INFORMATIQUE THÉORIQUE ET APPLICATIONS

HANS-JOACHIM BÖCKENHAUER
SEBASTIAN SEIBERT

Improved lower bounds on the approximability of the traveling salesman problem

Informatique théorique et applications, tome 34, nº 3 (2000), p. 213-255

<http://www.numdam.org/item?id=ITA_2000__34_3_213_0>

Theoretical Informatics and Applications
Theoret. Informatics Appl. **34** (2000) 213–255

IMPROVED LOWER BOUNDS ON THE APPROXIMABILITY OF THE TRAVELING SALESMAN PROBLEM

HANS-JOACHIM BÖCKENHAUER[1] AND SEBASTIAN SEIBERT[1]

Abstract. This paper deals with lower bounds on the approximability of different subproblems of the Traveling Salesman Problem (TSP) which is known not to admit any polynomial time approximation algorithm in general (unless $\mathcal{P} = \mathcal{NP}$). First of all, we present an improved lower bound for the Traveling Salesman Problem with Triangle Inequality, Δ-TSP for short. Moreover our technique, an extension of the method of Engebretsen [11], also applies to the case of relaxed and sharpened triangle inequality, respectively, denoted Δ_β-TSP for an appropriate β. In case of the Δ-TSP, we obtain a lower bound of $\frac{3813}{3812} - \varepsilon$ on the polynomial-time approximability (for any small $\varepsilon > 0$), compared to the previous bound of $\frac{5381}{5380} - \varepsilon$ in [11]. In case of the Δ_β-TSP, for the relaxed case ($\beta > 1$) we present a lower bound of $\frac{3803+10\beta}{3804+8\beta} - \varepsilon$, and for the sharpened triangle inequality ($\frac{1}{2} < \beta < 1$), the lower bound is $\frac{7611+10\beta^2+5\beta}{7612+8\beta^2+4\beta} - \varepsilon$. The latter result is of interest especially since it shows that the TSP is $\mathcal{APX}$-hard even if one comes arbitrarily close to the trivial case that all edges have the same cost.

AMS Subject Classification. 68Q25, 68R10.

1. INTRODUCTION

The Traveling Salesman Problem (TSP) is one of the hardest optimization problems in $\mathcal{NPO}$. (Recall that $\mathcal{NPO}$ is the class of optimization problems whose underlying threshold languages belong to $\mathcal{NP}$. For a formal definition of the class $\mathcal{NPO}$ see *e.g.* [15].) It is intractable (provided $\mathcal{P} \neq \mathcal{NP}$) in general because it does not admit any polynomial time $p(n)$-approximation algorithm for any polynomial

Keywords and phrases: Approximation algorithms, Traveling Salesman Problem.

[1] Lehrstuhl für Informatik I (Algorithmen und Komplexität), RWTH Aachen, 52056 Aachen, Germany; e-mail: {hjb,seibert}@i1.informatik.rwth-aachen.de

p in the input size n. On the other hand there are large subclasses of input instances of the TSP that admit polynomial-time approximation algorithms with a reasonable approximation ratio. The Euclidean TSP (also called geometric TSP) even admits a polynomial time approximation scheme [2, 3, 14] and the Δ-TSP (TSP with triangle inequality, also called metric TSP) can be approximated by Christofides algorithm with an approximation ratio of $\frac{3}{2}$ [10]. Generally, recent research has shown that the "relation" of an input instance of the TSP to the triangle inequality may be essential for estimating the hardness of this particular input instance. We say, for every $\beta \geq \frac{1}{2}$, that an input instance of the general TSP satisfies the β-triangle inequality if

$$\mathrm{cost}(\{u, w\}) \leqslant \beta \cdot (\mathrm{cost}(\{u, v\}) + \mathrm{cost}(\{v, w\}))$$

for all vertices u, v, w. By Δ_β-TSP we denote the TSP whose input instances satisfy the β-triangle inequality. Obviously, we have Δ_1-TSP= Δ-TSP. If $\beta > 1$ then we speak about the relaxed triangle inequality, and if $\frac{1}{2} \leqslant \beta < 1$ we speak about the sharpened triangle inequality. Note that at $\beta = \frac{1}{2}$, we reach the trivial case that all edges have the same cost, and $\beta < \frac{1}{2}$ is impossible.

Considering the relaxed triangle inequality, in [1,5,6] it has been proved that Δ_β-TSP can be approximated in polynomial time with approximation ratio $\min\{4\beta, \frac{3}{2}\beta^2\}$. Also, [5] contains a proof that there exists a small $\varepsilon > 0$ such that Δ_β-TSP cannot be approximated with approximation ratio $1 + \varepsilon \cdot \beta$ for $\beta > 1$, unless $\mathcal{P} = \mathcal{NP}$.

The sharpened triangle inequality was considered first in [7, 8]. There, some algorithms were developed to obtain an approximation ratio between 1 and 1.5, depending on β. More precisely, methods were developed to adopt algorithms for the metric TSP to the case of the sharpened triangle inequality, and also a new algorithm, the so called Cycle Cover Algorithm has been proposed. This leads to a combined upper bound of approximability of $\min\left\{1 + \frac{2\beta-1}{3\beta^2-2\beta+1}, \frac{2}{3} + \frac{1}{3} \cdot \frac{\beta}{1-\beta}\right\}$ which tends to 1 for $\beta \to \frac{1}{2}$, and to 1.5 for $\beta \to 1$.

In this paper, we will show improved, respectively first, lower bounds for these problems as follows. In the case of Δ-TSP, we get a lower bound of $\frac{3813}{3812} - \varepsilon$, for Δ_β-TSP, $\beta > 1$, we get $\frac{3803+10\beta}{3804+8\beta} - \varepsilon$, and for Δ_β-TSP, $\frac{1}{2} < \beta < 1$, we obtain $\frac{7611+10\beta^2+5\beta}{7612+8\beta^2+4\beta} - \varepsilon$ as lower bound, each one for an arbitrary small $\varepsilon > 0$. Note that for the case of the sharpened triangle inequality, this is the first lower bound at all. In view of the fact that the Euclidean TSP, in some sense an orthogonal subclass of the metric TSP, admits a polynomial time approximation scheme, it is somewhat surprising that the case of the sharpened triangle inequality is $\mathcal{APX}$-hard for β arbitrarily close to $\frac{1}{2}$.

Our proof is based on the idea of Engebretsen, who reduced a special case of a linear equations problem, called LinEq2-2(3), to the TSP subproblem with edge costs 1 and 2 only. This special case of the TSP was considered for the first time in [17]. We extend this proof technique by considering the reduction to input instances of Δ_β-TSP whose edge costs are from $\{1, m, l\}$, $1 < m < l$. (We use

$\{1, 2, 3\}$ for the metric TSP, and appropriate values, depending on β, for the other cases.) This modification requires some crucial changes in the construction of Engebretsen as well as some essentially new technical considerations.

Overall the proof is performed by a gap-preserving reduction from LinEq2-2(3), for which Berman and Karpinski have shown a lower bound of $\frac{332}{331} - \varepsilon$, for an arbitrary small $\varepsilon > 0$.

This paper is organized as follows. In the next section, we give the definitions and state the results. Section 3 is devoted to the proof of the central theorem, and we conclude in Section 4.

2. Definitions and results

Let B be an optimization problem, L_B the language of all inputs for B, and $\mathcal{A}$ an algorithm producing a feasible solution $\mathcal{A}(x)$ for any $x \in L_B$. For an input x, we denote by $opt(x)$ the cost of an optimal solution. Let $\alpha \in \mathbb{R}^{\geq 1}$. $\mathcal{A}$ is an *α-approximation algorithm* for B if for all inputs $x \in L_B$,

$$\max \left\{ \frac{\mathrm{opt}(x)}{\mathrm{cost}(\mathcal{A}(x))}, \frac{\mathrm{cost}(\mathcal{A}(x))}{\mathrm{opt}(x)} \right\} \leqslant \alpha.$$

The *Traveling Salesman Problem with β-triangle inequality*, Δ_β-TSP for short, is the following approximation problem.

Input: A complete graph $G = (V, E)$, $V = \{v_1, \dots, v_n\}$ and a cost function $\mathrm{cost} : E \to \mathbb{Q}$ which obeys the β-triangle inequality

$$\mathrm{cost}(\{u, w\}) \leqslant \beta \cdot (\mathrm{cost}(\{u, v\}) + \mathrm{cost}(\{v, w\})) \text{ for all } u, v, w \in V.$$

Output: A Hamiltonian tour in G, given by a permutation π on $\{1, \dots, n\}$.

Goal: Minimize $\sum_{i=1}^{n} \mathrm{cost}(\{v_{\pi_i}, v_{\pi_{(i+1) \bmod n}}\})$.

The $\Delta_{\{1,m,l\}}$-TSP is the subproblem of Δ-TSP where all edge costs are from the set $\{1, m, l\}$.

Next, we give the definition of LinEq2-2(3) for which we give gap-preserving reductions to Δ_β-TSP.

Input: A set $\{e_1, \dots, e_n\}$ of linear equations modulo 2 over a set of variables $\{x_1, \dots, x_m\}$ where each equation contains exactly two variables, and each variable occurs exactly three times.

Output: An assignment of binary values to the variables $\{x_1, \dots, x_m\}$.

Goal: Maximize the number of equations from $\{e_1, \dots, e_n\}$ which hold under the given assignment.

Finally, we have to clarify what type of reduction we will use between these two problems. We will use a gap-preserving reduction as introduced in [4]. The notion of gap problem is only implicit there. Also, the definition has been used in some modified ways since. We will use the following definition as in [15].

Let A be an optimization problem with input language L_A. Let size(x) denote the size of an input in an appropriate way. Moreover let $0 < c < s$ be two constants. The *gap problem* $(c,s)-A$ is the following decision problem.

Input: An instance $x \in L_A$ such that either

1. $\frac{\mathrm{opt}(x)}{\mathrm{size}(x)} \leqslant c$, or
2. $\frac{\mathrm{opt}(x)}{\mathrm{size}(x)} \geq s$.

Output: The decision which of the two cases applies.

As size(x), we use the number of vertices in a TSP instance, and the size of a LinEq2-2(3) instance is the number of equations.

Note the following difference to viewing languages as decision problems. Here, we don't have a "positive" and a "negative" answer but two different answers which can be handled symmetrically. Consequently, when speaking about $\mathcal{NP}$-hardness of a gap problem, it doesn't matter to which answer of the gap problem the positive answer of a language in $\mathcal{NP}$ is mapped.

Let $(c,s)-A$ and $(c',s')-B$ be two gap problems. A *gap-preserving reduction from* $(c,s)-A$ *to* $(c',s')-B$ is a polynomial time algorithm computing a mapping $y : L_A \to L_B$ such that one of the following two cases applies.

1. for all $x \in L_A$:
   $\frac{\mathrm{opt}_A(x)}{\mathrm{size}_A(x)} \leqslant c \Rightarrow \frac{\mathrm{opt}_B(y(x))}{\mathrm{size}_B(y(x))} \leqslant c'$, and
   $\frac{\mathrm{opt}_A(x)}{\mathrm{size}_A(x)} \geq s \Rightarrow \frac{\mathrm{opt}_B(y(x))}{\mathrm{size}_B(y(x))} \geq s'$;
2. for all $x \in L_A$:
   $\frac{\mathrm{opt}_A(x)}{\mathrm{size}_A(x)} \leqslant c \Rightarrow \frac{\mathrm{opt}_B(y(x))}{\mathrm{size}_B(y(x))} \geq s'$, and
   $\frac{\mathrm{opt}_A(x)}{\mathrm{size}_A(x)} \geq s \Rightarrow \frac{\mathrm{opt}_B(y(x))}{\mathrm{size}_B(y(x))} \leqslant c'$.

We write $(c,s)-A \leqslant_{gp} (c',s')-B$.

With the previous definition, we are able to deal with gap problems coming from both, maximization and minimization problems, without further distinction. Please remember that all one wants to use the reduction for is to show $\mathcal{NP}$-hardness of gap problems. This in turn implies a lower bound on the approximability of the underlying optimization problem.

Theorem 2.1. [4] *Let $(c,s)-A$ and $(c',s')-B$ be two gap problems. If $(c,s)-A$ is $\mathcal{NP}$-hard, and $(c,s)-A \leqslant_{gp} (c',s')-B$ then*

1. *$(c',s')-B$ is $\mathcal{NP}$-hard, and*
2. *there is no polynomial time approximation algorithm for B having a ratio better than $\frac{s'}{c'}$, unless $\mathcal{P} = \mathcal{NP}$.*

The basis of our hardness proofs will be the following result of Berman and Karpinski, stated here in our terminology.

Theorem 2.2. [9] *For every arbitrary small $\varepsilon_1, \varepsilon_2 > 0$, $\left(\frac{331}{336} + \varepsilon_1, \frac{332}{336} - \varepsilon_2\right)-$LinEq2-2(3) is $\mathcal{NP}$-hard.*

From this, we will present a gap-preserving reduction to the three variants of Δ_β-TSP studied here (*i.e.* cases $\frac{1}{2} < \beta < 1$, $\beta = 1$, $\beta > 1$). To give a uniform treatment, we will first clarify how we can get, in each of theses cases, instances which use only three different edge costs.

We call a triple of edge costs $1 < m < l$ *admissible for* Δ_β-TSP if $m \leqslant \beta(1+1)$, $l \leqslant \beta(1+m)$, and $1+l \leqslant 2m$. The first two conditions mean that triangles consisting of two edges with cost 1 and one edge with cost m satisfy the β-triangle inequality, as well as triangles that consist of one edge of each cost. The last condition assures that the cost of a Hamiltonian tour will not be increased if we substitute two edges of cost m by one edge each of cost 1 and l. This fact will be needed frequently in the proof.

Theorem 2.3. *Let $\beta > \frac{1}{2}$, and $1 < m < l$ be admissible edge costs for Δ_β-TSP. Then for arbitrary small $\varepsilon_1, \varepsilon_2 > 0$*

$$\left(\tfrac{331}{336} + \tfrac{68}{3}\varepsilon_2, \tfrac{331}{336} - \tfrac{34}{3}\varepsilon_1\right)\text{-LinEq2-2(3)} \leqslant_{gp} \left(\tfrac{7612+4l}{7616} + \varepsilon_1, \tfrac{7611+5l}{7616} - \varepsilon_2\right)\text{-}\Delta_\beta\text{-TSP}.$$

It is obvious that we can use edge costs $1, 2, 3$ for case $\beta = 1$. Also, one can easily check that for $\frac{1}{2} < \beta < 1$ the costs $1, 2\beta$, and $2\beta^2 + \beta$ are admissible, as are the costs $1, 2\beta$, and $4\beta - 1$ in case $\beta > 1$. Moreover, these are the maximal admissible edge costs for the respective cases.

Using these values, from Theorems 2.1, 2.2, and 2.3, we immediately obtain lower bounds on the approximability.

Theorem 2.4. *Unless $\mathcal{P} = \mathcal{NP}$, for every small $\varepsilon > 0$, the following holds.*

1. *There is no polynomial time $\left(\frac{7611+10\beta^2+5\beta}{7612+8\beta^2+4\beta} - \varepsilon\right)$-approximation algorithm for Δ_β-TSP, $\frac{1}{2} < \beta < 1$;*
2. *there is no polynomial time $\left(\frac{3813}{3812} - \varepsilon\right)$-approximation algorithm for Δ-TSP;*
3. *there is no polynomial time $\left(\frac{3803+10\beta}{3804+8\beta} - \varepsilon\right)$-approximation algorithm for Δ_β-TSP, $\beta > 1$.*

3. Proof of the central theorem

This section is devoted to the proof of Theorem 2.3. In the following, when referring to the three edge costs $1, m, l$ we will usually work with the values $1, 2, 3$. This is not only because these are the values for the most prominent case, the metric TSP. We also believe that the reader will get a better intuition, because these values correspond in a close way to the distance in a skeleton graph as you will see below. All one has to keep in mind is that, when replacing two edges of cost 2 by one of cost 1 and 3, each, we just use that fact that the costs are admissible, especially $2m \leqslant 1 + l$.

