x Q_\nu)^2(\mathrm{diam}_\xi Q_\nu)^2$ for $(x,\xi) \in Q_\nu$

(2) or $\displaystyle\max_{|\alpha|+|\beta| = 2} \|D^\alpha_x D^\beta_\xi a\|_{L^\infty(Q_\nu)} \geqslant c(\mathrm{diam}_x Q_\nu)^{2-|\alpha|}(\mathrm{diam}_\xi Q_\nu)^{2-|\beta|}$

and (1) is not satisfied ;

(3) or $(\mathrm{diam}_x Q_\nu)^2(\mathrm{diam}_\xi Q_\nu)^2 \sim 1$.

The cases (1) and (3) are easy to handle. The implicit function theorem and conjugation with Fourier integral operators reduce the second case to the case when

$$a(x,\xi) = \xi_1^2 + V(x,\xi') \qquad \xi' = (\xi_2,\dots,\xi_n) .$$

Here $V(x,\xi')$ is in general a pseudo-differential operator of order 2 where the first variable x_1 acts only as a parameter. The simple situation we shall study arises when $n = 2$ and $V(x,\xi')$ is a differential operator without first order terms. Changing the notation from $(x,\xi) \in \mathbb{R}^2 \times \mathbb{R}^2$ to $(t,x;\tau,\xi) \in \mathbb{R}^2 \times \mathbb{R}^2$ we may then write $a(x,\xi)$ as

$$a(x,\xi) = \tau^2 + p(t,x)\xi^2 + V(t,x)$$

Then $p(x,t)$ and $V(x,t)$ are polynomials of a fixed degree d and $p(t,x) \geqslant 0$ in view of Condition (+). It will suffice to show that the quadratic form

$$Q_R(u) = \iint_{I_t \times I_x} |\frac{\partial u}{\partial t}|^2 \, dxdt + \iint_{I_t \times I_x} p(x,t)\,|\frac{\partial u}{\partial x}|^2 dxdt + \iint_{I_t \times I_x} V(x,t)dxdt$$

is positive (up to admissible errors) for all rectangles $R = I_t \times I_x \subset \mathbb{R}^2$ of a fixed size, say 1×1.

We introduce the following terminology

(1) A rectangle $I_t \times I_x$ will be said to be "natural" if

$$|I_t|^{-2} \sim \|p\|_{L^\infty(I_t \times I_x)} |I_x|^{-2}$$

(2) A "stopping" rectangle $I_t \times I_x$ is a natural rectangle which satisfies in addition the condition

$$\|V\|_{L^\infty(I_t \times I_x)} \sim |I_t|^{-2} . \tag{5}$$

Consider now a fixed rectangle R of sizes 1×1. Unless $a(x,t) \equiv 0$, R can be decomposed into a disjoint union of natural rectangles $I_t \times I_x$ with $|I_t| = 1$ since $a(x,t)$ can vanish identically in t only for a finite number of values of x. Next observe that for a small natural rectangle $I_t \times I_x$

(1) the maximum of $p(t,x)$ on each of the two rectangles obtained by cutting I_t in two decreases at most by a multiplicative constant depending on d ;

(2) $p(t_o,x)$ remains of the same size for all $x \in I_x$ if $p(t_o,x_o) \sim \|p\|_{L^\infty(I_t \times I_x)}$ for some x_o. Indeed the size of $p(t,x)$ does not change in $|x - x_o| \leqslant \|p\|^{1/2}_{L^\infty(I_t \times I_x)}$ since $p(t,x)$ is positive and $\|p\|^{1/2}_{L^\infty(I_t \times I_x)} \sim |I_x|/|I_t| \geqslant |I_x|$

for $|I_t|$ small since $I_t \times I_x$ is natural.

To obtain stopping rectangles it thus suffices to keep cutting I_t in two and I_x in a bounded number of equal intervals, and among the new rectangles retain those for which $\|V\|_{L^\infty(\text{Rectangle})} \leqslant |I_t|^{-2}$.

Finally we note that for any fixed $\omega > 0$ Condition $(+)_\varepsilon$ with ε small enough will imply that

$$\min_{I_t \times I_x} V(t,x) \geqslant -\omega |I_t|^{-2} \tag{6}$$

if $I_t \times I_x$ is a stopping rectangle.

Now given a stopping rectangle $I_t \times I_x$ let $I_t^o \times I_x \subset I_t \times I_x$ be a rectangle with $|I_t^o|/|I_t| \sim 1$ and $p(t,x) \sim \|p\|_{L^\infty(I_t \times I_x)}$ for all $(t,x) \in I_t^o \times I_x$;

then the presence of $\|D_t u\|^2_{L^2(I_t^o \times I_x)}$ and $\|p\|_{L^\infty(I_t \times I_x)} \|D_x u\|^2_{L^2(I_t^o \times I_x)}$ in the quadratic form $Q_{I_t^o \times I_x}(u)$ implies that $Q_{I_t^o \times I_x}(u)$ is bounded below by

$$c_d' \left(\iint_{I_x I_t^o} \min\{c_d |I_t|^{-2}, V(x,t)\} dxdt\right) \|u\|^2_{L^2(I_t^o \times I_x)}$$

and thus by

$$c_d'' |I_t|^{-2} \|u\|^2_{L^2(I_t^o \times I_x)}$$

in view of (5) and (6). But the positivity of $Q_{I_t \times I_x}(u)$ follows easily, since $V(x,t)$ is not that negative and the L^2 norm of u on $(I_t \setminus I_t^o) \times I_x$ can be controlled by $\|u\|^2_{L^2(I_t^o \times I_x)}$ and $\|D_t u\|^2_{L^2(I_t \times I_x)}$.

This completes the sketch of the proof of the simple case $\tau^2 + p(t,x)\xi^2 + V(t,x)$. More precise statements and complete proofs for the conjecture when $n = 2$ will appear elsewhere.

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