First, we will give a short outline of the proof.

For a given LinEq2-2(3) instance we first construct an undirected graph G_0 which consists of $68n + 1$ vertices, if the given LinEq2-2(3) instance has $3n$ equations and $2n$ variables. This graph consists of one equation gadget for each equation of the LinEq2-2(3) instance and one variable cluster for each variable. These components are connected to each other in such a way that every variable cluster is connected *via* exactly two edges to every equation gadget belonging to an equation in which this variable occurs. We call these edges the connector edges. Then we construct a $\Delta_{\{1,2,3\}}$-TSP instance G from G_0 by setting the edge costs for all edges in G_0 to one and setting all other edge costs to the maximal possible value from $\{2,3\}$ such that the triangle inequality is still satisfied. This construction is described in detail in Section 3.1.

We will show the following correspondence between the optimal assignment for the LinEq2-2(3) instance and the optimal Hamiltonian tour for the constructed TSP instance. We will prove that an optimal Hamiltonian tour in G uses e edges of cost $\neq 1$ iff an optimal assignment for the LinEq2-2(3) instance satisfies all but e equations. The main technical difficulty lies in additionally showing that all these expensive edges must have a cost of 3, *i.e.* they connect vertices of distance at least 3 in G_0.

To prove this claim we will show that any Hamiltonian tour in G can be transformed in such a way that

- the costs do not increase;
- the modified tour does not use edges of cost 2;
- every vertex incident to an edge of the tour of cost 3 lies inside an equation gadget;
- every equation gadget contains 0 or 2 vertices that are incident to an edge of the tour of cost 3, and (1)
- for every six connector edges belonging to the three occurrences of the same variable either all or none of these edges are used by the tour.

The idea behind the construction of an assignment for the LinEq2-2(3) instance from a given Hamiltonian tour of the form as claimed above is that a variable is assigned the value 1 iff all connector edges belonging to the occurrences of this variable are used by the tour. Then an equation gadget without vertices incident to an edge of cost 3 of the tour corresponds to a satisfied equation and an equation gadget with two such vertices corresponds to an equation that is not satisfied.

To present a more detailed overview of the proof we need the details of the construction and some more definitions which we will give in Section 3.1. For this reason we postpone the overview of the structure of the proof to Section 3.2.

3.1. The construction

The complete construction of G_0 is shown schematically for a sample instance with 9 equations and 6 variables in Figure 1.

For each equation in I we will construct an **equation gadget**, represented by a shaded box in Figure 1. For simplicity, only the vertices a, b, c, e, f, h, l, m from Figure 2 are shown in Figure 1.

Depending on the type of the equation we will construct two types of gadgets. The equation gadgets for equations of the type $x + y = 0$ are called **equation gadgets of type 0** (*cf.* Fig. 2a), and the equation gadgets for equations of the

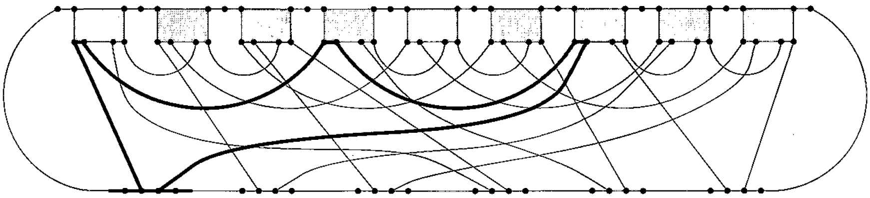

FIGURE 1. The complete construction of G_0 for a sample instance with 9 equations and 6 variables.

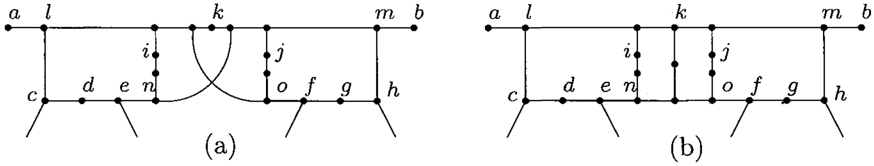

FIGURE 2. The equation gadget of type 0 is shown in (a), the equation gadget of type 1 is shown in (b).

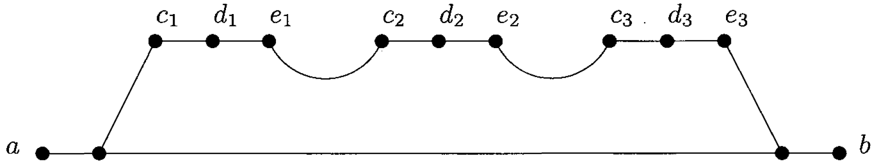

FIGURE 3. The variable cluster.

type $x + y = 1$ are called **equation gadgets of type 1** (*cf.* Fig. 2b)[1]. Every equation gadget contains exactly six vertices that are connected to the rest of the graph. The vertices a and b are used to link all equation gadgets in a long chain in arbitrary order. This linking is done by identifying vertex b of one gadget with vertex a of the next gadget. The other four vertices are used to connect the equation gadget to the variable clusters which are defined as follows.

For each variable in I we construct a **variable cluster** consisting of four vertices connected as a path and a second path using just vertices from three of the equation gadgets. In each of the three gadgets three vertices are used. The structure of a variable cluster is shown in Figure 3. Furthermore, in Figure 1 one of the variable clusters is drawn with bold lines.

The equation gadgets and variable clusters are linked in the following way: the vertices c, d, e and f, g, h are used to connect the equation gadget with the variable clusters, they are called the **connector vertices** of this gadget. The edges from the connector vertices to the vertices outside this gadget are called **connector edges**. The connector vertices c, d, e (f, g, h respectively) together

[1]Note that every equation modulo 2 with exactly two variables can be written either as $x + y = 0$ or as $x + y = 1$.

with the corresponding connector edges and the edges $\{c, d\}$ and $\{d, e\}$ ($\{f, g\}$ and $\{g, h\}$ respectively) are called a **connector** of the equation gadget. The edges $\{c, d\}$ and $\{d, e\}$ ($\{f, g\}$ and $\{g, h\}$ respectively) are called the **internal edges** of the connector.

The vertices a and b of every variable cluster are used to link all variable clusters in a long chain in arbitrary order. In contrast to the chain of the equation gadgets here the vertex b of one variable cluster is connected to the vertex a of the next variable cluster *via* an additional edge. The vertices $\{c_1, d_1, e_1\}$, $\{c_2, d_2, e_2\}$, $\{c_3, d_3, e_3\}$ are identified with the connector vertices $\{c, d, e\}$ ($\{f, g, h\}$ respectively) of the three connectors corresponding to the three occurrences of this variable in the equations.

Finally, the first equation gadget in the chain and the first variable cluster in the chain are connected *via* one additional edge, and also the last equation gadget and the last variable cluster are connected *via* an additional edge.

From this graph G_0, we construct an instance of $\Delta_{\{1,2,3\}}$-TSP by defining a complete graph G with the vertex set $V(G_0)$ and a cost function cost : $E(G) \to \{1, 2, 3\}$. This cost function is defined by

$$\operatorname{cost}(\{x, y\}) = \begin{cases} 1 & \text{if } \{x, y\} \in E(G_0) \\ 2 & \text{if } \{x, y\} \notin E(G_0) \text{ and there exists a } z \in V(G) \text{ such that} \\ & \{x, z\} \in E(G_0) \text{ and } \{y, z\} \in E(G_0) \\ 3 & \text{otherwise.} \end{cases}$$

This means that edges from G_0 have cost 1 and all other edges have the maximal possible cost from $\{2, 3\}$ such that the triangle inequality is still satisfied.

In the sequel, we will need some more definitions for our proof. We start with the following notation describing how the connectors are traversed by a given Hamiltonian tour.

Definition 3.1. Given a Hamiltonian tour in G, we say that a connector is **traversed** if both connector edges are traversed by the tour, **untraversed** if none of the connector edges is traversed, and **semitraversed** otherwise.

Every Hamiltonian tour H in G defines a path cover P in G_0, *i.e.* a set of pairwise vertex-disjoint paths that cover all vertices in $V(G_0)$. Every edge of cost 1 in H corresponds to an edge on a path from P, and every edge of cost ≥ 2 corresponds to a pair of endpoints of different paths in P (unless the path cover consists of exactly one path). This correspondence between Hamiltonian tours in G and path covers in G_0 leads to the following definition.

Definition 3.2. Given a Hamiltonian tour H in G, a vertex x is called an **endpoint** of H if at least one of the edges of H incident to x is not in $E(G_0)$, it is called a **double endpoint**, if both edges of H incident to x are not in $E(G_0)$. We say that two vertices x and y are **connected by the tour *H***, if there exists a path from x to y in G_0 that uses only edges from H. We call a pair of endpoints (x, y) **critical**, if x and y are not connected by the tour and $\operatorname{cost}(\{x, y\}) \leqslant 2$.

Observe that the tour does not contain any edge of cost 2, if there are no critical pairs of endpoints in it. An edge of cost 2 can result only from a pair of endpoints (x, y) at distance 2 in G_0. But if that pair is not critical, it is connected by the tour. Only if x and y are the only endpoints, the edge $\{x, y\}$ can form, together with the connection in G_0, a Hamiltonian circle in G. This special case – that the Hamiltonian tour in G already defines a Hamiltonian path in G_0 – will be handled separately in Section 3.7. Otherwise, the edge $\{x, y\}$ would close a circle containing not all vertices, which is impossible in a Hamiltonian tour.

3.2. Overview of the proof

In this section, we will give an overview of the structure of the proof of Theorem 2.3. In Section 3.1, we have constructed a TSP instance from a given LinEq2-2(3) instance. To establish the gap in our reduction, we will prove the following main lemma.

Lemma 3.3. *Let I be a* LinEq2-2(3) *instance and let G be the corresponding TSP instance as constructed in Section 3.1. The cost of an optimal Hamiltonian tour in G is $|V(G)| + (l-1)e$ if and only if in the* LinEq2-2(3) *instance I at most all but e equations can be satisfied at the same time.*

Note that when we use in the metric case edge cost $l = 3$, the cost of the optimal tour will be $|V(G)| + 2e$.

One direction of the proof of Lemma 3.3 is straightforward. Starting from an assignment to the variables of the LinEq2-2(3) instance, we obtain a tour of the claimed maximal cost as follows. It traverses the graph of Figure 1 essentially along the outer cycle, taking some detours through variable clusters and equation gadgets. If a variable is set to 1 in the given assignment, the tour uses the edges of the variable cluster which visit all three gadgets of those equations where the variable occurs. Otherwise it uses the the shortcut below in Figure 3.

Thus, for a satisfied equation of type $x + y = 1$, in the corresponding gadget exactly one of the connectors is visited as part of a variable cluster. The rest of that gadget can be traversed as shown in Figure 4a.

Similarly, in a gadget for a satisfied equation of type $x + y = 0$, both or none of the connectors are traversed as part of a variable cluster. Thus, it admits a traversal as depicted in Figures 4b and c.

It remains to include the vertices of equation gadgets for unsatisfied equations in the tour. Here, the traversal of the variable clusters implies that, for a gadget of type 1, both or none of the connectors are left open. And for a gadget of type 0, exactly one of the connectors remains open. In this case, we add pieces to the tour as depicted in Figures 4d–f.

Only in the last step, we have used edges which are not part of G_0 (the dashed edges in Fig. 4). These connect vertices having distance at least 3 in G_0, thus they have cost 3. All other edges are part of G_0, *i.e.*, they have cost 1 in G.

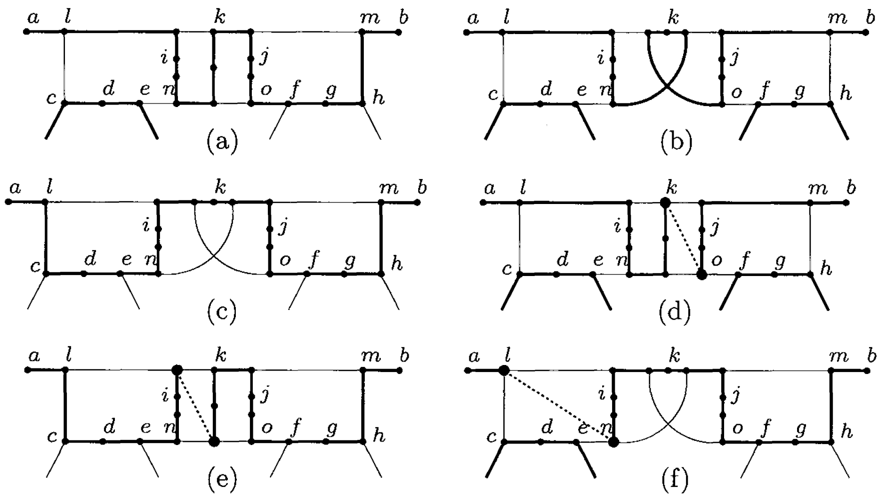

FIGURE 4. The traversal of equation gadgets.

Overall, an assignment leaving e equations unsatisfied results in a tour of cost $|V| + 2e$.

For the opposite direction, we have to show that an arbitrary Hamiltonian tour through G can be modified, without increasing the cost, into a tour which has the structure of a tour constructed from an assignment as above. Then, an assignment can be inferred from that tour having the claimed quality in a direct reversal of the above procedure.

The mentioned transformation of an arbitrary tour consists of a lengthy procedure with many case distinctions. In the following we will give an overview of the steps that are necessary to prove the hard direction of Lemma 3.3.

Recall that to prove this claim we have to show that any Hamiltonian tour in G can be transformed in such a way that condition (1) holds.

The proof will be organized as follows:

1. given an arbitrary Hamiltonian tour in G that is not the extension of a Hamiltonian path in G_0, we will transform the tour without increasing its cost in the following way;
   (a) *eliminate all critical pairs of endpoints at distance* 1 *from the tour.* This will be done in Section 3.3;
   (b) *transform the tour locally to take a certain way through an equation gadget, without using any critical pair of endpoints inside the gadget.* We will describe this local transformations of the tour inside one equation gadget in Section 3.4, where we will also show that the described way of traversing an equation gadget is optimal in the sense that no other traversal of the gadget can be of lower cost. We will describe these local transformations first for equation gadgets without semitraversed connectors, and after that for equation gadgets with semitraversed connectors;

(c) *eliminate all critical pairs of endpoints from the tour.* The necessary transformations will be described in Section 3.5;

(d) *eliminate all semitraversed connectors from the tour.* We will present this transformations in Section 3.6. After performing this step, the tour is of the form as claimed in condition (1);

2. given a Hamiltonian tour in G with exactly two endpoints, show that the edge connecting these two endpoints cannot have cost 2. We will prove this claim in Section 3.7. This means that in this special case, there is no modification to be done;
3. construct the assignment to the variables of the underlying LinEq2-2(3) instance from the tour. The construction of the assignment will be described in Section 3.8.

3.3. Elimination of critical pairs of endpoints at distance one

In this section, we will show that we can transform any Hamiltonian tour without increasing its cost in such a way that the modified tour does not contain any critical pair of endpoints at distance 1.

Lemma 3.4. *Suppose that we have a Hamiltonian tour H with two endpoints x and y at distance* 1 *that are not connected by the tour. Then there exists a Hamiltonian tour H' with two endpoints less than H and with* $\text{cost}(H') \leqslant \text{cost}(H)$.

Proof. Let H be a Hamiltonian tour with two endpoints x and y at distance 1 in G_0 such that x and y are not connected by the tour. Since x and y are endpoints there exists a vertex w at distance ≥ 2 from x in G_0 and a vertex w' at distance ≥ 2 from y in G_0 such that $\{x, w\}$ and $\{y, w'\}$ are edges of H.

If the tour H is of the form $xwP_1yw'P_2x$ for some paths P_1 and P_2 in G, then we can obtain the claimed Hamiltonian tour H' from H by replacing the edges $\{x, w\}$ and $\{y, w'\}$ by the edges $\{x, y\}$ and $\{w, w'\}$. This transformation is shown in Figure 5a. Since the distance between w and w' can be at most 3, we have

$$\begin{aligned}
\text{cost}(H') &= \text{cost}(H) - \text{cost}(\{x, w\}) - \text{cost}(\{y, w'\}) \\
&\quad + \text{cost}(\{x, y\}) + \text{cost}(\{w, w'\}) \\
&\leqslant \text{cost}(H) - 2 - 2 + 1 + 3 \\
&= \text{cost}(H),
\end{aligned}$$

and obviously H' has two endpoints less than H.

If the tour H is of the form $xwP_1w'yP_2x$ for some paths P_1 and P_2 in G, we have the situation as shown in the left part of Figure 5b. Note that the existence of an edge $\{z, z'\}$ of cost ≥ 2 on the path P_2 is due to the fact that x and y are not connected by the tour. Observe that we can assume that the distance between x and u in G_0 is 1, and also the distance between y and v in G_0 is 1. Otherwise an analogous transformation as in the first case would be possible.

If one of the edges $\{x, w\}$, $\{y, w'\}$, or $\{z, z'\}$ of H has cost 3, we can apply the transformation as shown in Figure 5b. By this transformation we save the

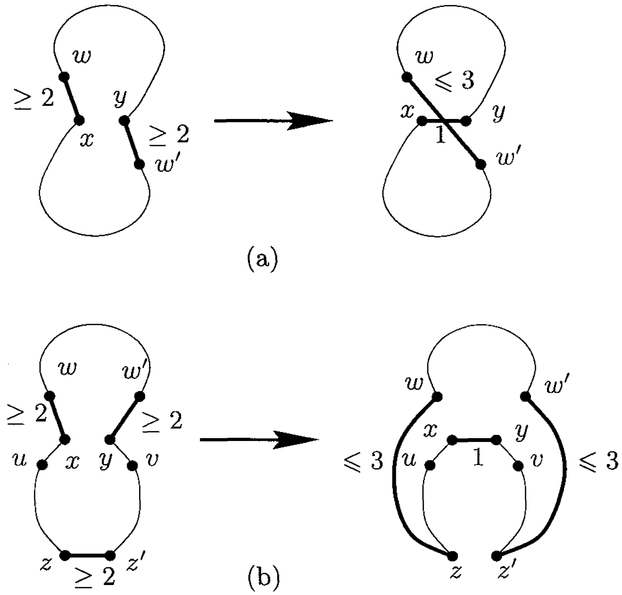

FIGURE 5. The transformations in the first two cases of the proof of Lemma 3.4.

costs of the edges $\{x, w\}$, $\{y, w'\}$, or $\{z, z'\}$ which sum up to at least 7, but we have to add the costs of the edges $\{w, z\}$, $\{w', z'\}$, and $\{x, y\}$ which sum up to at most 7. Thus, the resulting tour H' has no higher costs than H, and obviously H' has two endpoints less than H. Note that we can see the change in edge costs as replacing two edges of cost 2 by one of cost 1 and one of cost at most 3, besides one additional replacement of an edge of cost 3 by one of less or equal cost. In the following, we will see that this is a general scheme of the replacements we will perform.

At this point it remains to analyze the case that all three edges $\{x, w\}$, $\{y, w'\}$, and $\{z, z'\}$ of H have costs of 2 and that furthermore no edge of cost 3 exists on the path P_2. This leads to the situation in G_0 as shown in Figure 6 where the Hamiltonian tour H is drawn with bold lines.

In the following we will distinguish three cases according to whether the vertices u and u', and the vertices v and v' respectively, are identical or not. We will start with the case that $u = u'$ and $v = v'$. In this case we can transform the Hamiltonian tour H as shown in Figure 7a. Here, we replace two edges of cost 2 by one of cost 1 and at most 3 each, besides one additional replacement of an edge of cost 2 by one of less or equal cost Thus, the costs of the obtained Hamiltonian tour H' do not exceed the costs of H. Furthermore, the three endpoints x, w, and w' of H are not endpoints of H', but H' has only one additional endpoint, namely s.

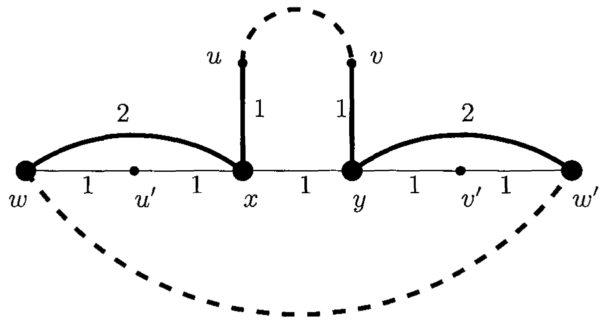

FIGURE 6. The situation in the critical case of the proof of Lemma 3.4.

Now we will analyze the case that $u \neq u'$. If one of the edges incident to u' that is used by the tour H has costs ≥ 2, we can use the transformation as shown in Figure 7b. Here, we replace two edges of cost 2 by one of cost 1 and one of cost at most 3, besides one additional replacement of an edge of cost 1 by one of equal cost. Furthermore, the endpoints w, u', and x of H are no endpoints in H', and only the vertex t might be an additional endpoint of H', if the edge $\{u', t\}$ has cost 1.

If both edges of H incident to u' are of cost 1, then the degree of u' in G_0 has to be 3, since the edge $\{u', x\}$ is not used by H. In the sequel we will distinguish two cases.

If $v = v'$, then we have the situation as in the left part of Figure 7c, and we can transform the tour as shown in this figure. Again, we replace two edges of cost 2 by one of cost 1 and one of cost at most 3, besides one additional replacement of an edge of cost 1 by one of equal cost. Furthermore the endpoints x and w' of H are not endpoints of H', and no new endpoint is added by the transformation.

If $v \neq v'$, we can assume without loss of generality that also both edges of H incident to v' are of cost 1, otherwise we could use a transformation analogous to the one shown in Figure 7b. This implies that also the degree of v' has to be 3 in G_0. From the facts $u \neq u'$ and $v \neq v'$ we can conclude that also the vertices x and y are of degree 3 in G_0.

In this case we can use a transformation as shown in Figure 7d. Note that this transformation does not reduce the number of endpoints in the Hamiltonian tour, but it moves one endpoint from the vertex x to the vertex s. As in the previous cases the cost of the Hamiltonian tour does not increase since we replace two edges of cost 2 by one of cost 1 and one of cost at most 3, besides one additional replacement of an edge of cost 1 by one of equal cost.

It remains to show that this transformation can only finitely often lead to a similar situation again, but at some point it will lead to a situation as already solved in the previous cases. If the transformation leads to a similar situation again, this is because there exists another endpoint r of the tour at distance 1 to s. In this case we will leave the tour through s unchanged, but we will use an analogous transformation to remove the endpoint from r. To show that a sequence

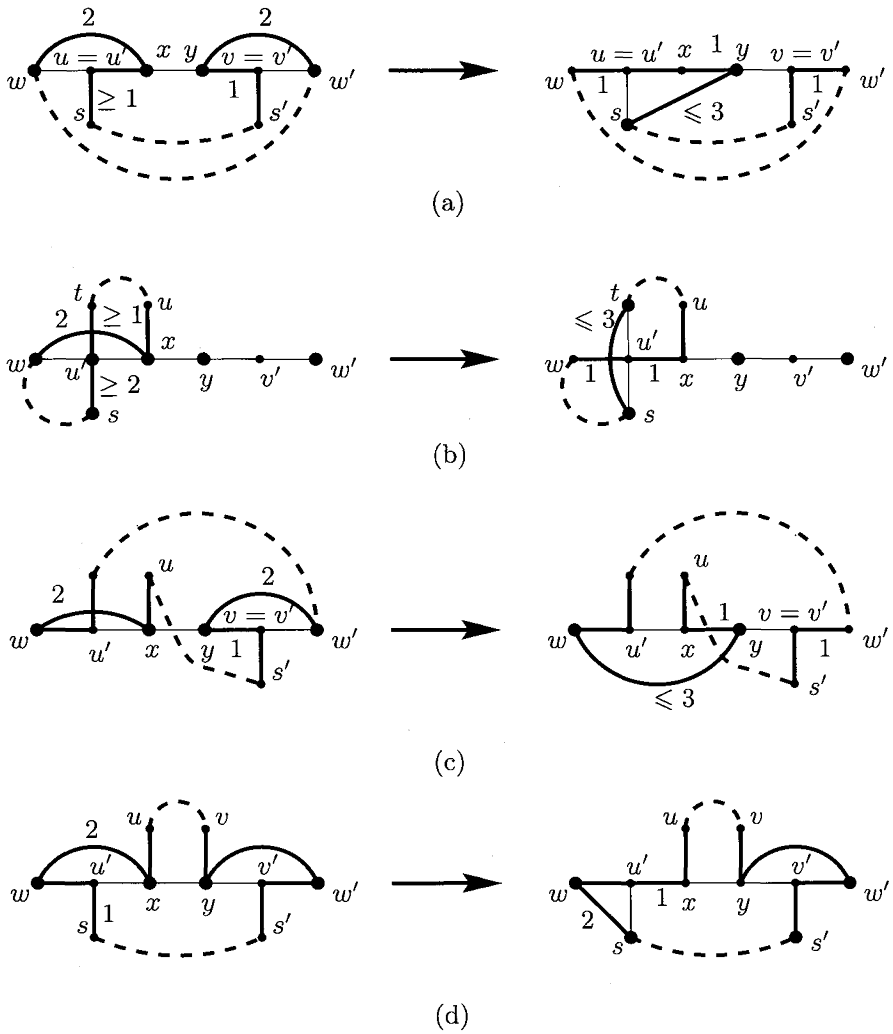

FIGURE 7. The transformations in the critical case of the proof of Lemma 3.4.

of transformations of this type will always be finite, it suffices to show that each transformation produces one additional vertex that cannot play the role of the vertex s. For the proof consider a sequence of k transformations of this type, where the vertices are denoted as shown in Figure 7d with the number of the transformation as subscript (*i.e.* as $x_i, y_i, s_i, \ldots$ for the situation before the i-th transformation).

Now we will show that $x_1 \neq s_i$ for all $i \geq 2$. Without loss of generality we can assume that $x_1 \neq s_2$[2]. Assume that $x_1 = s_i$ for some $i \geq 3$, and consider the vertices v_1, v_1', w_1, s_1, and the other neighbor of u_1. These are all the vertices at distance 2 from x_1 in G_0. This means that one of them has to be equal to x_i. Since every degree 3 vertex in G_0 has at least one neighbor of degree 2, the vertices u_1, v_1, w_1 have to be of degree 2. This immediately implies $v_1 \neq x_i$ and $w_1 \neq x_i$. It furthermore implies that the other neighbor of u_1 is not equal to x_i, since the degree of the common neighbor u_i' of x_i and s_i has to be 3. Furthermore, we know $v_1' \neq x_i$ because y_1 is an endpoint and thus $y_1 \neq u_i'$. Finally, $s_1 \neq x_i$ since r_1 is no longer an endpoint of the tour after the second transformation and therefore $r_1 \neq y_i$. □

According to Lemma 3.4, we can in the following assume that a Hamiltonian path does not contain any pair of endpoints at distance 1 that are not connected by the tour. This implies that every critical pair consists of two endpoints at distance 2.

In the following sections, we will mainly consider the graph G_0, and thus all distances, paths, and other graph-theoretic notations are always understood with respect to G_0, unless otherwise noted.

3.4. Local transformation of the tour inside the equation gadgets

In this section, we will show that an optimal tour can be assumed to traverse every equation gadget in a special way without any critical pair of endpoints inside the gadget. This proof will be divided into four lemmas. In Lemma 3.6 (Lem. 3.7 respectively) we will show the claim for a gadget of type 0 (type 1 respectively) without semitraversed connectors. In Lemma 3.8 we will prove the claim for an equation gadget containing exactly one semitraversed connector, and in Lemma 3.9 for an equation gadget with two semitraversed connectors.

To describe the local properties of a Hamiltonian tour we will need the following definition.

Definition 3.5. Let H and H' be two Hamiltonian tours in G, let X be an equation gadget of G_0, and let $\tilde{X}$ be the subgraph of G induced by the vertices of X. We say that H' can be constructed from H by a **local transformation with respect to X**, if H and H' do not differ on the edges from $E(G) - E(\tilde{X})$, *i.e.* if the tour H is only changed inside the gadget $\tilde{X}$ to obtain H', but not on the connector edges and not outside $\tilde{X}$.

Recall that, according to Lemma 3.4, we can in the following assume that any Hamiltonian tour does not contain two endpoints at distance 1 that are not connected by the tour.

[2]If $x_1 = s_2$ we just transform the right part of the tour as shown in Figure 7d instead of the left part. It is impossible that $x_1 = s_2$ and $y_1 = s_2'$ since x_1 is adjacent to y_1, but s_2 is not adjacent to s_2'.

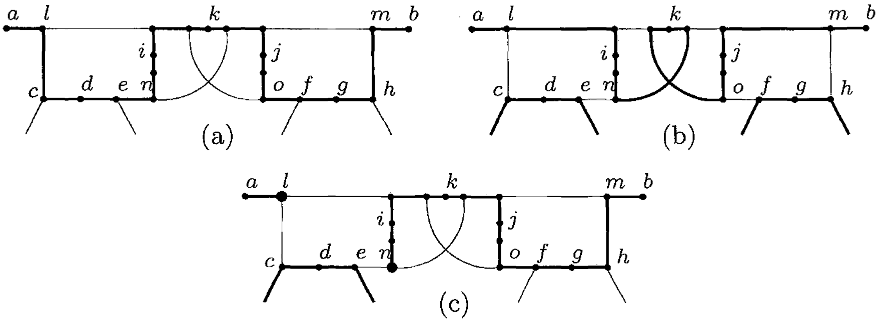

FIGURE 8. The traversal of an equation gadget of type 0 without semitraversed connectors.

Lemma 3.6. *Suppose that we have a Hamiltonian tour H traversing an equation gadget X of type 0 in such a way that there are no semitraversed connectors in it.*

(a) *If X has zero or two traversed connectors, it is possible to modify the tour H by a local transformation with respect to X such that there are no endpoints in the gadget.*

(b) *If X has exactly one traversed connector, it is possible to modify the tour H by a local transformation with respect to X such that the modified tour has exactly two endpoints inside the gadget X, and these endpoints are at distance ≥ 3 in G_0.*

(c) *If X has exactly one traversed connector, it is not possible to modify the tour H by a local transformation with respect to X such that the modified tour has less than two endpoints inside X or exactly two endpoints inside X that form a critical pair.*

Proof. The Hamiltonian tours claimed in (a) are shown in Figures 8a and b, the Hamiltonian tour claimed in (b) is shown in Figure 8c.

Now it remains to prove (c). Since the equation gadget is symmetric, it suffices to show the claim for the case that the connector with the vertices $\{c, d, e\}$ is traversed.

First, we will show the nonexistence of a tour with exactly two endpoints at distance 2 in the gadget that are not connected by the tour.

Assume that we have a critical pair of endpoints (x, y). If there exists a common neighbor z of x and y that has degree 2, then z obviously has to be another endpoint of the tour. Therefore to prove the nonexistence of exactly two unconnected endpoints of distance 2 in the gadget it is sufficient to prove the nonexistence of a Hamiltonian path from a to b in a slightly modified gadget. This modified gadget can be constructed from the original one by removing the vertices $\{c, d, e\}$ and extending it by exactly one edge connecting two vertices x and y at distance 2 such that the common neighbor of x and y has degree 3. Note that the vertices f and h can be treated as degree 2 vertices since the second connector is untraversed, and also the vertex l (n respectively) can be treated as a degree 2 vertex since the edge $\{c, l\}$ ($\{e, n\}$ respectively) cannot be used by the tour.

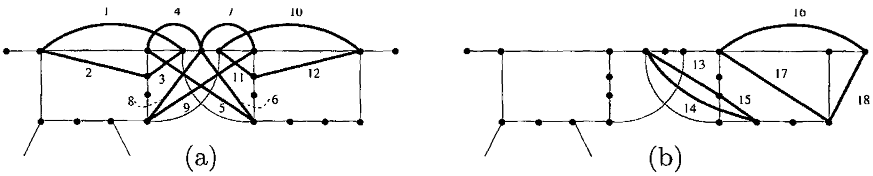

FIGURE 9. The possibilities of choosing an additional edge in the proof of Lemma 3.6 (c).

In Figure 9 the possibilities of choosing such an edge are shown. In Figure 10 for each of these 18 cases those edges are shown that have to be traversed by the tour (*e.g.* because they are incident to a vertex of degree 2).

As an example we will analyze Case 1 and Case 5 in detail: in Case 1 of Figure 10 all edges of G_0 that are drawn with bold lines have to be traversed by the Hamiltonian path from a to b because they either belong to the traversed left connector or are incident to a vertex of degree 2. Since the right connector is untraversed, the vertices f and h can be treated as vertices of degree 2. If one of these edges would be untraversed, one of the degree 2 vertices would be an additional endpoint. But from the figure it is clear that, if all these edges are used, either the vertex n and the upper neighbor of i or the upper neighbors of i and j have to be additional endpoints. Thus, it is impossible to construct a Hamiltonian path from a to b using only the edges of G_0 and the one additional edge of cost 2.

In Case 5 of Figure 10 also all edges of G_0 that are drawn with bold lines have to be traversed by the Hamiltonian path because they either belong to the traversed left connector or are incident to a vertex of degree 2, where f and h are again treated as degree 2 vertices. In this case we get an immediate contradiction since the additional edge from the upper neighbor of i to the vertex o cannot be used by the Hamiltonian path since the vertex o is already the neighbor of two degree 2 vertices.

In all other cases an analogous analysis leads to a contradiction, too.

It remains to show that it is impossible to construct a tour with less than two endpoints. Since the tour has to enter or to leave the gadget *via* the vertices a and b and one connector is traversed and the other one is untraversed, the number of endpoints inside the gadget has to be even. Thus, we only have to show that there does not exist a tour without endpoints inside the gadget. Such a Hamiltonian tour without endpoints in the gadget must use all edges incident to vertices of degree 2, where f and h can again be treated as degree 2 vertices. This leads to the situation as shown in Figure 11 where all these mandatory edges are drawn bold. From this figure it is immediately clear that such a Hamiltonian tour cannot exist. □

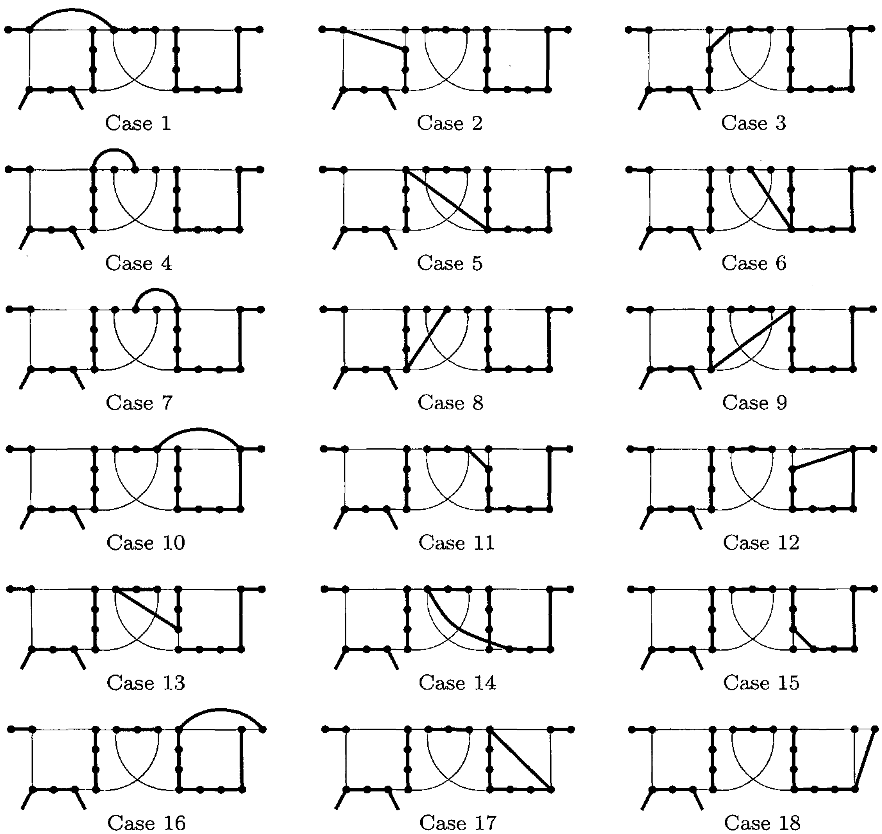

FIGURE 10. The case analysis in the proof of Lemma 3.6 (c).

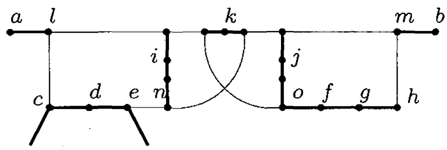

FIGURE 11. The situation in the proof of Lemma 3.6 (c), if we assume that there are no endpoints inside the gadget.

Lemma 3.7. *Suppose that we have a Hamiltonian tour H traversing an equation gadget X of type 1 in such a way that there are no semitraversed connectors in it.*

(a) *If there is exactly one traversed connector, it is possible to modify the tour H by a local transformation with respect to X such that the modified tour has no endpoints inside the gadget X.*

(b) *If there are zero or two traversed connectors, it is possible to modify the tour H by a local transformation with respect to X such that the modified tour*

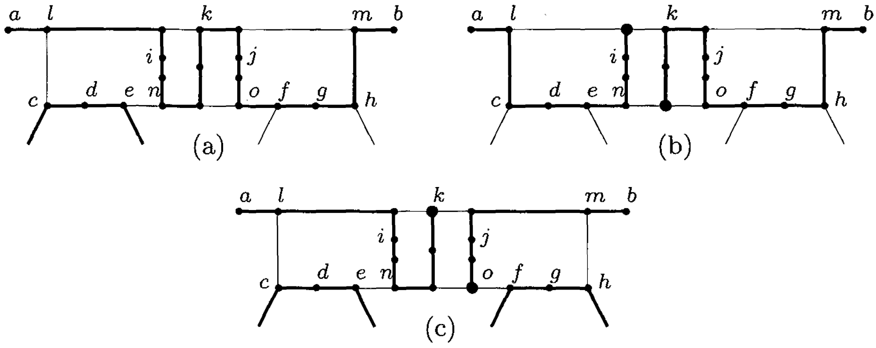

FIGURE 12. The traversal of an equation gadget of type 1 without semitraversed connectors.

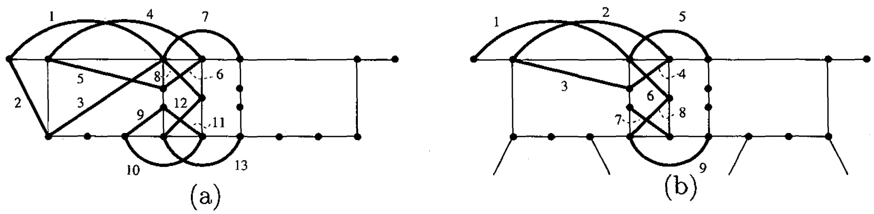

FIGURE 13. The possibilities of choosing an additional edge in the proof of Lemma 3.7 (c).

has exactly two endpoints inside X, and these endpoints are at distance ≥ 3 in G_0.

(c) *If there are zero or two traversed connectors, it is not possible to modify the tour H by a local transformation with respect to X such that the modified tour has less than two endpoints inside X or exactly two endpoints inside X which form a critical pair.*

Proof. The Hamiltonian tour claimed in (a) is shown in Figure 12a, the Hamiltonian tours claimed in (b) are shown in Figures 12b and c.

Now it remains to prove (c). The nonexistence of a tour with exactly two endpoints at distance 2 not connected by the tour can be shown by a similar case analysis as in the proof of Lemma 3.6 (c). The possible cases for using an additional edge of length 2 are shown in Figure 13a for the case that both connectors are untraversed, and in Figure 13b for the case that both connectors are traversed. All cases not shown in Figure 13 can be handled symmetrically. In Figure 14 for each of these cases those edges are shown that have to be traversed by the tour (*e.g.* because they are incident to a vertex of degree 2) for the case that both connectors are untraversed, and in Figure 15 these edges are shown for each of the cases, if both connectors are traversed. As in the proof of Lemma 3.6, in each of the cases shown in Figure 14 and in Figure 15 the tour needs at least one more endpoint in the gadget, which gives the contradiction.

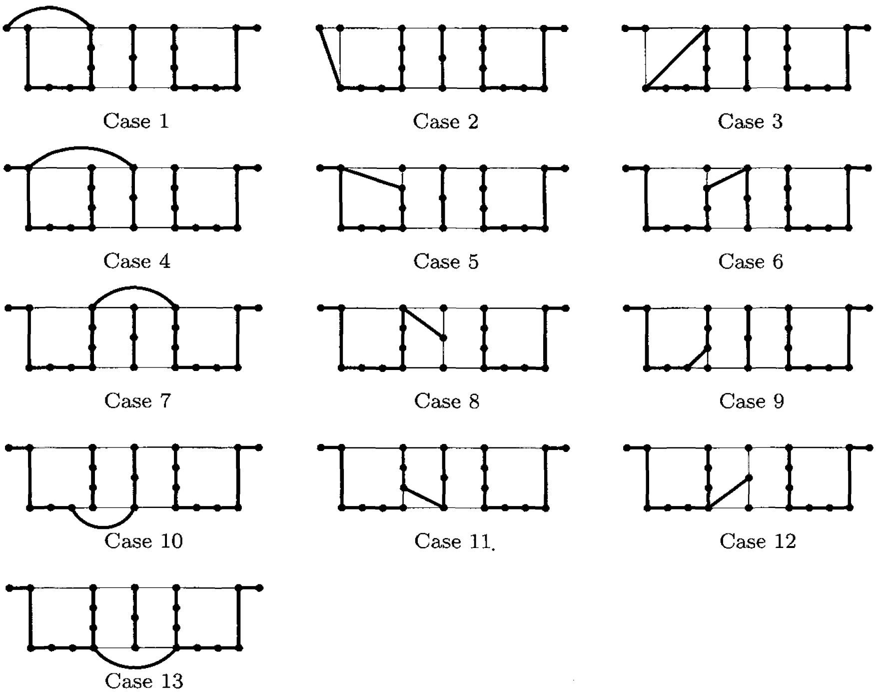

FIGURE 14. The case analysis in the proof of Lemma 3.7 (c), if both connectors are untraversed.

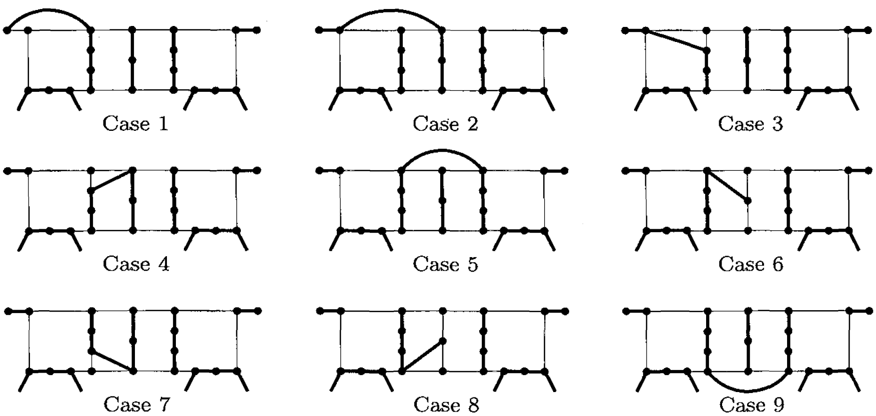

FIGURE 15. The case analysis in the proof of Lemma 3.7 (c), if both connectors are traversed.

It remains to show that it is impossible to construct a tour with less than two endpoints. Since the tour has to enter or to leave the gadget *via* the vertices a and b and both connectors are either traversed or untraversed, the number of endpoints

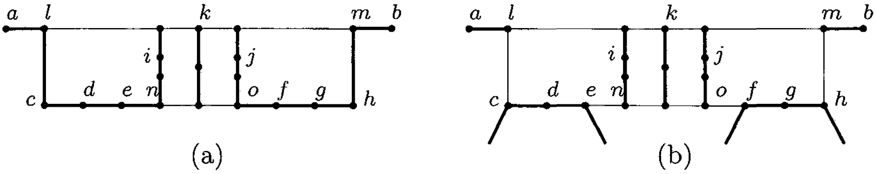

FIGURE 16. The situation in the proof of Lemma 3.7 (c), if we assume that there are no endpoints inside the gadget. In (a) the situation is shown for the case of two untraversed connectors, and in (b) it is shown for the case of two traversed connectors.

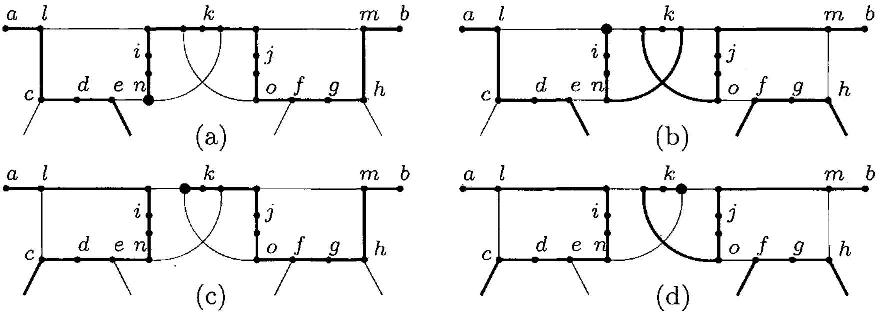

FIGURE 17. The traversal of an equation gadget of type 0 with exactly one semitraversed connector.

inside the gadget has to be even. Thus, we only have to show that there does not exist a tour without endpoints inside the gadget. Such a Hamiltonian tour without endpoints in the gadget must use all edges incident to vertices of degree 2, where again as in the proof of Lemma 3.6 the connector vertices of the untraversed connectors can be treated as degree 2 vertices. This leads to the situation as shown in Figure 16 where all these mandatory edges are drawn bold. From this figure it is immediately clear that such Hamiltonian tour cannot exist. □

Lemma 3.8. *Suppose that we have a Hamiltonian tour H traversing an equation gadget X in such a way that there is exactly one semitraversed connector in it.*

(a) *It is possible to modify the tour H by a local transformation with respect to X in such a way that there is exactly one endpoint in the gadget X.*
(b) *It is not possible to modify the tour H by a local transformation with respect to X in such a way that there is no endpoint in the gadget X.*

Proof. In case of a type 0 equation gadget the Hamiltonian tours claimed in (a) are shown in Figure 17, in case of a type 1 equation gadget the tours claimed in (a) are shown in Figure 18.

It remains to show the claim (b), *i.e.* that it is impossible to construct a tour without an endpoint in the gadget. The tour has to enter (or to leave) the gadget *via* the vertices a and b and *via* the semitraversed connector. This implies that the number of endpoints inside the gadget has to be odd. □

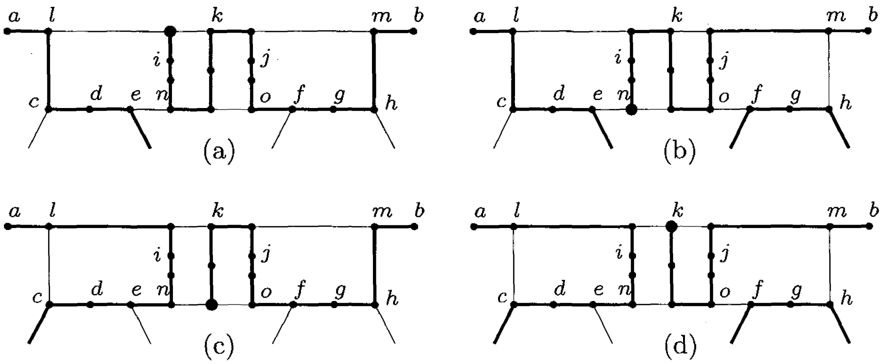

FIGURE 18. The traversal of an equation gadget of type 1 with exactly one semitraversed connector.

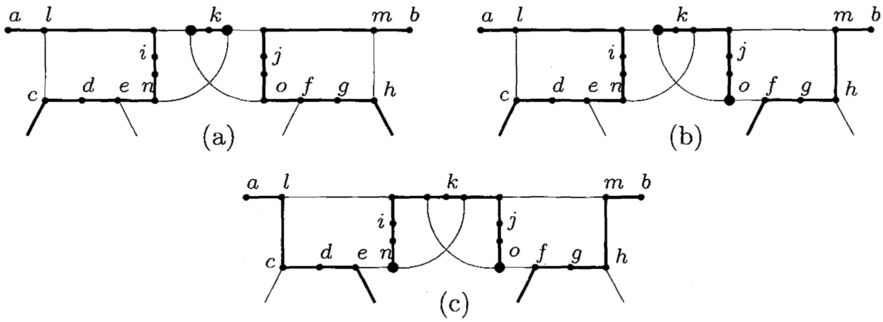

FIGURE 19. The traversal of an equation gadget of type 0 with two semitraversed connectors.

Lemma 3.9. *Suppose that we have a Hamiltonian tour H traversing an equation gadget X in such a way that there are two semitraversed connectors in it.*

(a) *It is possible to modify the tour H by a local transformation with respect to X in such a way that there are exactly two endpoints inside the gadget X which are connected by the tour.*

(b) *It is not possible to modify the tour H by a local transformation with respect to X in such a way that the modified tour contains less than two endpoints inside X or exactly two endpoints inside X that form a critical pair.*

Proof. The Hamiltonian tours claimed in (a) are shown in Figure 19 for type 0 equation gadgets and in Figure 20 for type 1 equation gadgets.

It remains to show the claim (b). The nonexistence of a tour with exactly two endpoints at distance 2 that are not connected by the tour can be shown by a similar case analysis as in the proof of Lemma 3.6 (c). The possible cases of adding an edge of cost 2 are shown in Figure 21 for a gadget of type 0 and in Figure 22 for a gadget of type 1. All cases not shown in the figures are handled symmetrically. The case analysis for the case that the connector edges incident to the vertices c and h are used by the tour is shown in Figure 23 for a gadget

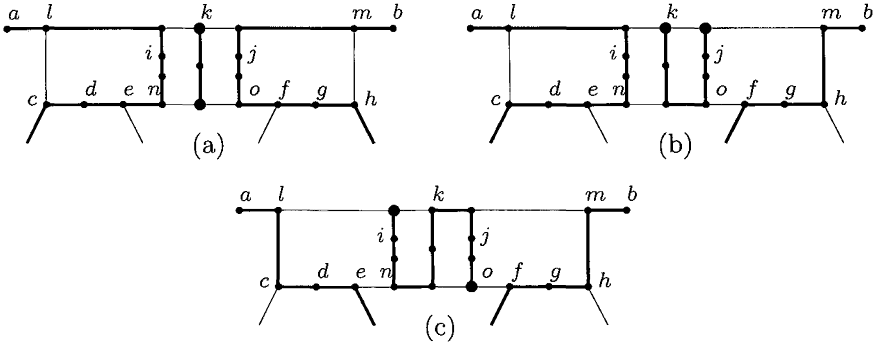

FIGURE 20. The traversal of an equation gadget of type 1 with two semitraversed connectors.

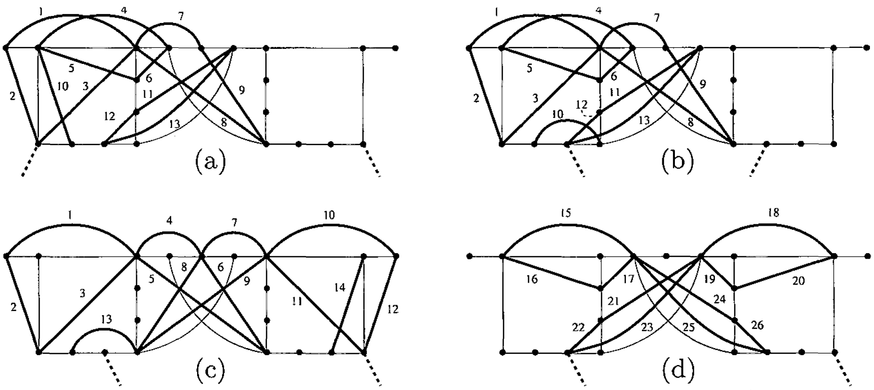

FIGURE 21. The possibilities of choosing an additional edge for a type 0 equation gadget in the proof of Lemma 3.9 are shown in (a) for the case that the connector edges incident to c and h are used, in (b) for the case that the connector edges incident to e and f are used, and in (c) and (d) for the case that the connector edges incident to e and h are used.

of type 0 and in Figure 26 for a gadget of type 1. The case analysis for the case that the connector edges incident to the vertices e and f are used by the tour is shown in Figure 24 for a gadget of type 0 and in Figure 27 for a gadget of type 1. The case analysis for the case that the connector edges incident to the vertices e and h are used by the tour is shown in Figure 25 for a gadget of type 0 and in Figure 28 for a gadget of type 1. The cases in which the connector edges incident to the vertices c and f are used by the tour can be handled symmetrically.

Now we will show that it is impossible to construct a tour without endpoints in the gadget.

Since the tour has to enter or to leave the gadget at the vertices a and b and at the two connectors, the number of endpoints in the gadget is even.

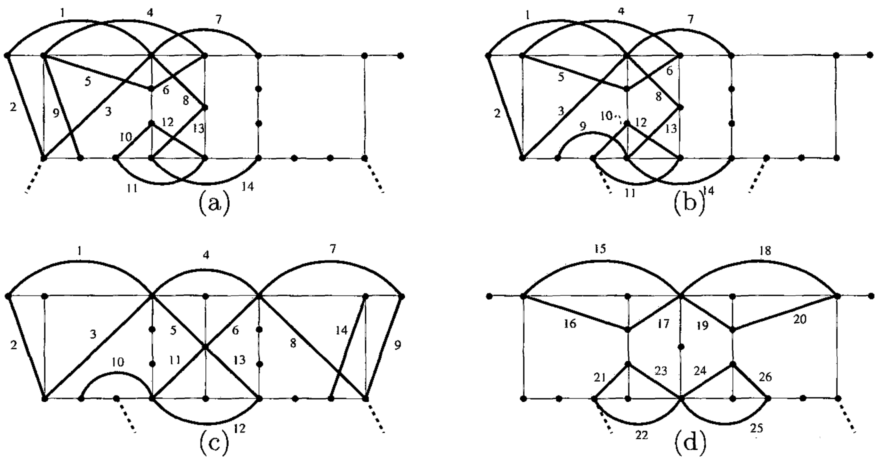

FIGURE 22. The possibilities of choosing an additional edge for a type 1 equation gadget in the proof of Lemma 3.9 are shown in (a) for the case that the connector edges incident to c and h are used, in (b) for the case that the connector edges incident to e and f are used, and in (c) and (d) for the case that the connector edges incident to e and h are used.

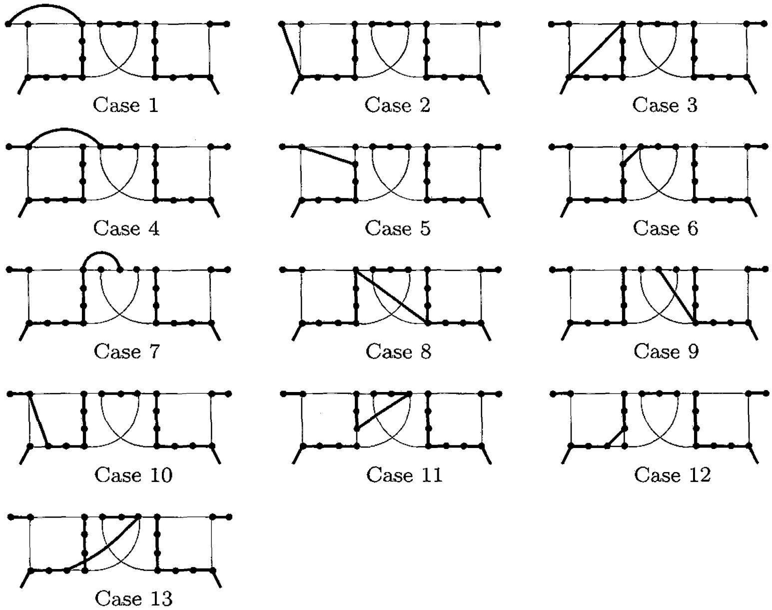

FIGURE 23. The case analysis for a type 0 equation gadget in the proof of Lemma 3.9, if the connector edges incident to c and h are used.

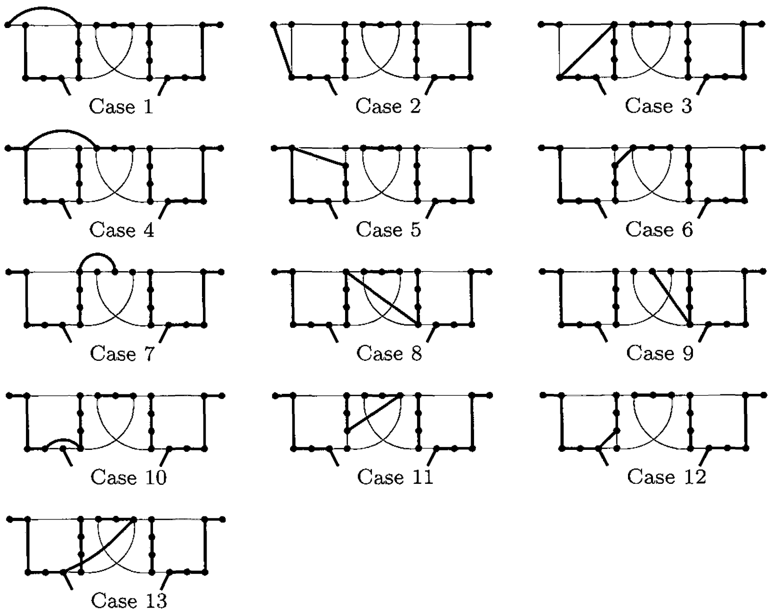

FIGURE 24. The case analysis for a type 0 equation gadget in the proof of Lemma 3.9, if the connector edges incident to e and f are used.

If not all internal edges of the connectors are used, we have at least one endpoint in the gadget, namely the vertex d or g. Together with the observation above we have at least two endpoints in the gadget. Therefore we can assume without loss of generality that all internal edges of the connectors are traversed by the tour.

Assume that there exists a Hamiltonian tour without endpoints in the gadget. If the tour enters the gadget *via* the vertex c (h respectively), it has to follow the route as shown in Figure 19a for type 0 gadgets and in Figure 20a for type 1 gadgets. Otherwise the lower neighbor of the vertex i (the lower neighbor of j respectively) would be an endpoint. If the tour enters the gadget *via* the vertex e (f respectively), it has to follow the route as shown in Figure 19c for type 0 gadgets and in Figure 20c for type 1 gadgets. This implies that at least the vertex k cannot lie on those paths of the Hamiltonian tour entering the gadget *via* the connectors. This contradicts our assumption since k has to be traversed by the tour. □

3.5. Elimination of critical pairs of endpoints at distance two

Due to the above lemmas we know that, for any Hamiltonian tour, there exists a modified tour of the same or smaller cost that does not contain a critical pair of endpoints inside the same equation gadget. Now we will show that no critical

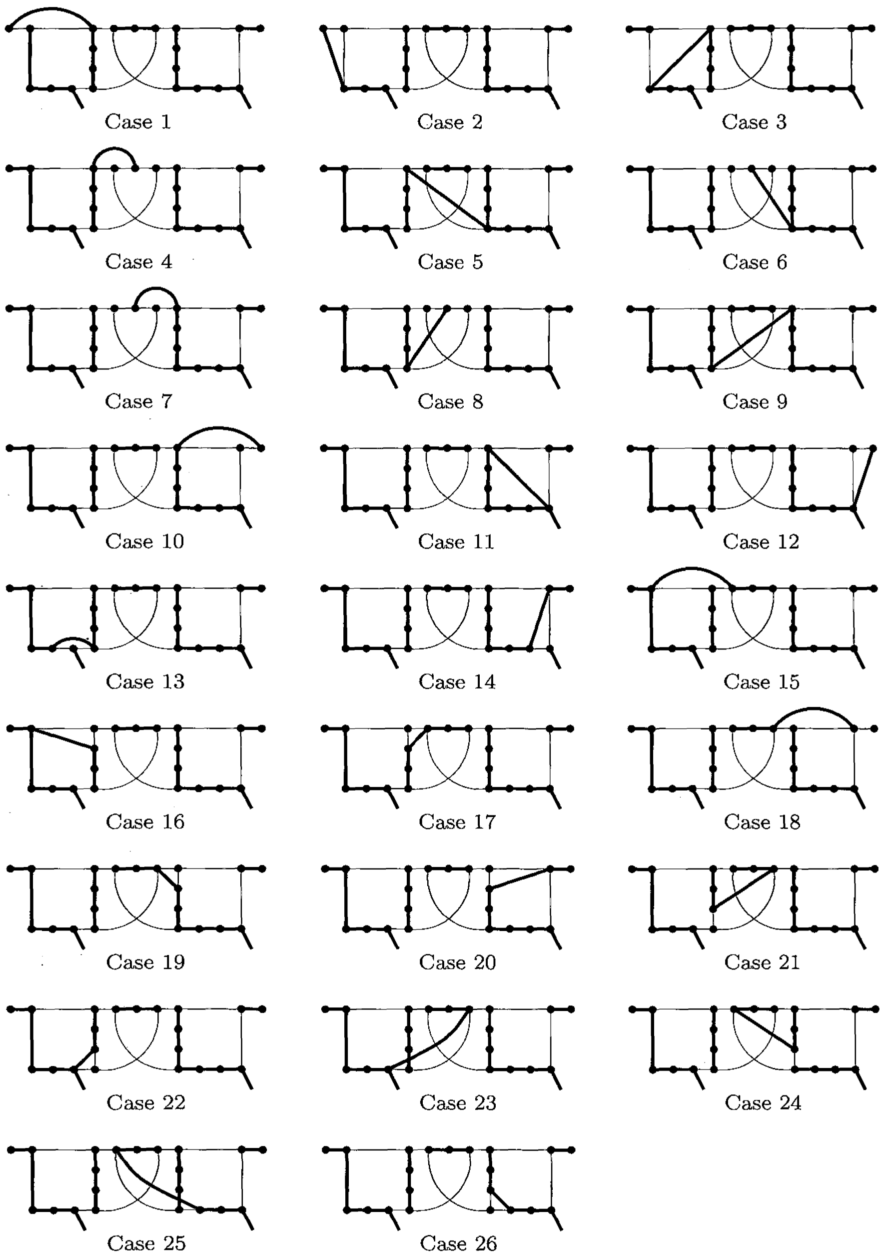

FIGURE 25. The case analysis for a type 0 equation gadget in the proof of Lemma 3.9, if the connector edges incident to e and h are used.

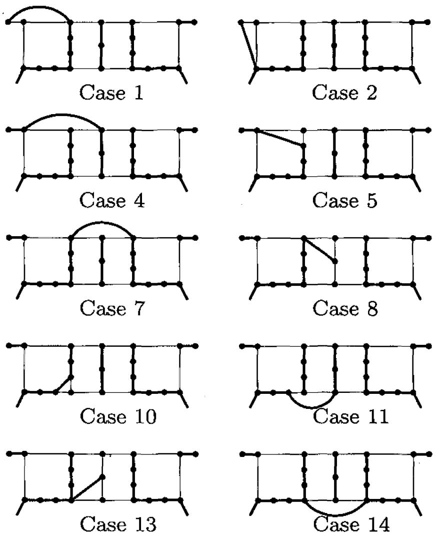

FIGURE 26. The case analysis for a type 1 equation gadget in the proof of Lemma 3.9, if the connector edges incident to c and h are used.

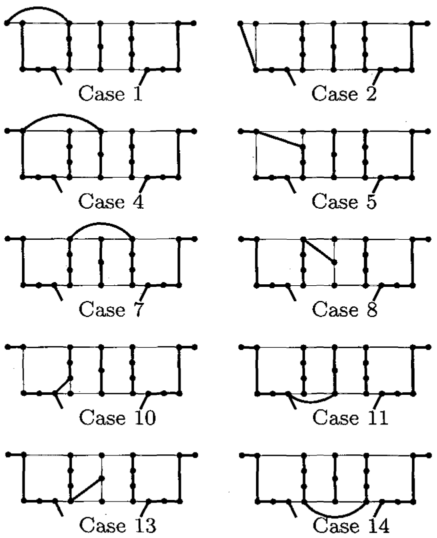

FIGURE 27. The case analysis for a type 1 equation gadget in the proof of Lemma 3.9, if the connector edges incident to e and f are used.

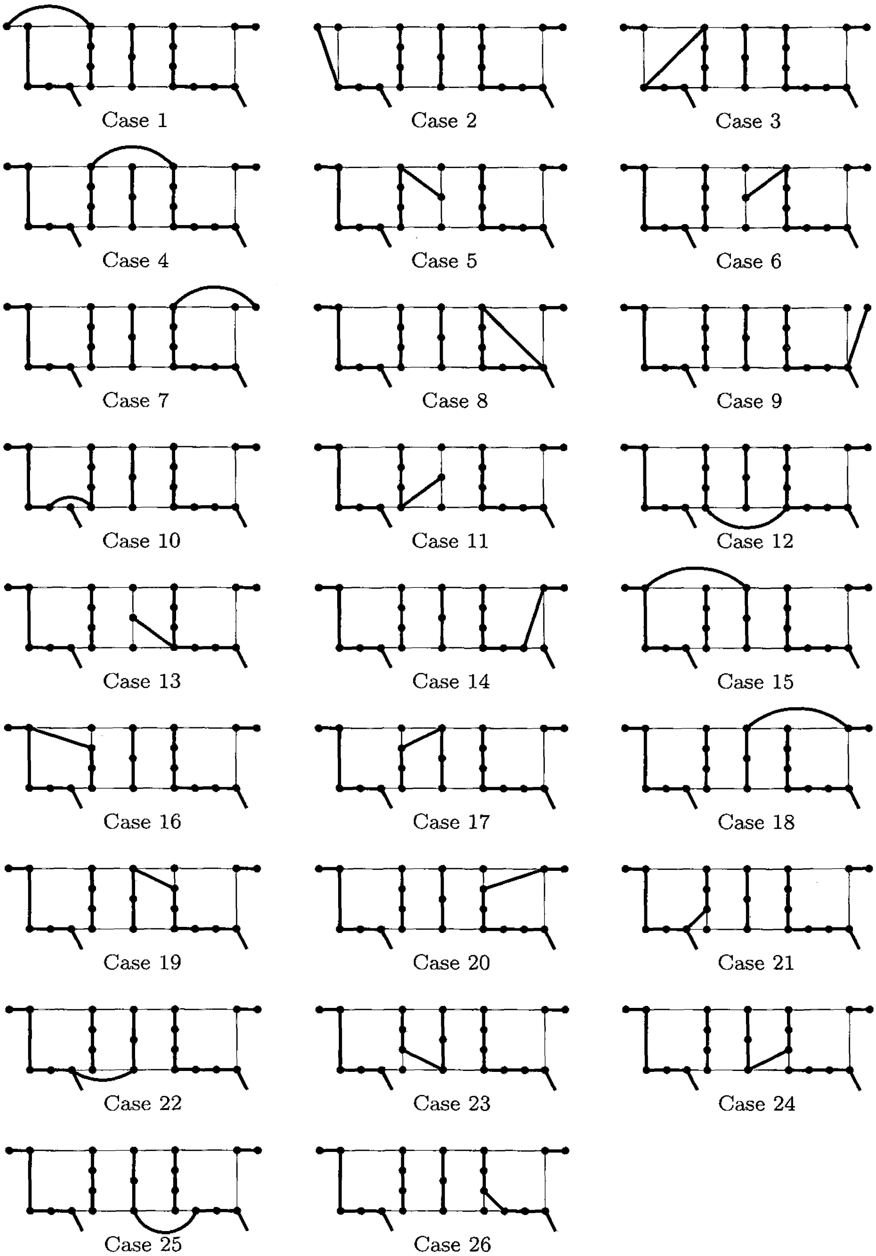

FIGURE 28. The case analysis for a type 1 equation gadget in the proof of Lemma 3.9, if the connector edges incident to e and h are used.

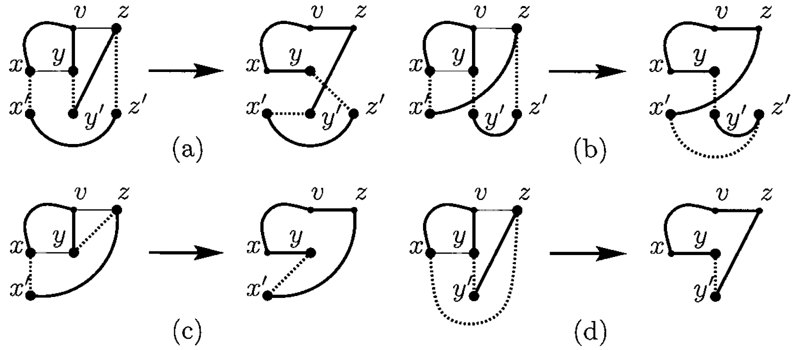

FIGURE 29. The transformations for the proof of Lemma 3.10.

pair of endpoints exists in G_0. Therefore we first prove a technical lemma about paths of the tour ending at adjacent vertices.

Lemma 3.10. *Suppose that we have a Hamiltonian tour that connects two adjacent endpoints x and y via a path in G_0 containing at least one additional neighbor v of y. If there is another endpoint z that is adjacent to v, we can modify the tour without increasing its cost such that the modified tour contains two endpoints less, unless all edges of the tour outside G_0 incident to x, y, and z are distinct and have costs of 2.*

Proof. Let x' be the neighbor of x *via* an edge of cost ≥ 2 in the tour, let y' be the neighbor of y *via* an edge of cost ≥ 2 in the tour, and let z' be the neighbor of z *via* an edge of cost ≥ 2 in the tour. If the Hamiltonian tour is of the form $xP_1vyy'P_2zz'P_3x'x$ for some paths P_1, P_2, and P_3 in G, we can perform the transformation as shown in Figure 29a, provided that at least one of the edges $\{x,x'\},\{y,y'\},\{z,z'\}$ has cost 3. If the Hamiltonian tour is of the form $xP_1vyy'P_2z'zP_3x'x$ for some paths P_1, P_2, and P_3 in G, we can perform the transformation as shown in Figure 29b. If the endpoints z and y' are identical, the tour has to be of the form $xP_1vyzP_2x'x$ for some paths P_1 and P_2 in G, and we perform the transformation as shown in Figure 29c. Finally, if the endpoints z and x' are identical, then the tour is of the form $xP_1vyy'P_2zx$ for some paths P_1 and P_2 in G, and we can perform the transformation as shown in Figure 29d. □

Now we are able to prove the following lemma:

Lemma 3.11. *Suppose that we have a Hamiltonian tour of G. Then we can modify this tour without increasing its cost such that the modified tour does not contain any critical pair of endpoints.*

Proof. By Lemmas 3.6–3.9 we know that there does not exist a critical pair of endpoints inside one equation gadget.

We will now show that, if the tour through an equation gadget contains one of the vertices in $\{c, d, e, f, g, h, l, m, n, o\}$ as an endpoint[3], then there exists a tour of the same or smaller cost in which there is no other endpoint at distance 2 to this vertex that is not connected *via* the tour. We will prove this claim by distinguishing five cases depending on whether c, d, e, l, or n is an endpoint of the tour. The remaining cases can be handled symmetrically.

Case 1: Assume that the tour contains c as an endpoint.

Let us first consider the case that the vertices c and d are both endpoints and are connected by the tour. We have to distinguish two cases.

We first assume that the connector edge incident to c is not traversed by the tour. If the path from c to d consists only of the edge $\{c, d\}$, this contradicts Lemmas 3.6, 3.7, 3.8, or 3.9, respectively, since the gadget has to contain at least one endpoint more than the number proved in these lemmas. This is immediate for the case of at most one semitraversed connector, and it is shown in Figures 30a to d for the case that both connectors are semitraversed. If the path from c to d does not consist of the edge $\{c, d\}$, we have one of the situations as shown in Figures 30e to j. In all these cases the tour needs at least three endpoints in the gadget which contradicts Lemmas 3.6, 3.7, or 3.9, respectively. In all cases not shown in Figure 30 even the existence of the two endpoints c and d contradicts the Lemmas 3.6, 3.7, or 3.8, respectively.

Let us now assume that the connector edge $\{c, x\}$ incident to c is used by the tour. The vertex x cannot be an endpoint of the tour, since it has to lie on the path from c to d. If one vertex y outside the gadget that is at distance 2 from c, *i.e.* that is a neighbor of x, is an endpoint of the tour, we can apply Lemma 3.10 to remove two endpoints from the tour. Since only one of the neighbors of x can be an endpoint of the tour (otherwise x would have to be an endpoint at distance 1 to c that is not connected to c which contradicts Lem. 3.4), the vertex c can be connected to no other endpoint at distance 2 than to y, and thus Lemma 3.10 is applicable.

Now we consider the case that there does not exist a path of the tour from c to d, *i.e.* that d is no endpoint of the tour. Then the edge $\{c, d\}$ has to be used by the tour since d is a vertex of degree 2. Furthermore the connector edge $\{c, x\}$ incident to c is not used by the tour. If a neighbor $y \neq c$ of x is an endpoint of the tour, then x has to be an endpoint, too. But this is a contradiction since in this case x would have to be connected by the tour to c as well as to y.

Thus, the claim holds for the vertex c and with an analogous argument also for the vertex h.

Case 2: Assume that the tour contains e as an endpoint.

Let us first consider the case that the connector edge $\{e, x\}$ is not used by the tour. If in this case another neighbor y of x is an endpoint, too, then also x has to

[3]Note that the vertices a and b already belong to the neighboring equation gadgets, and all other vertices of the gadget have distance ≥ 3 to vertices outside the gadget.

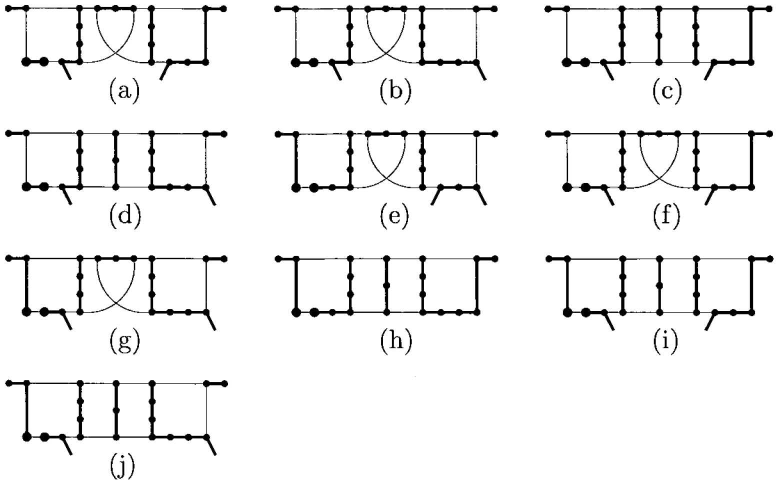

FIGURE 30. The case analysis in the proof of Lemma 3.11, if c and d are endpoints of the tour, and the connector edge incident to c is not used. The figures (a) to (d) show the cases, in which the edge $\{c, d\}$ is used, and (e) to (j) show the cases, in which the edge $\{c, d\}$ is not used by the tour.

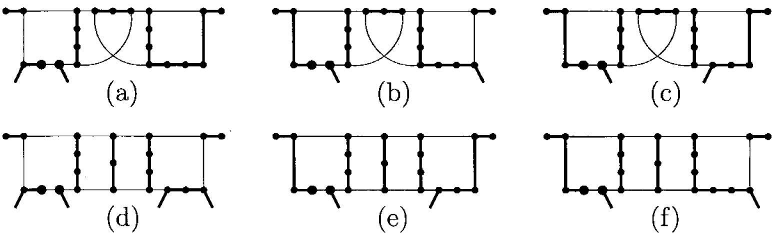

FIGURE 31. The case analysis in the proof of Lemma 3.11, if e and d are endpoints of the tour, and the connector edge incident to e is used by the tour.

be an endpoint and x has to be connected as well to e as to y by the tour which is a contradiction.

Thus, in the following we can assume that the connector edge $\{e, x\}$ is used by the tour. This implies that also d has to be an endpoint and therefore has to be connected to e *via* a path of the tour. Furthermore the edge $\{c, d\}$ has to be used since d is a vertex of degree 2. (Otherwise d would be a double endpoint which would contradict Lems. 3.6, 3.7, 3.8, or 3.9 respectively.) But this results in one of the situations as shown in Figure 31 which again contradict Lemmas 3.6, 3.7, or 3.9, respectively. In the cases not shown in Figure 31 already the existence of two endpoints inside the gadget gives a contradiction.

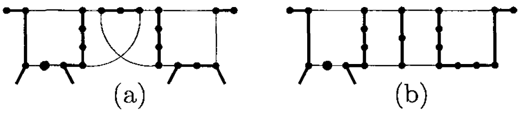

FIGURE 32. The cases in the proof of Lemma 3.11, if d is a double endpoint.

Thus, the claim holds for the vertex e and with an analogous argument also for the vertex f.

Case 3: Assume that the tour contains d as an endpoint.

Let us first assume that d is a double endpoint of the tour. Then the connector containing d has to be traversed, otherwise c or e would be another endpoint which would contradict Lemmas 3.6, 3.7, 3.8, or 3.9, respectively. The two remaining possible situations, namely a type 0 gadget with two traversed connectors and a type 1 gadget with this connector traversed and the other untraversed are treated in Figure 32, where it is shown that both cases contradict Lemma 3.6 or Lemma 3.7, respectively, since at least one additional endpoint is needed.

Therefore we can assume in the following that d is a single endpoint. If c is another endpoint of the tour, this leads to a contradiction as already shown in Figure 30. If e is another endpoint of the tour and the connector edge incident to e is used, this leads to a contradiction as shown in Figure 31. If e is another endpoint and the connector edge incident to e is not used, this again contradicts Lemmas 3.6, 3.7, 3.8, or 3.9, respectively, as shown in Figures 33a to d for the case that the edge $\{d, e\}$ is used, and in Figures 33e to j for the case that the edge $\{d, e\}$ is not used. Thus, we can assume that neither c nor e is an endpoint of the tour. We will in the following distinguish two cases depending on which of the edges incident to d is used.

Let us first consider the case that the edge $\{c, d\}$ is used. We have to show that neither the neighbor x_c of c *via* the connector edge nor the neighbor x_e of e *via* the connector edge can be an endpoint of the tour not connected to d. If x_c is an endpoint of the tour, then the edge $\{c, l\}$ has to be used, unless d and x_c are connected by the tour. Since also x_c has a degree 2 vertex y_c as a neighbor, y_c has to be also an endpoint of the tour that is furthermore connected to x_c. This situation can be transformed as shown in Figure 34a such that the critical pair of endpoints is moved to the inside of the gadget without increasing the cost of the tour and without affecting any other endpoints. But this is a contradiction to Lemma 3.8 and Lemma 3.9, since we have proved the nonexistence of a critical pair of endpoints inside one gadget there. If x_e is an endpoint of the tour, then also one of its neighbors y_e has to be an endpoint since it is a degree 2 vertex. Furthermore x_e and y_e have to be connected by the tour. Thus, we have the situation as shown in Figure 35a, and Lemma 3.10 is applicable, since it is not possible to connect the endpoint d to another endpoint than x_e *via* an edge of cost 2, since x_c cannot be an endpoint in this case as shown above.

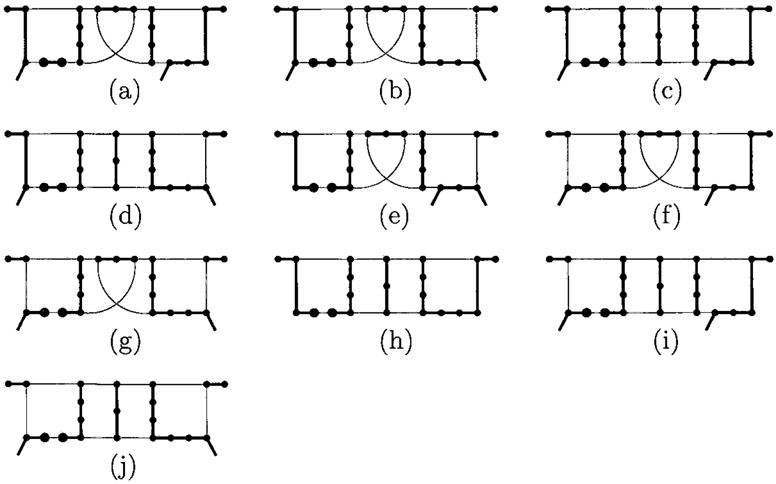

FIGURE 33. The case analysis in the proof of Lemma 3.11, if e and d are endpoints of the tour, and the connector edge incident to e is not used by the tour.

Let us now consider the case that the edge $\{d, e\}$ is used. We have again to show that neither the neighbor x_c of c *via* the connector edge nor the neighbor x_e of e *via* the connector edge can be an endpoint not connected to d. This can be shown similarly as in the preceding case. If x_e is an endpoint of the tour, then the edge $\{e, n\}$ has to be used by the tour unless d and x_e are connected by the tour. In this case the transformation shown in Figure 34b can be applied. By this transformation the critical pair of endpoints is moved to the inside of the gadget which again is contradicting Lemmas 3.8 and 3.9. To show the claim for vertex x_c we use the same argument as for the vertex x_e in the previous case: if x_c is an endpoint, then also its neighbor y_c of degree 2 has to be an endpoint connected to x_c which results in the situation as shown in Figure 35b. With the same argument as in the previous case Lemma 3.10 is applicable in this case.

Thus, the claim holds for the vertex d and with an analogous argument also for the vertex g.

Case 4: Assume that the tour contains l as an endpoint.

Let m' be the other neighbor of a, *i.e.* the vertex m in the preceding equation gadget of the chain (or the first vertex of the first variable cluster in the chain, if the considered equation gadget is the first one in the chain). Then (l, m') cannot be a critical pair of endpoints since this would imply that a is a double endpoint of the tour which would result in three endpoints inside the gadget. It remains to show that the neighbor x of c *via* the connector edge cannot be an endpoint of the tour that is not connected to l. We will distinguish two cases.

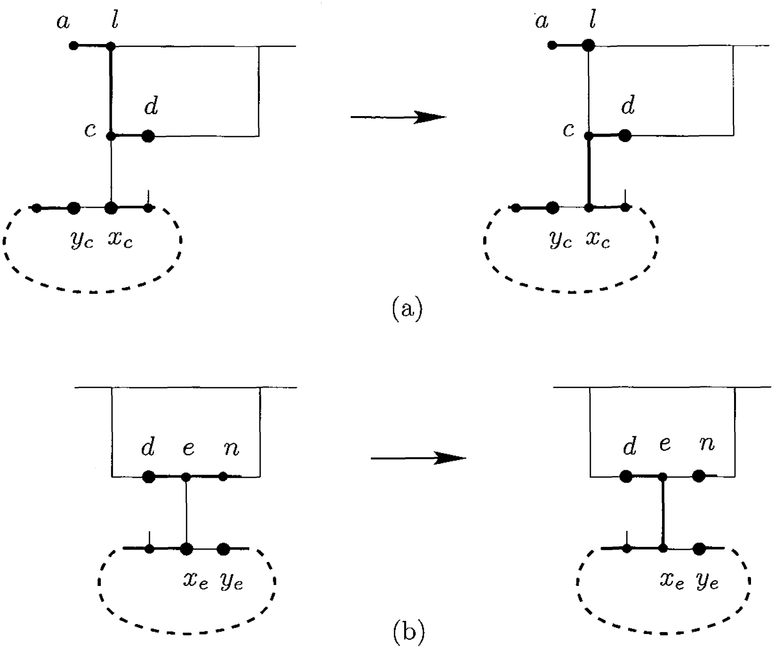

FIGURE 34. Figure (a) shows the transformation in the proof of Lemma 3.11, if d and x_c are endpoints and the edge $\{c, d\}$ is used, (b) shows the transformation, if d and x_e are endpoints and the edge $\{d, e\}$ is used.

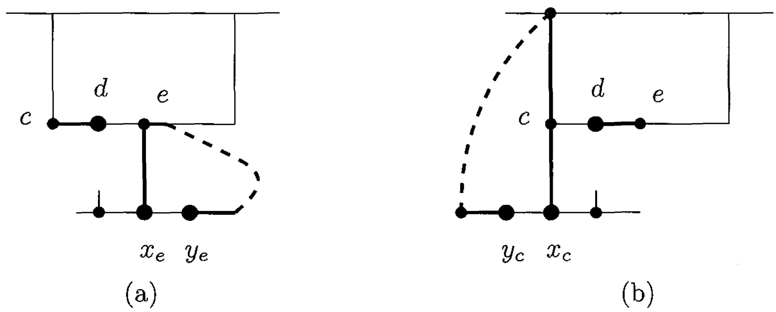

FIGURE 35. Figure (a) shows the situation in the proof of Lemma 3.11, if d and x_e are endpoints and the edge $\{c, d\}$ is used, (b) shows the situation, if d and x_c are endpoints and the edge $\{d, e\}$ is used.

Let us first assume that a is an endpoint of the tour, too. As shown in Figure 36 this leads to a contradiction to Lemmas 3.6, 3.7, 3.8, or 3.9, respectively, in all cases.

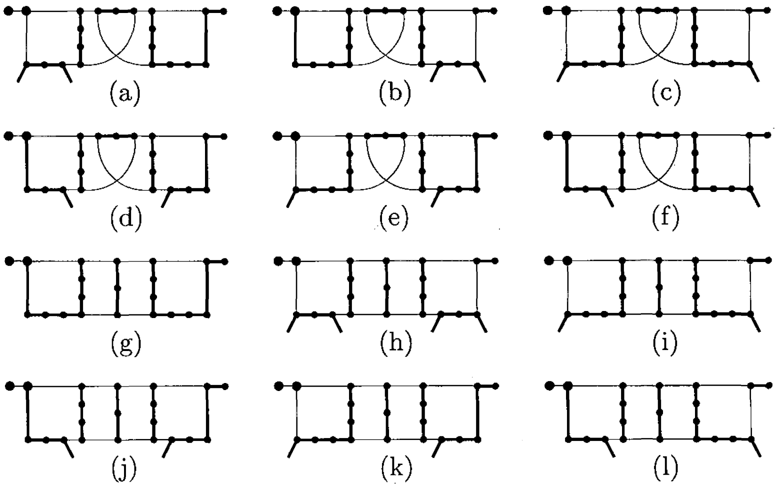

FIGURE 36. The case analysis in the proof of Lemma 3.11, if a and l are endpoints. (a–f) show the cases for a type 0 gadget depending on how the connectors are traversed, and (g–l) show the cases for a type 1 gadget.

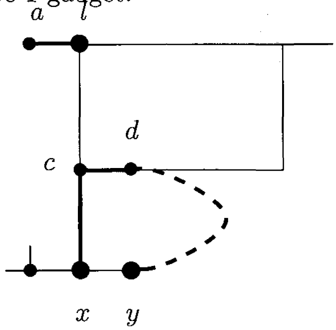

FIGURE 37. The situation in the proof of Lemma 3.11, if l and x are endpoints and the edge $\{a, l\}$ is used.

Now we assume that a is no endpoint of the tour. Then the edges $\{m', a\}$ and $\{a, l\}$ are used by the tour. If x is an endpoint of the tour, then also its neighbor y of degree 2 is an endpoint, and x and y are connected by the tour. This implies that c cannot be another endpoint since it would have to be connected to x, too. Thus, we have the situation as shown in Figure 37, and Lemma 3.10 can be applied because the only possible endpoint at distance 2 to l that is not connected to l is x[4]. Thus, the claim holds for the vertex l and with an analogous argument also for the vertex m.

Case 5: Assume that the tour contains n as an endpoint.

[4]If the vertex m of the previous gadget in the chain would be an endpoint not connected to l, then a would have to be a double endpoint at distance 1 to l contradicting Lemma 3.4.

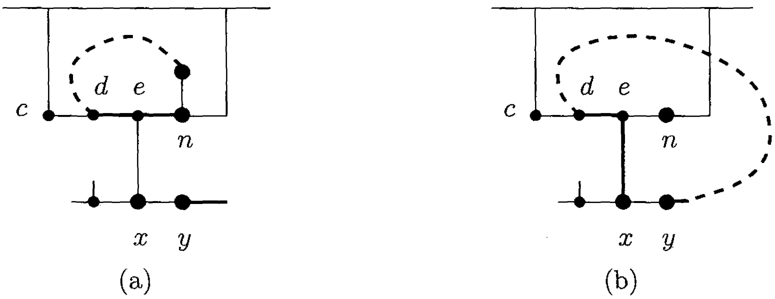

FIGURE 38. The situation in the proof of Lemma 3.11, if n and x are endpoints is shown in (a) for the case that the connector edge incident to e is not used, and in (b) for the case that this connector edge is used.

Then we have to show that the neighbor x of e *via* the connector edge cannot be an endpoint that is not connected to n. Assume that x is an endpoint. If its neighbor y of degree 2 is no endpoint of the tour then either e or the neighbor of degree 2 of n has to be an endpoint. But if e is an endpoint, then it has to be connected to both x and n which is not possible, and if the degree 2 neighbor of n is an endpoint, then Lemma 3.10 is applicable since this neighbor of n cannot be connected to an endpoint *via* an edge of cost 2. Thus, we can assume in the following that y is an endpoint of the tour, and x and y are connected by the tour. This implies that e cannot be an endpoint since it would have to be connected to x, too. We distinguish two cases depending on which of the edges incident to e are used.

Let us first assume that the connector edge $\{e, x\}$ is not used. Then the edges $\{d, e\}$ and $\{e, n\}$ are used. Therefore that neighbor of n, which has degree 2, has to be an endpoint of the tour. The resulting situation is shown in Figure 38a. Obviously, Lemma 3.10 is applicable in this case since this neighbor of n cannot be connected to an endpoint *via* an edge of cost 2.

Now we assume that the connector edge $\{e, x\}$ is used by the tour. This results in the situation shown in Figure 38b, in which again Lemma 3.10 can be applied because the only endpoint at distance 2 from n in G_0 is x.

Thus, the claim holds for the vertex n and with an analogous argument also for the vertex o.

Up till now we have shown that a critical pair of endpoints cannot contain any vertex from an equation gadget. Now it remains to show that there cannot be two endpoints at distance 2 outside the equation gadgets that are not connected by the tour. Consider two adjacent variable clusters as shown in Figure 39. Without loss of generality we can assume that a_2 is an endpoint of the tour. We have to prove that neither y_1 nor y_2 is an endpoint of the tour that is not connected to a_2. All other cases can be handled symmetrically. It is impossible that y_1 is an endpoint not connected to a_2 *via* the tour, since b_1 would also be a (possibly double) endpoint in that case that would have to be connected to both y_1 and a_2

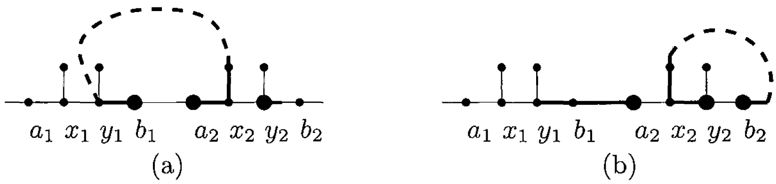

FIGURE 39. The situation in the proof of Lemma 3.11, if the considered endpoints lie outside the equation gadgets.

by the tour. If y_2 is an endpoint not connected to a_2 *via* the tour, then x_2 cannot be another endpoint since it would have to be connected to both a_2 and y_2. Thus, we have one of the situations as shown in Figure 39 depending on which of the edges incident to x_2 are used. In both cases Lemma 3.10 can be applied since there are no endpoints at distance 2 from a_2 or y_2 inside the equation gadgets as already proved above. Thus, in the case as shown in Figure 39a, y_2 is the only possible endpoint at distance 2 from a_2, and in the case as shown in Figure 39b, a_2 is the only possible endpoint at distance 2 from y_2. □

3.6. ELIMINATION OF SEMITRAVERSED CONNECTORS

In this section we will show how to eliminate semitraversed connectors from the tour.

Lemma 3.12. *Suppose that we have a Hamiltonian tour traversing a variable cluster in such a way that there are some semitraversed connectors in it. Then it is possible to modify the tour without increasing its cost in such a way that there are no semitraversed connectors in the cluster. Furthermore this transformation does not create any critical pair of endpoints.*

Proof. According to Lemma 3.8 and Lemma 3.9 we can assume that every semitraversal of a connector uses both internal edges of the connector.

When transforming a semitraversed connector into a traversed or an untraversed one we will in some cases save one endpoint in the corresponding equation gadget and in the other cases we will get an additional endpoint inside the gadget. The possible cases, depending on the type of the gadget and the traversal of the other connector of the gadget, are shown in Figure 40. In this figure, S stands for a semitraversed connector, T stands for a traversed connector, and U stands for an untraversed connector. The changes in the number of endpoints inside the gadget as claimed in Figure 40 follow directly from Lemmas 3.6, 3.7, 3.8, and 3.9.

First we consider the case that there is exactly one semitraversed connector in the variable cluster. If this is the middle connector of the cluster, we have the situation as shown in the left part of Figure 41. We can move this semitraversal to one of the outer connectors of the cluster in the following way: if the transformation of this semitraversed connector to an untraversed one saves one endpoint inside the corresponding equation gadget, we transform the tour on the cluster as shown in Figure 41a. If we have to transform the semitraversed connector to a traversed

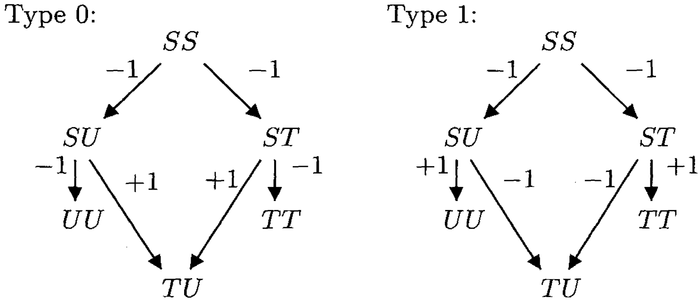

FIGURE 40. The change in the number of endpoints inside an equation gadget, if a semitraversed connector is changed into a traversed or untraversed one.

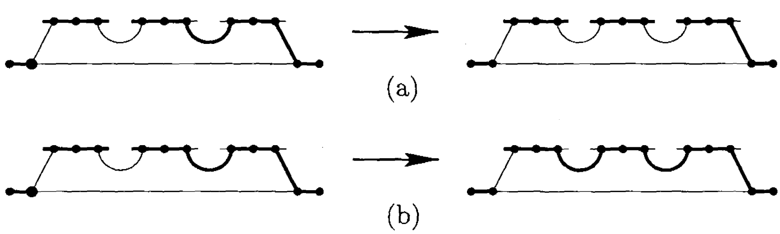

FIGURE 41. In (a) it is shown how to move the semitraversal from the middle connector of the cluster to the outer connector, if the middle connector has to get untraversed, and in (b) this move is shown for the case that the middle connector has to get traversed.

one to save one endpoint in the corresponding equation gadget, we perform the transformation as shown in Figure 41b. In both cases we have one endpoint less in the equation gadget corresponding to the middle connector of the variable cluster, but we get an additional endpoint in one of the equation gadgets corresponding to the outer connectors of the cluster. This transformation obviously does not create any critical pair of endpoints according to Lemmas 3.6, 3.7, 3.8, and 3.9. If the semitraversed connector of the cluster is one of the outer connectors, the cluster is traversed by the tour in one of the two ways shown in Figure 42. By moving the endpoint in the variable cluster into the equation gadget containing the semitraversed connector, we can make this connector untraversed in the case shown in Figure 42a or traversed in the case as shown in Figure 42b without changing the number of endpoints. According to Lemmas 3.6, 3.7, and 3.8 this transformation does not create any critical pair of endpoints.

Now we consider the case that the variable cluster contains exactly two semitraversed connectors. Then we have one of the four situations as shown in Figure 43. In the case (a) we transform the tour in such a way that all connectors of the clusters are traversed, if this saves an endpoint in the gadget containing the connector C_1. This results in at most one additional endpoint in the gadget containing C_2

FIGURE 42. The traversal of the variable cluster, if the first connector is semitraversed.

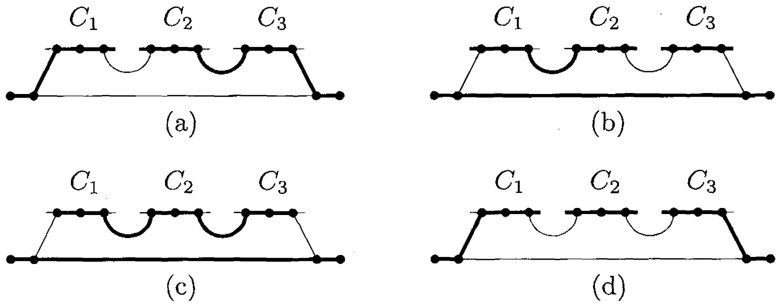

FIGURE 43. The possible traversals of the variable cluster, if exactly two connectors are semitraversed.

and therefore the number of endpoints does not increase. Otherwise we transform the tour to the one shown in the left part of Figure 41. This gives one endpoint less in the gadget containing C_1 and one additional endpoint in the cluster, and we have reduced the number of semitraversals by at least one.

In the case (b) we transform the tour into the one shown in the left part of Figure 41, if this saves an endpoint in the gadget containing the connector C_1. This results in one additional endpoint in the variable cluster, and therefore the total number of endpoints remains unchanged. Otherwise we transform the tour in such a way that all connectors become untraversed. This gives one endpoint less in the gadget containing C_1 and at most one additional endpoint in the gadget containing C_2. Thus, we have reduced the number of semitraversals by at least one without increasing the number of endpoints.

In the case (c) we transform the tour in such a way that either all connectors become traversed, if this reduces the number of endpoints in the gadget containing C_1 (C_3 respectively) by one. This leads to at most one additional endpoint in the gadget containing C_3 (C_1 respectively). Otherwise we transform the tour in such a way that all connectors are untraversed. This saves one endpoint in each of the gadgets containing C_1 or C_3, and gives at most two additional endpoints in the gadget containing C_2.

In the case (d) we transform the tour in such a way that either all connectors become untraversed, if this reduces the number of endpoints in the gadget containing C_1 (C_3 respectively) by one. This leads to at most one additional endpoint in the gadget containing C_3 (C_1 respectively). Otherwise we transform the tour in such a way that all connectors are traversed. This saves one endpoint in each

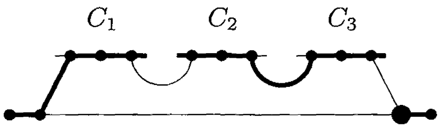

FIGURE 44. The traversal of the variable cluster, if three connectors are semitraversed.

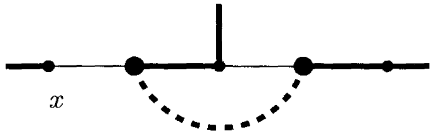

FIGURE 45. The situation in the proof of Lemma 3.13, if both endpoints lie outside the equation gadgets.

of the gadgets containing C_1 or C_3, and gives at most two additional endpoints in the gadget containing C_2.

Finally, we consider the case that the variable cluster contains three semitraversed connectors. Then we have the situation shown in Figure 44. We change the tour to that one shown in Figure 43a. This saves the endpoint in the cluster not belonging to an equation gadget and it gives at most one additional endpoint in the equation gadget containing the connector C_3.

Obviously none of these transformations creates a critical pair of endpoints. □

Note that after eliminating all semitraversals from the tour, either all connectors of a variable cluster are traversed or all connectors of the cluster are untraversed.

3.7. A Hamiltonian tour with exactly two endpoints

In this section we will deal with the case that a given Hamiltonian tour in G has exactly two endpoints. We will show that also in this case the tour does not contain an edge of cost 2 and that for each variable cluster either all connectors are traversed or all connectors are untraversed.

Lemma 3.13. *Any Hamiltonian tour in G with exactly two endpoints does not contain an edge of cost* 2.

Proof. Let H be a Hamiltonian tour in G with exactly two endpoints. We will in the sequel distinguish three cases.

Let us first assume that both endpoints lie inside the same equation gadget. Then we can show analogously to the proof of Lemmas 3.6, 3.7, 3.8, and 3.9 that the endpoints have to be at distance 3 from each other.

Secondly, we assume that both endpoints lie outside the equation gadgets at distance 2 from each other. Then we have the situation as shown in Figure 45. This immediately gives a contradiction since x has to be a vertex of degree 2 and therefore a third endpoint of the tour.

Finally, if there is an equation gadget that contains exactly one of the two endpoints of the tour, this gadget has to contain a semitraversed connector. We can reduce this case to one of the above cases by eliminating this semitraversal in the same way as in the proof of Lemma 3.12. □

Lemma 3.14. *Given a Hamiltonian tour in G with exactly two endpoints, for each variable cluster either all connectors are traversed or all connectors are untraversed.*

Proof. Let H be a Hamiltonian tour in G with exactly two endpoints. If there exists a variable cluster in G_0 for which not all connectors are traversed (not all connectors are untraversed respectively), this variable cluster has to contain a semitraversed connector. This semitraversal can be removed analogously as in the proof of Lemma 3.12. □

3.8. Construction of the assignment

In this section, we will show how to construct an assignment to the variables of the underlying LinEq2-2(3) instance from a Hamiltonian tour in G.

Lemma 3.15. *Given a Hamiltonian tour with $2e$ endpoints (where a double endpoint counts as two), we can construct an assignment to the variables of the corresponding* LinEq2-2(3) *instance leaving at most e equations unsatisfied.*

Proof. Given a Hamiltonian tour, we can by Lemma 3.12 and Lemma 3.14 construct a new tour, without increasing its cost, that does not contain two endpoints at distance 2 not connected by the tour and such that for each variable cluster either all or no connector edges are traversed. Then we can construct an assignment as follows: if the connector edges in a variable cluster are traversed by the tour, the corresponding variable is assigned the value one, otherwise it is assigned the value zero. By Lemma 3.6 and Lemma 3.7, this assignment has the property that there are two endpoints in the equation gadgets corresponding to unsatisfied equations, and zero endpoints in the equation gadgets corresponding to satisfied equations. Thus, the assignment leaves at most e equations unsatisfied, if there are $2e$ endpoints. □

Now we are ready to prove the lower bound on the approximation ratio. Remember that we have started from a $\left(\frac{331}{336} + \frac{68}{3}\varepsilon_2, \frac{331}{336} - \frac{34}{3}\varepsilon_1\right)$-LinEq2-2(3) instance with $336n$ equations.

We remind of the fact that in general, we use edge costs $1, m, l$, where in the previous description the special case of $1, 2, 3$ was used for better intuition.

Thus, if we construct an instance of $\Delta_{\{1,m,l\}}$-TSP from an LinEq2-2(3) instance as described above, the graph in the $\Delta_{\{1,m,l\}}$-TSP instance contains $68n+1$ vertices given that the LinEq2-2(3) instance contains $2n$ variables and $3n$ equations. Since the Hamiltonian tour traversing the $\Delta_{\{1,m,l\}}$-TSP instance has to take an edge of cost l between two endpoints, by Lemma 3.15 a tour with cost $68n + (l-1)e + 1$ corresponds to an LinEq2-2(3) instance with $3n$ equations from which at most e

are unsatisfied. Analogously, a tour with cost $7616n + (l-1)e + 1$ corresponds to an LinEq2-2(3) instance with $336n$ equations from which at most e are unsatisfied.

Since we take inputs from $\left(\frac{331}{336} + \frac{68}{3}\varepsilon_2, \frac{331}{336} - \frac{34}{3}\varepsilon_1\right)$-LinEq2-2(3), the minimal number e of unsatisfied equations (from $336n$) is either above $(5 - \frac{68}{3}\varepsilon_2 336)n = (5 - 7616\varepsilon_2)n$ or below $(4 + \frac{34}{3}\varepsilon_1 336)n = (4 + 3808\varepsilon_1)n$.

In the first case, the cost of an optimal tour is above $7616n + (l+1)(5 - 7616\varepsilon_2)n + 1$, that is above $7611n + 5ln - 7616\varepsilon_2 n$. In the second case, the cost of an optimal tour is below $7616n + (l+1)(4 + 3808\varepsilon_1)n + 1$, that is below $7612n + 4ln - 7616\varepsilon_1 n$ for large n.

Overall, a gap-preserving reduction from $\left(\frac{331}{336} + \frac{68}{3}\varepsilon_2, \frac{331}{336} - \frac{34}{3}\varepsilon_1\right)$-LinEq2-2(3) to $\left(\frac{7612+4l}{7616} + \varepsilon_1, \frac{7611+5l}{7616} - \varepsilon_2\right)$-$\Delta_\beta$-TSP is established.

This completes the proof of Theorem 2.3.

4. Conclusion

We have shown lower bounds on the approximability of the Δ_β-TSP for every non-trivial choice of β. In case of the metric TSP, this is an improvement over the previously known highest lower bound.

For the case of the relaxed triangle inequality, as a lower bound only the existence of a very small ε was known such that $1 + \beta\varepsilon$ is a lower bound [5]. Here, we have given the first concrete lower bound. Since this tends to $\frac{5}{4}$ for $\beta \to \infty$, one goal for future research is clearly to look for such a concrete bound which grows linearly in β.

Finally, in case of the sharpened triangle inequality, our results show the somewhat surprising fact that this special case of the TSP is $\mathcal{APX}$-hard even if one comes arbitrarily close to the trivial case of all edges having the same cost.

Very recently, after this paper was submitted, Papadimitriou and Vempala [16] showed that it is $\mathcal{NP}$-hard to approximate the Δ-TSP within $\frac{129}{128} - \varepsilon$ for an arbitrary small $\varepsilon > 0$, but it is left open whether their rather complicated construction can be transferred to the case of the Δ_β-TSP for $\beta \neq 1$.

Still, in all these cases, there is much room for improvement to close the gap between upper and lower bounds.

References

[1] T. Andreae and H.-J. Bandelt, Performance guarantees for approximation algorithms depending on parametrized triangle inequalities. *SIAM J. Discrete Math.* **8** (1995) 1–16.

[2] S. Arora, Polynomial Time Approximation Schemes for Euclidean Traveling Salesman and Other Geometric Problems. *J. ACM* **45** (1998) 753–782.

[3] S. Arora, Nearly linear time approximation schemes for Euclidean TSP and other geometric problems, in *Proc. 38th Ann. Symp. on Foundations of Comp. Sci. (FOCS '97)*, IEEE (1997) 554–563.

[4] S. Arora and C. Lund, Hardness of Approximations, Chapter 10 of [12], pp. 399–446.

[5] M.A. Bender and C. Chekuri, Performance guarantees for the TSP with a parameterized triangle inequality. *Inform. Process. Lett.* **73** (2000) 17–21.

[6] H.-J. Böckenhauer, J. Hromkovič, R. Klasing, S. Seibert and W. Unger, Towards the Notion of Stability of Approximation Algorithms and the Traveling Salesman Problem (Extended Abstract), edited by G.C. Bongiovanni, G. Gambosi and R. Petreschi, *Algorithms and Complexity, Proc. 4th Italian Conference CIAC 2000.* Springer, *Lecture Notes in Comput. Sci.* **1767** (2000) 72–86. (Full version in: *Electronic Colloquium on Computational Complexity* (http://www.eccc.uni-trier.de/eccc/), Report No. 31 (1999).)

[7] H.-J. Böckenhauer, J. Hromkovič, R. Klasing, S. Seibert and W. Unger, An Improved Lower Bound on the Approximability of Metric TSP and Approximation Algorithms for the TSP with Sharpened Triangle Inequality (Extended Abstract), edited by H. Reichel and S. Tison, *STACS 2000, Proc. 17th Ann. Symp. on Theoretical Aspects of Comp. Sci.*, Springer, *Lecture Notes in Comput. Sci.* **1770** (2000) 382–394.

[8] H.-J. Böckenhauer, J. Hromkovič, R. Klasing, S. Seibert and W. Unger, Approximation Algorithms for the TSP with Sharpened Triangle Inequality. *Inform. Process. Lett.* **75** (2000) 133–138.

[9] P. Berman and M. Karpinski, On some tighter inapproximability results. Technical Report TR98-029, *Electronic Colloquium on Computational Complexity* (1998) http://www.eccc.uni-trier.de/eccc/

[10] N. Christofides, *Worst-case analysis of a new heuristic for the traveling salesman problem*, Technical Report 388. Graduate School of Industrial Administration, Carnegie-Mellon University, Pittsburgh (1976).

[11] L. Engebretsen, An explicit lower bound for TSP with distances one and two. Extended abstract, edited by C. Meinel and S. Tison, *STACS 99, Proc. 16th Ann. Symp. on Theoretical Aspects of Comp. Sci.* Springer, *Lecture Notes in Comput. Sci.* **1563** (1999) 373–382. Full version in: *Electronic Colloquium on Computational Complexity* (http://www.eccc.uni-trier.de/eccc/), Revision 1 of Report No. 46 (1999).

[12] D.S. Hochbaum, *Approximation Algorithms for NP-hard Problems.* PWS Publishing Company (1996).

[13] E.L. Lawler, J.K. Lenstra, A.H.G. Rinnooy Kan and D.B. Shmoys, *The Traveling Salesman Problem.* John Wiley & Sons (1985).

[14] I.S.B. Mitchell, Guillotine subdivisions approximate polygonal subdivisions: Part II – a simple polynomial-time approximation scheme for geometric k-MST, TSP and related problems. Technical Report, Dept. of Applied Mathematics and Statistics, Stony Brook (1996).

[15] E.W. Mayr, H.J. Prömel and A. Steger, *Lectures on Proof Verification and Approximation Algorithms.* Springer, *Lecture Notes in Comput. Sci.* **1967** (1998).

[16] Ch. Papadimitriou and S. Vempala, On the approximability of the traveling salesman problem, in *Proc. 32nd Ann. Symp. on Theory of Comp. (STOC '00)*, ACM (2000).

[17] Ch. Papadimitriou and M. Yannakakis, The traveling salesman problem with distances one and two. *Math. Oper. Res.* **18** (1993) 1–11.

Communicated by J. Hromkovič.
Received May 21, 2000. Accepted August 24, 2000.